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Fourier Truncation Regularization Method for a Three-Dimensional Cauchy Problem of the Modified Helmholtz Equation with Perturbed Wave Number

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Abstract: In this paper, the Cauchy problem of the modified Helmholtz equation (CPMHE) with perturbed wave number is considered. In the sense of Hadamard, this problem is severely ill-posed. The Fourier truncation regularization method is used to solve this Cauchy problem. Meanwhile, the corresponding error estimate between the exact solution and the regularized solution is obtained. A numerical example is presented to illustrate the validity and effectiveness of our methods.

Keywords: modified Helmholtz equation; ill-posed problem; Cauchy problem; Fourier truncation method

MSC: 35R25; 47A52; 35R30

1. Introduction

Modified Helmholtz equation which is pointed out in [1] usually occurs in applications of physics and engineering, such as wave propagation and scattering [2], structural vibration [3], implicit marching schemes for the heat equation [1], the Navier–Stokes equations [4], etc. Regarding the inverse problem for the modified Helmholtz equation, there are a lot of research results. In [5,6] the authors used the simplified Tikhonov regularization method and the quasi-reversibility regularization method to identify the unknown source of the modified Helmholtz equation. The Cauchy problems for the modified Helmholtz equation is typical ill-posed problem according to the sense of Hadamard. Many researchers have used lots of regularization solutions to solve it, such as the Landweber method [7], the method of fundamental solutions (MFS) [8–10], the plane wave method [11], the Tikhonov type regularization method [12], a fourth-order modified method [13] and the quasi-boundary value method [14,15]. From these references, there are two defects: one is that the equation is homogeneous; the other is that the measurable data is only one. In [16], the authors used the truncation method for the cauchy problem of Helmholtz equation. In [17], the authors used the truncation method for the cauchy problem of Helmholtz equation using three measurable data. However, in [16,17], the authors considered the Cauchy problem of the Helmholtz equation.

In this work, we consider a three-dimensional Cauchy problem of the modified Helmholtz equation with perturbed wave number as follows:

$$\begin{cases} \Delta u(x,y,z) - k^2 u(x,y,z) = S(x,y,z), & (x,y)\in\mathbb{R}^2, z\in(0,1), \\ u_z(x,y,0) = f(x,y), & (x,y)\in\mathbb{R}^2, \\ u(x,y,0) = g(x,y), & (x,y)\in\mathbb{R}^2, \end{cases} \tag{1}$$

where Δ is the Laplace operator, $(f, g, k) \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times \mathbb{R}^+$ *and* $S \in L^2(\mathbb{R}^2 \times (0,1))$ are given data. However, $g(x,y)$, $f(x,y)$ and k can be measured, there exists measured value with error. The measure data are g_ε, f_ε and k_ε, and satisfy

$$\|g_\varepsilon - g\|_{L^2(\mathbb{R}^2)} \le \varepsilon,\ \|f_\varepsilon - f\|_{L^2(\mathbb{R}^2)} \le \varepsilon,\ |k_\varepsilon - k| \le \varepsilon, \tag{2}$$

where $\|\cdot\|$ denotes $L^2(R^2)$ norm and $\varepsilon > 0$ is a noise level.

We solve the ill-posed problem (1) by using the Fourier truncation regularization method in the Fourier domain. Let $\widehat{f}(\xi)$ denote the Fourier transform of the function $f(x,y)$ which is defined by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{-i(\xi_1 x + \xi_2 y)} f(x,y) dxdy,\ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{3}$$

We assume the exact data (f_{ex}, g_{ex}, k_{ex}) has the following priori bound:

Assume five positive numbers A, B, C, D, E satisfy

$$A \le k_{ex} \le B,\ \int_{R^2} |\widehat{g}_{ex}|^2 d\xi \le C^2,\ \int_{R^2} |\widehat{f}_{ex}|^2 d\xi \le D^2,\ \int_{R^2} (\int_0^1 \widehat{S}(\xi, z) dz)^2 d\xi \le E^2. \tag{4}$$

Recently, Fourier regularization method has been effectively applied to solve different inverse problem: The sideways heat equation [18,19], a more general sideways parabolic equation [20], numerical differentiation [21], a posteriori Fourier method for solving ill-posed problems [22], the unknown source in the Poisson equation [23], the time-dependent heat source for heat equation [24], the heat source problem for time fractional diffusion equation [25,26], the semi-linear backward parabolic problems [27], the unknown source for time-fractional diffusion equation in bounded domain [28], the Cauchy problem for the Helmholtz equation [29], the a posteriori truncation method for some Cauchy problems associated with Helmholtz-type equations [30], the Cauchy problem of the inhomogeneous Helmholtz equation [31].

This paper is organized as follows. Some auxiliary results are given in Section 2. In Section 3, the Fourier truncation regularization method is used to solve this problem and the Hölder error estimate between the exact solution and the regularized solution are obtained. An example is given in Section 4 which is used to show our method effective. There is a brief conclusion in Section 5.

2. Some Auxiliary Results

In this section, we give some lemmas that are essential to prove our main conclusion.

Lemma 1. *For* $\alpha \in (0,1)$, $b > 1$ *and* $0 < x < \frac{1}{b} ln \frac{4}{\sqrt{\alpha}}$, *there holds:*

$$(i)\ 0 < coshzx \le 4^{\frac{z-b}{b}} (1 - \frac{\alpha}{4} e^{bx})^{-1} \delta(b, \alpha, z) \alpha^{-\frac{z}{b}},$$

$$(ii)\ 0 < \frac{sinhzx}{x} \le 4^{\frac{z-b}{b}} (1 - \frac{\alpha}{4} e^{bx})^{-1} \delta(b, \alpha, z) \alpha^{-\frac{z}{b}},$$

where $\delta(b, \alpha, z) = (4 - \alpha)^{\frac{b-z}{b}}$, $\forall z \in [0,1]$.

Proof. For $0 < x < \frac{4}{\alpha}$, we can get $1 - \frac{\alpha}{4} x \ge 0$. For all $m, n \ge 0$, using the inequality $mn \le (\frac{m+n}{2})^2$, we obtain

$$\frac{\alpha}{4} x (1 - \frac{\alpha}{4} x) \le (\frac{\frac{\alpha}{4} x + 1 - \frac{\alpha}{4} x}{2})^2,$$

then we can get

$$x(1 - \frac{\alpha}{4} x) \le \alpha^{-1},\ 0 < x < \frac{4}{\alpha}. \tag{5}$$

For $0 < x < \frac{1}{b} ln \frac{4}{\sqrt{\alpha}}$ *and* $\alpha \in (0,1)$, we obtain $1 - \frac{\alpha}{4} e^{bx} > 0$.

Using inequalities $coshzx \leq e^{zx}$, $\frac{sinhzx}{x} \leq e^{zx}$, we obtain $(1 - \frac{\alpha}{4} e^{bx}) coshzx \leq (1 - \frac{\alpha}{4} e^{bx}) e^{zx}$ and $(1 - \frac{\alpha}{4} e^{bx}) \frac{sinhzx}{x} \leq (1 - \frac{\alpha}{4} e^{bx}) e^{zx}$.

According to (5), we obtain

$$\begin{aligned} (1 - \frac{\alpha}{4} e^{bx}) coshzx &\leq (1 - \frac{\alpha}{4} e^{bx}) e^{zx} \\ &= (1 - \frac{\alpha}{4} e^{bx})^{\frac{b-z}{b}} [e^{bx}(1 - \frac{\alpha}{4} e^{bx})]^{\frac{z}{b}} \\ &\leq 4^{\frac{z-b}{b}} (4-\alpha)^{\frac{b-z}{b}} \alpha^{-\frac{z}{b}}. \end{aligned}$$

Then, we get

$$coshzx \leq 4^{\frac{z-b}{b}} (1 - \frac{\alpha}{4} e^{bx})^{-1} \delta(b, \alpha, z) \alpha^{-\frac{z}{b}},$$

$$\frac{sinhzx}{x} \leq 4^{\frac{z-b}{b}} (1 - \frac{\alpha}{4} e^{bx})^{-1} \delta(b, \alpha, z) \alpha^{-\frac{z}{b}},$$

where $\delta(b, \alpha, z) = (4-\alpha)^{\frac{b-z}{b}}$, $\forall z \in [0,1]$.

□

Lemma 2 ([21])**.** *Let* $b > a > 0$, $\varphi_1(x) = coshzx$ *and* $\psi_1(x) = \frac{sinhzx}{x}$, $z \in [0,1]$. *The following inequalities hold:*

$$|cosh(zb) - cosh(za)| \leq (b-a) cosh(zc_1),$$

$$|\frac{sinh(zb)}{b} - \frac{sinh(za)}{a}| \leq (b-a) \frac{sinh(zc_2)}{c_2},$$

in which $c_1, c_2 \in [a, b]$.

3. Fourier Truncation Regularization Method and Error Estimate

Using the Fourier transform, we obtain the solution of problem (1) according to (f_{ex}, g_{ex}, k_{ex}) and $(f_{ex}, g_{ex}, k_\varepsilon)$, respectively, in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times \mathbb{R}^+$ as follows:

$$\widehat{u}^{ex}_{(f_{ex}, g_{ex}, k_{ex})}(\xi, z) = \widehat{g}_{ex}(\xi) cosh(zr_{ex}) + \widehat{f}_{ex}(\xi) \frac{sinh(zr_{ex})}{r_{ex}} + \int_0^z \frac{\widehat{S}(\xi, t) sinh((z-t) r_{ex})}{r_{ex}} dt, \tag{6}$$

$$\widehat{u}^{ex}_{(f_{ex}, g_{ex}, k_\varepsilon)}(\xi, z) = \widehat{g}_{ex}(\xi) cosh(zr_\varepsilon) + \widehat{f}_{ex}(\xi) \frac{sinh(zr_\varepsilon)}{r_\varepsilon} + \int_0^z \frac{\widehat{S}(\xi, t) sinh((z-t) r_\varepsilon)}{r_\varepsilon} dt, \tag{7}$$

We get the regularization solution of (1) by using the Fourier truncation regularization method as follows:

$$\widehat{u}^{re}_{(f_{ex}, g_{ex}, k_\varepsilon)}(\xi, z) = \Big(\widehat{g}_{ex}(\xi) cosh(zr_\varepsilon) + \widehat{f}_{ex}(\xi) \frac{sinh(zr_\varepsilon)}{r_\varepsilon} + \int_0^z \frac{\widehat{S}(\xi, t) sinh((z-t) r_\varepsilon)}{r_\varepsilon} dt\Big) \chi_{max}, \tag{8}$$

$$\widehat{u}^{re}{}_{(f_\varepsilon, g_\varepsilon, k_\varepsilon)}(\xi, z) = \Big(\widehat{g}_\varepsilon(\xi) cosh(zr_\varepsilon) + \widehat{f}_\varepsilon(\xi) \frac{sinh(zr_\varepsilon)}{r_\varepsilon} + \int_0^z \frac{\widehat{S}(\xi, t) sinh((z-t) r_\varepsilon)}{r_\varepsilon} dt\Big) \chi_{max}, \tag{9}$$

where $\mathcal{X}_{max}$ denotes the norm in domain $[-\xi_{max}, \xi_{max}]$ defined by

$$\mathcal{X}_{max} = \begin{cases} 1, |\ \xi\ | \leq \xi_{max}, \\ 0, |\ \xi\ | \geq \xi_{max}. \end{cases} \tag{10}$$

Now we give the error estimate between the regularization solution and the exact solution.

Theorem 1. *Let $\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\xi,z)$ be given by (6), let $\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\xi,z)$ be given by (7). The corresponding regularization solutions are given by (8) and (9). Let (f_{ex},g_{ex},k_{ex}) satisfy (4). Choosing*

$$\xi_{max}=\sqrt{(ln\frac{1}{\varepsilon})^2-k_\varepsilon^2}, \tag{11}$$

we will obtain the following error estimate:

$$\|u^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\|\le\sqrt{2}\varepsilon^{2-z}+F\varepsilon^b, \tag{12}$$

where $F=(2\sqrt{6}+2\sqrt{3}+6)\delta(b,\alpha,z)\alpha^{-\frac{z}{b}}\sqrt{C^2+D^2+E^2}$.

Proof. By the triangle inequality and Parseval formula, we obtain

$$\begin{aligned}&\|u^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\|\\=&\|\widehat{u}^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\|\\\le&\|\widehat{u}^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-\widehat{u}^{re}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\|+\|\widehat{u}^{re}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\|\\&+\|\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\|.\end{aligned} \tag{13}$$

We firstly give an estimate for the first term. Due to (8) and (9) and Lemma 1, we have

$$\begin{aligned}&\|\widehat{u}^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-\widehat{u}^{re}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\|^2\\=&\int_{R^2}|(\widehat{g}_{ex}(\xi)cosh(zr_{ex})+\widehat{f}_{ex}(\xi)\frac{sinh(zr_{ex})}{r_{ex}}+\int_0^z\frac{\widehat{S}(\xi,t)sinh((z-t)r_{ex})}{r_{ex}}dt)\chi_{max}\\&-(\widehat{g}_{ex}(\xi)cosh(zr_{ex})+\widehat{f}_{ex}(\xi)\frac{sinh(zr_{ex})}{r_{ex}}+\int_0^z\frac{\widehat{S}(\xi,t)sinh((z-t)r_{ex})}{r_{ex}}dt)|^2d\xi\\=&\int_{|\xi|<\xi_{max}}|cosh(zr_\varepsilon)(\widehat{g}_\varepsilon(\xi)-\widehat{g}_{ex}(\xi))+\frac{sinh(zr_\varepsilon)}{r_\varepsilon}(\widehat{f}_\varepsilon(\xi)-\widehat{f}_{ex}(\xi))|^2d\xi\\&+\int_{|\xi|>\xi_{max}}(\widehat{g}_{ex}(\xi)cosh(zr_{ex})+\widehat{f}_{ex}(\xi)\frac{sinh(zr_{ex})}{r_{ex}}+\int_0^z\frac{\widehat{S}(\xi,t)sinh((z-t)r_{ex})}{r_{ex}}dt)^2d\xi\\\le&max\{\sup_{|\xi|<\xi_{max}}\{cosh(zr_\varepsilon)\}^2,\sup_{|\xi|<\xi_{max}}\{\frac{sinh(zr_\varepsilon)}{r_\varepsilon}\}^2\}\int_{-\xi_{max}}^{\xi_{max}}(|\widehat{g}_\varepsilon(\xi)-\widehat{g}_{ex}(\xi)|+|\widehat{f}_\varepsilon(\xi)-\widehat{f}_{ex}(\xi)|)^2d\xi\\&+2max\{\sup_{|\xi|>\xi_{max}}(cosh(zr_{ex}))^2,\sup_{|\xi|>\xi_{max}}(\frac{sinh(zr_{ex})}{r_{ex}})^2\}\int_{|\xi|>\xi_{max}}|\widehat{g}_{ex}(\xi)+\widehat{f}_{ex}(\xi)+\int_0^z\widehat{S}(\xi,t)dt|^2d\xi\\\le&\sup_{|\xi|<\xi_{max}}e^{2zr_\varepsilon}\cdot2(\|\widehat{g}_\varepsilon-\widehat{g}_{ex}\|^2+\|\widehat{f}_\varepsilon-\widehat{f}_{ex}\|^2)\\&+6(\sup_{|\xi|>\xi_{max}}4^{2\frac{z-b}{b}}(1-\frac{\alpha}{4}e^{br_{ex}})^{-2}\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}}(\int_{|\xi|>\xi_{max}}|\widehat{g}_{ex}(\xi)|^2d\xi+\int_{|\xi|>\xi_{max}}|\widehat{f}_{ex}(\xi)|^2d\xi\\&\qquad+\int_{|\xi|>\xi_{max}}[\int_0^z|\widehat{S}(\xi,t)dt]^2d\xi)\\\le&2e^{2z\sqrt{|\xi_{max}|^2+k_\varepsilon^2}}\varepsilon^2+24e^{-2b\sqrt{|\xi_{max}|^2+k_{ex}^2}}\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}}(C^2+D^2+E^2)\\\le&2e^{2z\sqrt{|\xi_{max}|^2+k_\varepsilon^2}}\varepsilon^2+24e^{-2b\sqrt{|\xi_{max}|^2+k_\varepsilon^2}}\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}}(C^2+D^2+E^2).\end{aligned}$$

Hence, we get

$$\|\widehat{u}^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-\widehat{u}^{re}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\|\le\sqrt{2}e^{z\sqrt{|\xi_{max}|^2+k_\varepsilon^2}}\varepsilon^2+2\sqrt{6}e^{-b\sqrt{|\xi_{max}|^2+k_\varepsilon^2}}\delta(b,\alpha,z)\alpha^{-\frac{z}{b}}\sqrt{C^2+D^2+E^2}. \tag{14}$$

Then, we will give an estimate for the second term. Combining (7) and (8), Lemma 1 and using the inequalities $coshzx \le e^{zx}$, $\frac{sinhzx}{x} \le e^{zx}$, $(a+b)^2 \le 2(a^2+b^2)$, $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ where $x > 0, 0 \le z \le 1$, we have

$$
\begin{aligned}
&\|\widehat{u}^{re}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\|^2 \\
&= \int_{-\infty}^{\infty} |(\widehat{g}_{ex}(\xi)cosh(zr_\varepsilon) + \widehat{f}_{ex}(\xi)\frac{sinh(zr_\varepsilon)}{r_\varepsilon} + \int_0^z \frac{\widehat{S}(\xi,t)sinh((z-t)r_\varepsilon)}{r_\varepsilon}dt|^2(1-\mathcal{X}_{max})d\xi \\
&= \int_{|\xi|>\xi_{max}} |(\widehat{g}_{ex}(\xi)cosh(zr_\varepsilon) + \widehat{f}_{ex}(\xi)\frac{sinh(zr_\varepsilon)}{r_\varepsilon} + \int_0^z \frac{\widehat{S}(\xi,t)sinh((z-t)r_\varepsilon)}{r_\varepsilon}dt|^2 d\xi \\
&\le max\{\sup_{|\xi|>\xi_{max}} \{cosh(zr_\varepsilon)\}^2, \sup_{|\xi|>\xi_{max}} \{\frac{sinh(zr_\varepsilon)}{r_\varepsilon}\}^2\} \int_{|\xi|>\xi_{max}} |\widehat{g}_{ex}(\xi) + \widehat{f}_{ex}(\xi) + \int_0^z \widehat{S}(\xi,t)dt|^2 d\xi \\
&\le 3 \sup_{|\xi|>\xi_{max}} (4^{2\frac{z-b}{b}}(1-\frac{\alpha}{4}e^{br_\varepsilon})^{-2}\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}})(\int_{|\xi|>\xi_{max}} |\widehat{g}_{ex}(\xi)|^2 d\xi + \int_{|\xi|>\xi_{max}} |\widehat{f}_{ex}(\xi)|^2 d\xi \\
&\quad + \int_{|\xi|>\xi_{max}} [\int_0^z |\widehat{S}(\xi,t)\frac{e^{(z-t)r_{ex}}}{r_{ex}}dt]^2 d\xi) \\
&\le 12e^{-2b\sqrt{|\xi_{max}|^2+k_\varepsilon^2}})\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}}(C^2+D^2+E^2).
\end{aligned}
$$

Hence

$$
\|\widehat{u}^{re}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\| \le \sqrt{12}e^{-b\sqrt{|\xi_{max}|^2+k_\varepsilon^2}})\delta(b,\alpha,z)\alpha^{-\frac{z}{b}}\sqrt{C^2+D^2+E^2}. \quad (15)
$$

Now we estimate the third term. By using (6) and (7), Lemma 2, $(a+b)^2 \le 2(a^2+b^2)$ and $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we have

$$
\begin{aligned}
&\|\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\|^2 \\
&= \int_{-\infty}^{\infty} |(\widehat{g}_{ex}(\xi)(cosh(zr_\varepsilon) - cosh(zr_{ex})) + \widehat{f}_{ex}(\xi)(\frac{sinh(zr_\varepsilon)}{r_\varepsilon} - \frac{sinh(zr_{ex})}{r_{ex}}) \\
&+ \int_0^z \widehat{S}(\xi,t)(\frac{sinh((z-t)r_\varepsilon)}{r_\varepsilon} - \frac{sinh((z-t)r_{ex})}{r_{ex}})dt|^2 d\xi \\
&\le \int_{-\infty}^{\infty} |\widehat{g}_{ex}(\xi)(r_\varepsilon - r_{ex})cosh(zc_1) + \widehat{f}_{ex}(\xi)(r_\varepsilon - r_{ex})\frac{sinh(zc_2)}{c_2} \\
&+ \int_0^z \widehat{S}(\xi,t)(r_\varepsilon - r_{ex})\frac{sinh((z-t)c_3)}{c_3}dt|^2 d\xi \\
&\le (r_\varepsilon - r_{ex})^2(\int_{-\infty}^{\infty} |\widehat{g}_{ex}(\xi)cosh(zr_\varepsilon) + \widehat{f}_{ex}(\xi)\frac{sinh(zr_\varepsilon)}{r_\varepsilon} + \frac{sinh((zr_\varepsilon)}{r_\varepsilon}\int_0^z \widehat{S}(\xi,t)dt|^2 d\xi \\
&\le 36e^{-2b\sqrt{|\xi_{max}|^2+k_\varepsilon^2}})\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}}(C^2+D^2+E^2).
\end{aligned}
$$

Hence, according to $\sqrt{a^2+b^2} \le a+b$, we obtain

$$
\begin{aligned}
&\|\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\|_{L^2(\mathbb{R}^2)} \\
&\le 6e^{-b\sqrt{|\xi_{max}|^2+k_\varepsilon^2}})\delta(b,\alpha,z)\alpha^{-\frac{z}{b}}\sqrt{C^2+D^2+E^2}.
\end{aligned} \quad (16)
$$

Using (14)–(16), we can obtain

$$
\|u^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\| \le \sqrt{2}e^{z\sqrt{|\xi_{max}|^2+k_\varepsilon^2}}\varepsilon^2 + ce^{-b\sqrt{|\xi_{max}|^2+k_\varepsilon^2}},
$$

where $F = (2\sqrt{6}+2\sqrt{3}+6)\delta(b,\alpha,z)\alpha^{-\frac{z}{b}}\sqrt{C^2+D^2+E^2}$.

Using (11), we obtain

$$\|u^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\| \leq \sqrt{2}\varepsilon^{2-z} + F\varepsilon^{b}. \tag{17}$$

□

4. Numerical Implementation

In this section, we give an example to show our method's effectiveness. We consider the following three-dimensional Cauchy problem of the modified Helmholtz equation:

$$\begin{cases} \Delta u(x,y,z) - k_{ex}^2 u(x,y,z) = S(x,y,z), & (x,y) \in \mathbb{R}^2, z \in (0,1), \\ u_z(x,y,0) = f_{ex}(x,y), & (x,y) \in \mathbb{R}^2, \\ u(x,y,0) = g_{ex}(x,y), & (x,y) \in \mathbb{R}^2, \end{cases} \tag{18}$$

where Δ denotes the Laplace operator, $f_{ex}(x,y) = g_{ex}(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$ and $S(x,y,z) = ze^{-\frac{1}{4}(x^2+y^2)}$, then we have

$$\widehat{f}_{ex}(\xi) = \widehat{g}_{ex}(\xi) = \frac{e^{-\frac{\xi_1^2}{2}}}{\sqrt{2\cdot\frac{1}{2}}}\frac{e^{-\frac{\xi_2^2}{2}}}{\sqrt{2\cdot\frac{1}{2}}} = e^{-\frac{1}{2}(\xi_1^2+\xi_2^2)} = e^{-\frac{1}{2}|\xi|^2},\ \widehat{S}(\xi,z) = ze^{-\frac{1}{2}|\xi|^2}, \tag{19}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2,\ \xi \in \mathbb{R}^2$ *and* $k_{ex} = 1,\ z \in (0,1)$. The measured data $(f_\varepsilon, g_\varepsilon, k_\varepsilon)$ is given by

$$f_\varepsilon(x,y) = (\frac{\varepsilon \cdot rand(\cdot)}{\sqrt{2\pi}} + 1)f_{ex}(x,y),$$

$$g_\varepsilon(x,y) = (\frac{\varepsilon \cdot rand(\cdot)}{\sqrt{2\pi}} + 1)g_{ex}(x,y),$$

$$k_\varepsilon = k_{ex} + \varepsilon \cdot rand(\cdot), \tag{20}$$

where $\varepsilon \in (0,1)$, $rand(\cdot)$ is determined on $[-1,1]$.

According to (19) and (20) we obtain

$$\begin{aligned} \|f_\varepsilon - f_{ex}\|_{L^2(\mathbb{R})^2} = {} & \|g_\varepsilon - g_{ex}\|_{L^2(\mathbb{R})^2} \\ & = \|\widehat{f}_\varepsilon - \widehat{f}_{ex}\|_{L^2(\mathbb{R})^2} \\ & = [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\varepsilon \cdot rand(\cdot))^2}{\pi} e^{-(\xi_1^2+\xi_2^2)} d\xi_1 d\xi_2]^{\frac{1}{2}} \leq \varepsilon, \\ |k_\varepsilon - k_{ex}| \leq \varepsilon. & \end{aligned}$$

From (19), the exact solution of problem (1) is given as follows:

$$\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\xi,z) = e^{-\frac{1}{2}|\xi|^2}[cosh(zr_{ex}) + \frac{sinh(zr_{ex})}{r_{ex}} + \int_0^z \frac{t \cdot sinh((z-t)r_{ex})}{r_{ex}}dt].$$

Let $\alpha = \varepsilon,\ b = 2$, we obtain the regularized solution as follows:

$$\widehat{u}^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\xi,z) = e^{-\frac{1}{2}|\xi|^2}[cosh(zr_{ex}) + \frac{sinh(zr_{ex})}{r_{ex}} + \int_0^z \frac{t \cdot sinh((z-t)r_{ex})}{r_{ex}}dt]\chi_{max}.$$

Choose $\varepsilon = 0.01, 0.001, 0.0001$, respectively and $z = 0$. The graphs of the exact solution $u^{ex}_{(f_{ex},g_{ex},k_{ex})}$ and the regularized solution $u^{\varepsilon}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$ are showed in Figure 1.

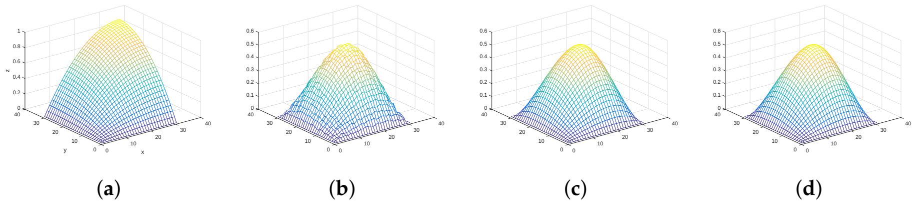

Figure 1. The exact solution $u^{ex}_{(f_{ex},g_{ex},k_{ex})}$ and the regularized solutions $u^{re}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$: (**a**) Exact solution; (**b**) $\varepsilon = 0.01$; (**c**) $\varepsilon = 0.001$; (**d**) $\varepsilon = 0.0001$.

5. Conclusions

In this paper, we solved the Cauchy problem of the three-dimensional modified Helmholtz equation with perturbed wave number k at $z = 0$. The Fourier truncation regularization method is proposed to obtain a regularization solution. The error estimate is obtained between the exact solution and the regularized solution. In future work, we will continue to study some source terms for the modified Helmholtz equation.

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