

# Geometric Characterizations of Canal Surfaces in Minkowski 3-Space III

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**Abstract:** Canal surfaces are defined and divided into nine types in Minkowski 3-space  $\mathbb{E}_1^3$ , which are obtained as the envelope of a family of pseudospheres  $\mathbb{S}_1^2$ , pseudohyperbolic spheres  $\mathbb{H}_0^2$ , or lightlike cones  $\mathbb{Q}^2$ , whose centers lie on a space curve (resp. spacelike curve, timelike curve, or null curve). This paper focuses on canal surfaces foliated by pseudohyperbolic spheres  $\mathbb{H}_0^2$  along three kinds of space curves in  $\mathbb{E}_1^3$ . The geometric properties of such surfaces are presented by classifying the linear Weingarten canal surfaces, especially the relationship between the Gaussian curvature and the mean curvature of canal surfaces. Last but not least, two examples are shown to illustrate the construction of such surfaces.

**Keywords:** Minkowski 3-space; canal surface; pseudohyperbolic sphere; linear Weingarten surface

## 1. Introduction

The concept of canal surface is the envelope of a moving sphere whose centers lie on a space curve, and their radius varies depending on this curve in Euclidean 3-space  $\mathbb{E}^3$ . Canal surfaces are useful for representing long thin objects, e.g., pipes, ropes, 3D fonts, or internal body organs in solid/surface modeling. Tori and tubes are the special types of the canal surfaces. Apart from being used in pure mathematics, canal surfaces are a kind of blending surface that plays an important role in computer aided geometric design, i.e., CAGD. Most studies on canal surfaces within the CAGD context is related to such surfaces with a rational spine curve and rational radius function. For example, the authors presented that each canal surface with a rational spine curve and rational radius function is a rational Pythagorean hodograph curve in Minkowski space [1,2].

The Lorentz–Minkowski space is the basic space model of quantum physics that plays an important role in general relativity. In recent years, with the development of the theory of relativity, physicians and geometers extended the topics in classical differential geometry of Riemannian manifolds to that of Lorentzian manifolds. It is clearly demonstrated by the fact that many works in Euclidean space have found their counterparts in Minkowski space [3]. At present, the properties of canal surfaces have been researched in  $\mathbb{E}^3$  [4,5]. As a natural idea, we can extend canal surfaces into spaces with an indefinite metric, such as Minkowski space. Similar to the generating process of canal surfaces in  $\mathbb{E}^3$ , a canal surface in Minkowski 3-space  $\mathbb{E}_1^3$  can be obtained as the envelope of a family of pseudospheres  $\mathbb{S}_1^2$ , pseudohyperbolic spheres  $\mathbb{H}_0^2$ , or lightlike cones  $\mathbb{Q}^2$  whose centers lie on a space curve (resp. spacelike curve, timelike curve, or null curve). The classification of canal surfaces was obtained by Uçum and Ilarslan in [6]. For convenience, the authors of this paper denoted the notations for all kinds of canal surfaces in  $\mathbb{E}_1^3$ . At the same time, the authors discussed canal surfaces foliated by pseudospheres along three kinds of space curves in  $\mathbb{E}_1^3$  [7]. The relationship between Gaussian curvature and mean curvature is revealed, which is an important tool for future research, such as the Weingarten canal surfaces or linear Weingarten canal surfaces. Weingarten

surfaces (resp. linear Weingarten surfaces) are attractive for use in CAGD, particularly in surface design due to the advantages of using these surfaces that can mitigate curvature computations and also admit simpler, more direct shape control procedures [8].

As a follow-up work of [7], in this paper we focus on the geometric properties of canal surfaces foliated by pseudohyperbolic spheres  $\mathbb{H}_0^2$  along three kinds of space curves in  $\mathbb{E}_1^3$ . We discuss canal surfaces purely by geometric arguments, thereby avoiding a cumbersome algorithmic procedure. The paper is organized as follows. In Section 2, we review the Frenet formulas of space curves and the definitions of canal surfaces in  $\mathbb{E}_1^3$ . We recall definitions of Weingarten surface and linear Weingarten surface in  $\mathbb{E}_1^3$ . In Section 3, the geometric properties of three types of canal surfaces are discussed, respectively. For each type of canal surface, the relationships between Gaussian curvature and mean curvature are presented (Theorems 1, 5, and 9). Different kinds of linear Weingarten canal surfaces are explored, the developable, minimal and umbilical canal surfaces are discussed at the same time. The applications of these surfaces in shape control are important hopefully motivated. Finally, some common results for canal surfaces are shown (Theorems 13 and 14).

## 2. Preliminaries

Let  $\mathbb{E}_1^3$  be a Minkowski 3-space with natural Lorentzian metric

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

in terms of the natural coordinate system  $(x_1, x_2, x_3)$ . It is well known that a vector  $v \in \mathbb{E}_1^3$  is said to be spacelike if  $\langle v, v \rangle > 0$  or  $v = 0$ ; timelike if  $\langle v, v \rangle < 0$ ; null (lightlike) if  $\langle v, v \rangle = 0$ , respectively. The norm of vector  $v$  is given by  $\|v\| = \sqrt{|\langle v, v \rangle|}$ . Due to the causal character of the tangent vector of a space curve, curves in Minkowski space can be divided into a spacelike curve, timelike curve, or null curve. At the same time, a surface is called a timelike surface, spacelike surface, or lightlike surface if its normal vector is spacelike, timelike, or lightlike. In  $\mathbb{E}_1^3$ , there exist three space forms, i.e., pseudosphere  $\mathbb{S}_1^2$ , pseudohyperbolic sphere  $\mathbb{H}_0^2$ , and lightlike cones  $\mathbb{Q}^2$ , which are complete semi-Riemannian manifolds with index 1.

Let  $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$  be vectors in  $\mathbb{E}_1^3$ . Then, their scalar product is given by

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3$$

and the exterior product by

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right),$$

where  $\{e_1, e_2, e_3\}$  is an orthonormal basis in  $\mathbb{E}_1^3$ .

Let  $c(s) : \mathbb{I} \rightarrow \mathbb{E}_1^3$  be a space curve with a moving Frenet frame  $\{T(s), N(s), B(s)\}$  consisting of tangent vector  $T$ , principal normal vector  $N$ , and binormal vector  $B$ , respectively.

Case 1. Let  $c = c(s)$  be a spacelike curve parameterized by arc length  $s$ . Due to the causal character of the normal vector, it can be divided into the following two cases:

Case 1.1. Let  $\langle c''(s), c''(s) \rangle \neq 0$ , then the following Frenet equations are satisfied

$$c'(s) = T(s), T'(s) = \kappa(s)N(s), N'(s) = -\varepsilon\kappa(s)T(s) + \tau(s)B(s), B'(s) = \tau(s)N(s),$$

where  $\langle T, T \rangle = 1$ ,  $\langle N, N \rangle = \varepsilon = \pm 1$ ,  $\langle B, B \rangle = -\varepsilon$ ,  $\langle T, N \rangle = \langle T, B \rangle = \langle B, N \rangle = 0$ . Functions  $\kappa(s)$  and  $\tau(s)$  are called the curvature and torsion of  $c(s)$ , respectively. When  $\varepsilon = 1$ ,  $c(s)$  is called the first-kind spacelike curve, and the second-kind spacelike curve when  $\varepsilon = -1$ .

Case 1.2. Let  $\langle c''(s), c''(s) \rangle = 0$ , the Frenet equations are given by

$$c'(s) = T(s), T'(s) = N(s), N'(s) = \kappa(s)N(s), B'(s) = -T(s) - \kappa(s)B(s),$$

where  $\langle T, T \rangle = \langle N, B \rangle = 1$ ,  $\langle N, N \rangle = \langle B, B \rangle = \langle T, N \rangle = \langle T, B \rangle = 0$ . Function  $\kappa(s)$  is also called the curvature function. Such kind of spacelike curve is said to be null-type spacelike.

Case 2. Let  $c = c(s)$  be a timelike curve parameterized by arc length  $s$ ; then, the following Frenet equations are satisfied:

$$c'(s) = T(s), T'(s) = \kappa(s)N(s), N'(s) = \kappa(s)T(s) + \tau(s)B(s), B'(s) = -\tau(s)N(s),$$

where  $\langle T, T \rangle = -1$ ,  $\langle N, N \rangle = \langle B, B \rangle = 1$ ,  $\langle T, N \rangle = \langle T, B \rangle = \langle B, N \rangle = 0$ . The functions  $\kappa(s)$  and  $\tau(s)$  are called the curvature and the torsion of  $c(s)$ , respectively.

Case 3. Let  $c = c(s)$  be a null curve with null arc-length parameter  $s$ , i.e.,  $\langle c''(s), c''(s) \rangle = 1$ . Then, we have

$$c'(s) = T(s), T'(s) = N(s), N'(s) = \kappa(s)T(s) - B(s), B'(s) = -\kappa(s)N(s),$$

where  $\langle T, T \rangle = \langle B, B \rangle = \langle T, N \rangle = \langle B, N \rangle = 0$ ,  $\langle T, B \rangle = \langle N, N \rangle = 1$ . Function  $\kappa(s)$  is called the null curvature of  $c(s)$ .

**Remark 1.** For null curves, there exist a variety of concepts where not all authors' terminologies coincide. The null curvature here expresses the same meaning as the pseudo torsion or the pseudo curvature in articles related to null curves.

Next, we recall the definition of canal surfaces in  $\mathbb{E}_1^3$  as the following:

**Definition 1.** [7] Surface  $\mathbb{M}$  in  $\mathbb{E}_1^3$  is called a canal surface that is formed as the envelope of a family of pseudohyperbolic spheres  $\mathbb{H}_0^2$  (resp. pseudospheres  $\mathbb{S}_1^2$  or lightlike cones  $\mathbb{Q}^2$ ) whose centers lie on a space curve  $c(s)$  framed by  $\{T, N, B\}$ . Then,  $\mathbb{M}$  can be parametrized by

$$x(s, \theta) = c(s) + \lambda(s, \theta)T(s) + \mu(s, \theta)N(s) + \omega(s, \theta)B(s), \quad (1)$$

where  $\lambda, \mu$  and  $\omega$  are differential functions of  $s$  and  $\theta$ ,  $\|x(s, \theta) - c(s)\|^2 = \epsilon r^2(s)$ , ( $\epsilon = \pm 1$  or  $0$ ). Curve  $c(s)$  is called the center curve (or spine curve), and  $r(s)$  is the radius function of  $\mathbb{M}$ .

Explicitly, if  $\mathbb{M}$  is foliated by pseudohyperbolic spheres  $\mathbb{H}_0^2$  (resp. pseudospheres  $\mathbb{S}_1^2$  or lightlike cones  $\mathbb{Q}^2$ ), then  $\epsilon = -1$  (resp.  $1$  or  $0$ ) and  $\mathbb{M}$  is said to be of the type  $\mathbb{M}_-$  (resp.  $\mathbb{M}_+$  or  $\mathbb{M}_0$ ). Canal surfaces of type  $\mathbb{M}_-$  can be divided into three types. In the case that  $c(s)$  is spacelike (resp. timelike or null), it is said to be of type  $\mathbb{M}_-^1$  (resp.  $\mathbb{M}_-^2$  or  $\mathbb{M}_-^3$ ). Furthermore,  $\mathbb{M}_-^1$  can be divided into  $\mathbb{M}_-^{11}$ ,  $\mathbb{M}_-^{12}$  and  $\mathbb{M}_-^{13}$  when  $c(s)$  is the first-kind spacelike curve, the second-kind spacelike curve, and the null-type spacelike curve, respectively. Similar to  $\mathbb{M}_-$ , canal surfaces  $\mathbb{M}_+$  (resp.  $\mathbb{M}_0$ ) can be divided into  $\mathbb{M}_+^1$ ,  $\mathbb{M}_+^2$  and  $\mathbb{M}_+^3$  (resp.  $\mathbb{M}_0^1$ ,  $\mathbb{M}_0^2$  or  $\mathbb{M}_0^3$ ). Naturally,  $\mathbb{M}_+^1$  (resp.  $\mathbb{M}_0^1$ ) can be divided into  $\mathbb{M}_+^{11}$ ,  $\mathbb{M}_+^{12}$  and  $\mathbb{M}_+^{13}$  (resp.  $\mathbb{M}_0^{11}$ ,  $\mathbb{M}_0^{12}$  or  $\mathbb{M}_0^{13}$ ).

**Remark 2.** [9] In particular, if center curve  $c(s)$  is a straight line, then Frenet frame  $\{T, N, B\}$  of  $c(s)$  can be regarded as a trivial orthogonal frame, and the canal surface is nothing but a surface of revolution. If the radius function is constant, then  $\mathbb{M}$  is a tube (or pipe) surface.

**Definition 2.** [10] For curvatures  $K$  and  $H$  of a surface  $\mathbb{M}$  in  $\mathbb{E}_1^3$ , if  $\mathbb{M}$  satisfies

$$W(K, H) = 0,$$

where  $W$  is the Jacobian determinant, then that is said to be a Weingarten surface.

**Definition 3.** [10] For curvatures  $K$  and  $H$  of a surface  $\mathbb{M}$  in  $\mathbb{E}_1^3$ , if  $\mathbb{M}$  satisfies

$$2aH + bK = c \quad (a, b, c \in \mathbb{R} \text{ and } (a, b, c) \neq (0, 0, 0)), \quad (2)$$

then that is said to be a linear Weingarten surface.

**Remark 3.** When  $a = 0$  or  $b = 0$  in (2), surface  $\mathbb{M}$  has a constant Gaussian curvature or constant mean curvature. Without loss of generality, we always assume  $c = 1$  in (2).

All surfaces we are dealing with are smooth, regular, and topologically connected unless otherwise stated.

### 3. Main Results

In this part, we focus on the geometric properties of different types of canal surfaces formed by the movement of pseudohyperbolic spheres  $\mathbb{H}_0^2$  along a space curve in  $\mathbb{E}_1^3$ .

#### 3.1. Canal Surfaces of Type $\mathbb{M}_-^{11}$ and $\mathbb{M}_-^{12}$

First, we assume  $\mathbb{M}$  is a canal surface formed by the movement of  $\mathbb{H}_0^2$  along a first kind spacelike curve  $c(s)$  in  $\mathbb{E}_1^3$ . i.e.,  $\mathbb{M}_-^{11}$ . According to the definition of  $\mathbb{M}_-^{11}$ , through detailed calculation, we get

$$\begin{cases} \lambda(s) = r(s)r'(s), \\ \mu(s, \theta) = r(s)\sqrt{1+r'^2(s)} \sinh \theta, \\ \omega(s, \theta) = r(s)\sqrt{1+r'^2(s)} \cosh \theta \end{cases}$$

in Equation (1). Then,  $\mathbb{M}_-^{11}$  can be parameterized by

$$x(s, \theta) = c(s) + r(s)(r'(s)T + \sqrt{1+r'^2(s)} \sinh \theta N + \sqrt{1+r'^2(s)} \cosh \theta B),$$

where  $c(s)$  is parameterized by arc length  $s$ . For convenience, we may assume  $r'(s) = \sinh \varphi$  for some smooth function  $\varphi = \varphi(s)$ . Then, canal surface  $\mathbb{M}_-^{11}$  can be rewritten by

$$x(s, \theta) = c(s) + r(s)(\sinh \varphi T + \cosh \varphi \sinh \theta N + \cosh \varphi \cosh \theta B). \quad (3)$$

Initially, we have

$$x_s = x_s^1 T + x_s^2 N + x_s^3 B, \quad x_\theta = r \cosh \varphi \cosh \theta N + r \cosh \varphi \sinh \theta B,$$

where

$$\begin{aligned} x_s^1 &= rr'' + \cosh^2 \varphi - r\kappa \cosh \varphi \sinh \theta, \\ x_s^2 &= r' \cosh \varphi \sinh \theta + rr'\kappa + rr'\varphi' \sinh \theta + r\tau \cosh \varphi \cosh \theta, \\ x_s^3 &= r' \cosh \varphi \cosh \theta + r\tau \cosh \varphi \sinh \theta + rr'\varphi' \cosh \theta. \end{aligned}$$

Then, quantities of the first fundamental form are given by

$$\begin{aligned} E = \langle x_s, x_s \rangle &= r^2(\kappa^2 \cosh^2 \varphi \sinh^2 \theta + r'^2 \kappa^2 + \varphi'^2 + \tau^2 \cosh^2 \varphi - 2\varphi' \kappa \sinh \theta \\ &\quad + 2r' \kappa \tau \cosh \varphi \cosh \theta) + \cosh^2 \varphi + 2(rr'' - r\kappa \cosh \varphi \sinh \theta), \\ F = \langle x_s, x_\theta \rangle &= r^2 r' \kappa \cosh \varphi \cosh \theta + r^2 \tau \cosh^2 \varphi, \\ G = \langle x_\theta, x_\theta \rangle &= r^2 \cosh^2 \varphi. \end{aligned} \quad (4)$$

and

$$EG - F^2 = r^2(rr'' - r\kappa \cosh \varphi \sinh \theta + \cosh^2 \varphi)^2. \quad (5)$$

Unit normal vector field  $n$  to  $\mathbb{M}_-^{11}$  is given by

$$n = \frac{x_s \times x_\theta}{\|x_s \times x_\theta\|} = \sinh \varphi T + \cosh \varphi \sinh \theta N + \cosh \varphi \cosh \theta B, \quad (6)$$

which point canal surface  $\mathbb{M}_-^{11}$  and  $\langle n, n \rangle = -1$  outwards.

Furthermore, by Equation (6), we have

$$\begin{aligned} n_s &= (r'' - \kappa \cosh \varphi \sinh \theta)T + (r'\kappa + r'\varphi' \sinh \theta + \tau \cosh \varphi \cosh \theta)N \\ &\quad + (\tau \cosh \varphi \sinh \theta + r'\varphi' \cosh \theta)B, \\ n_\theta &= \cosh \varphi \cosh \theta N + \cosh \varphi \sinh \theta B. \end{aligned}$$

Quantities of the second fundamental form are obtained by

$$\begin{aligned} L &= -\langle x_s, n_s \rangle = -r(\kappa^2 \cosh^2 \varphi \sinh^2 \theta + r'^2 \kappa^2 + \varphi'^2 + \tau^2 \cosh^2 \varphi - 2\varphi' \kappa \sinh \theta \\ &\quad + 2r'\kappa \tau \cosh \varphi \cosh \theta) - (r'' - \kappa \cosh \varphi \sinh \theta), \\ M &= -\langle x_\theta, n_s \rangle = -rr'\kappa \cosh \varphi \cosh \theta - r\tau \cosh^2 \varphi, \\ N &= -\langle x_\theta, n_\theta \rangle = -r \cosh^2 \varphi. \end{aligned} \quad (7)$$

From Equations (6) and (7), we have

**Proposition 1.** The quantities of the first and second fundamental forms of canal surface  $\mathbb{M}_-^{11}$  satisfy

$$L = \frac{E - P_1}{-r}, \quad M = \frac{F}{-r'}, \quad N = \frac{G}{-r}$$

and

$$EG - F^2 = r^2 P_1^2, \quad LN - M^2 = r P_1 Q_1, \quad (8)$$

where

$$\begin{aligned} P_1 &= rr'' - r\kappa \cosh \varphi \sinh \theta + \cosh^2 \varphi = rQ_1 + \cosh^2 \varphi, \\ Q_1 &= r'' - \kappa \cosh \varphi \sinh \theta. \end{aligned} \quad (9)$$

**Remark 4.** Due to regularity, we see that  $P_1 \neq 0$  everywhere by Equation (8).

By Proposition 1, Gaussian curvature  $K$  and mean curvature  $H$  of  $\mathbb{M}_-^{11}$  are given by, respectively,

$$K = -\frac{LN - M^2}{EG - F^2} = -\frac{Q_1}{rP_1}, \quad (10)$$

$$H = -\frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{2P_1 - \cosh^2 \varphi}{2rP_1}. \quad (11)$$

Second, for canal surface  $\mathbb{M}_{-}^{12}$ , according to the definition of  $\mathbb{M}_{-}^{12}$ , we get

$$\begin{cases} \lambda(s) = r(s)r'(s), \\ \mu(s, \theta) = r(s)\sqrt{1+r'^2(s)} \cosh \theta, \\ \omega(s, \theta) = r(s)\sqrt{1+r'^2(s)} \sinh \theta \end{cases}$$

in Equation (1). Then,  $\mathbb{M}_{-}^{12}$  can be parameterized by

$$x(s, \theta) = c(s) + r(s)(r'(s)T + \sqrt{1+r'^2(s)} \cosh \theta N + \sqrt{1+r'^2(s)} \sinh \theta B),$$

where  $c(s)$  is parameterized by arc length  $s$ . Here, we may assume that  $r'(s) = \sinh \varphi$  for smooth function  $\varphi = \varphi(s)$ . So, canal surface  $\mathbb{M}_{-}^{12}$  can be written by

$$x(s, \theta) = c(s) + r(s)(\sinh \varphi T + \cosh \varphi \cosh \theta N + \cosh \varphi \sinh \theta B). \quad (12)$$

With similar calculations to those of  $\mathbb{M}_{-}^{11}$ , we have the following conclusions.

**Proposition 2.** The quantities of the first and second fundamental forms of canal surface  $\mathbb{M}_{-}^{12}$  satisfy

$$L = \frac{E - P_2}{-r}, \quad M = \frac{F}{-r}, \quad N = \frac{G}{-r}$$

and

$$EG - F^2 = r^2 P_2^2, \quad LN - M^2 = r P_2 Q_2, \quad (13)$$

where

$$\begin{aligned} P_2 &= rr'' + r\kappa \cosh \varphi \cosh \theta + \cosh^2 \varphi = rQ_2 + \cosh^2 \varphi, \\ Q_2 &= r'' + \kappa \cosh \varphi \cosh \theta. \end{aligned} \quad (14)$$

**Remark 5.** Due to regularity, we see  $P_2 \neq 0$  everywhere by Equation (13).

By Proposition 2, Gaussian curvature  $K$  and mean curvature  $H$  of  $\mathbb{M}_{-}^{12}$  are given by, respectively,

$$K = -\frac{LN - M^2}{EG - F^2} = -\frac{Q_2}{rP_2}, \quad (15)$$

$$H = -\frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{2P_2 - \cosh^2 \varphi}{2rP_2}. \quad (16)$$

Based on the Gaussian curvature and mean curvature of  $\mathbb{M}_{-}^{11}$  and  $\mathbb{M}_{-}^{12}$ , it is obvious to obtain the following results.

**Theorem 1.** Gaussian curvature  $K$  and mean curvature  $H$  of canal surface  $\mathbb{M}_{-}^{11}$  ( $\mathbb{M}_{-}^{12}$ ) are related by

$$H = -\frac{1}{2}\left(Kr - \frac{1}{r}\right). \quad (17)$$

**Proof of Theorem 1.** For  $\mathbb{M}_{-}^{11}$ , from Equations (10) and (11), we can easily obtain the conclusion. For  $\mathbb{M}_{-}^{12}$ , we can refer to Equations (15) and (16).  $\square$

Next, we study canal surface  $\mathbb{M}_{-}^{11}$  ( $\mathbb{M}_{-}^{12}$ ) whose Gaussian curvature and mean curvature satisfy some particular conditions.

**Remark 6.** In the following, we just prove the results for  $\mathbb{M}_{-}^{11}$  and omit the proof for  $\mathbb{M}_{-}^{12}$  since it can be similarly done to those of  $\mathbb{M}_{-}^{11}$ , and the results are same.

**Theorem 2.** Let  $\mathbb{M}_{-}^{11}$  ( $\mathbb{M}_{-}^{12}$ ) be a linear Weingarten canal surface; then, it is an open part of the following surfaces:

1. a surface of revolution such as

$$x(s, \theta) = (r(s) \sinh \varphi(s) + s, r(s) \cosh \varphi(s) \sinh \theta, r(s) \cosh \varphi(s) \cosh \theta),$$

where  $r(s)$  is given by (19);

2. a tube with radius  $r = a$  ( $a > 0$ ).

**Proof of Theorem 2.** From Equation (2) with  $c = 1$  and Equation (17), we obtain

$$(br - ar^2)K = r - a.$$

By Equation (10), we get

$$-\frac{(br - ar^2)(r'' - \kappa \cosh \varphi \sinh \theta)}{r(rr'' - r\kappa \cosh \varphi \sinh \theta + \cosh^2 \varphi)} = r - a, \quad (18)$$

i.e.,

$$\kappa(r^2 - 2ar + b) \cosh \varphi \sinh \theta - (r - a)(1 + r'^2) - (r^2 - 2ar + b)r'' = 0.$$

Therefore, we get

$$\kappa(r^2 - 2ar + b) \cosh \varphi = 0 \quad \text{and} \quad (r - a)(1 + r'^2) + (r^2 - 2ar + b)r'' = 0.$$

Case 1: If  $r^2 - 2ar + b \neq 0$ , i.e.,  $a^2 - b < 0$ , then  $\kappa = 0$ . Thus,  $\mathbb{M}_{-}^{11}$  is a surface of revolution and its radial function satisfies

$$(r^2 - 2ar + b)r'' + (r - a)(1 + r'^2) = 0.$$

Solving the above equation, we get

$$s = c_2 \pm \int \sqrt{\frac{r^2 - 2ar + b}{c_1 - r^2 + 2ar - b}} dr, \quad (19)$$

where  $c_1 > r^2 - 2ar + b$ ,  $c_2 \in \mathbb{R}$ .

Since  $\kappa = 0$ , without loss of generality, we may assume the center curve is  $c(s) = (s, 0, 0)$  and  $T = (1, 0, 0)$ ,  $N = (0, 1, 0)$ ,  $B = (0, 0, 1)$ , respectively. Then, by Equation (3),  $\mathbb{M}_{-}^{11}$  can be expressed by

$$x(s, \theta) = (r(s) \sinh \varphi(s) + s, r(s) \cosh \varphi(s) \sinh \theta, r(s) \cosh \varphi(s) \cosh \theta), \quad (20)$$

where  $r(s)$  is given by Equation (19).

Case 2: If  $\kappa \neq 0$ , then  $r^2 - 2ar + b = 0$ . Hence,  $r = a$  is a nonzero constant.  $\mathbb{M}_{-}^{11}$  is a tube and  $a, b$  satisfy  $a^2 - b = 0$ .

Note that  $\mathbb{M}_{-}^{11}$  is a circular cylinder if  $\kappa = r^2 - 2ar + b \equiv 0$ .  $\square$

**Corollary 1.** Let  $\mathbb{M}_{-}^{11}$  ( $\mathbb{M}_{-}^{12}$ ) be a canal surface with nonzero constant Gaussian curvature. Then, it is a surface of revolution with positive constant Gaussian curvature, such as

$$x(s, \theta) = (r(s) \sinh \varphi(s) + s, r(s) \cosh \varphi(s) \sinh \theta, r(s) \cosh \varphi(s) \cosh \theta),$$

where  $r(s)$  is given by Equation (21).

**Proof of Corollary 1.** By Theorem 2 with  $a = 0$ , when  $\mathbb{M}_{-}^{11}$  has nonzero constant Gaussian curvature  $K = \frac{1}{b}$ , from  $a^2 - b < 0$ , then it is nothing but a surface of revolution with positive constant Gaussian curvature. It can be expressed by

$$x(s, \theta) = (r(s) \sinh \varphi(s) + s, r(s) \cosh \varphi(s) \sinh \theta, r(s) \cosh \varphi(s) \cosh \theta),$$

where  $r(s)$  satisfies

$$s = c_2 \pm \int \sqrt{\frac{r^2 + b}{c_1 - r^2 - b}} dr, \quad (c_1 > r^2 + b, c_2 \in \mathbb{R}). \quad (21)$$

□

**Corollary 2.** Canal surface  $\mathbb{M}_{-}^{11} (\mathbb{M}_{-}^{12})$  with nonzero constant mean curvature does not exist.

**Proof of Corollary 2.** By Theorem 2 with  $b = 0$ , it must be a surface of revolution. However, from  $a^2 - b < 0$ , then  $a^2 < 0$ , it is a contradiction. □

**Theorem 3.** A canal surface  $\mathbb{M}_{-}^{11} (\mathbb{M}_{-}^{12})$  is developable iff it is congruent to a part of a circular cylinder or a circular cone.

**Proof of Theorem 3.**  $\mathbb{M}_{-}^{11}$  is developable iff  $K \equiv 0$ . By (10), we have  $Q_1 \equiv 0$ . Then, we get

$$r'' - \kappa \cosh \varphi \sinh \theta = 0.$$

It follows that  $r'' = 0$  and  $\kappa = 0$  (if  $\cosh \varphi = 0$ , by (5),  $\mathbb{M}_{-}^{11}$  is degenerate). Then,  $r(s) = c_1 s + c_2$ , where  $c_1, c_2$  are constants. Therefore,  $\mathbb{M}_{-}^{11}$  is a circular cylinder ( $c_1 = 0$ ) or a circular cone ( $c_1 \neq 0$ ) in  $\mathbb{E}_1^3$ , respectively. The converse is obvious. □

**Theorem 4.** Canal surface  $\mathbb{M}_{-}^{11} (\mathbb{M}_{-}^{12})$  is minimal iff it is a part of a surface of revolution, such as

$$x(s, \theta) = (r(s) \sinh \varphi(s) + s, r(s) \cosh \varphi(s) \sinh \theta, r(s) \cosh \varphi(s) \cosh \theta),$$

where  $r(s)$  satisfies (22).

**Proof of Theorem 4.**  $\mathbb{M}_{-}^{11}$  is minimal iff  $H \equiv 0$ . From (11),  $H \equiv 0$  implies

$$2P_1 - \cosh^2 \varphi = 0.$$

By Equation (9), we get

$$2rr'' - 2r\kappa \cosh \varphi \sinh \theta + \cosh^2 \varphi = 0.$$

Therefore, one can obtain  $r\kappa \cosh \varphi = 0$  and  $2rr'' + \cosh^2 \varphi = 0$ . Since  $r \neq 0$ ,  $\cosh \varphi \neq 0$ , then  $\kappa = 0$  and  $\mathbb{M}_{-}^{11}$  is a surface of revolution. Solving  $2rr'' + \cosh^2 \varphi = 0$ , we get

$$s = c_2 \pm \int \sqrt{\frac{r}{c_1 - r}} dr, \quad (c_1 > r, c_2 \in \mathbb{R}). \quad (22)$$

The converse is obvious through direct calculations. □



### 3.2. Canal Surfaces $\mathbb{M}_{-}^2$

Let  $\mathbb{M}$  be a canal surface formed by the movement of  $\mathbb{H}_0^2$  along a timelike curve  $c(s)$  in  $\mathbb{E}_1^3$ , i.e.,  $\mathbb{M}_{-}^2$ . Then, by the definition of  $\mathbb{M}_{-}^2$  and Frenet equations, we obtain

$$\begin{cases} \lambda(s) = -r(s)r'(s), \\ \mu(s, \theta) = r(s)\sqrt{r'^2(s) - 1} \cos \theta, \\ \omega(s, \theta) = r(s)\sqrt{r'^2(s) - 1} \sin \theta \end{cases}$$

in Equation (1). Then,  $\mathbb{M}_{-}^2$  can be parameterized by

$$x(s, \theta) = c(s) + r(s)(-r'(s)T + \sqrt{r'^2(s) - 1} \cos \theta N + \sqrt{r'^2(s) - 1} \sin \theta B), \quad (23)$$

where  $c(s)$  is parameterized by arc length  $s$ . Without loss of generality, we assume  $-r'(s) = \cosh \varphi$  for some smooth function  $\varphi = \varphi(s)$ . Then,  $\mathbb{M}_{-}^2$  can be written by

$$x(s, \theta) = c(s) + r(s)(\cosh \varphi T + \sinh \varphi \cos \theta N + \sinh \varphi \sin \theta B). \quad (24)$$

**Remark 7.** From Equation (23), tube  $\mathbb{M}_{-}^2$  does not exist.

**Proposition 3.** The quantities of the first and second fundamental forms of canal surface  $\mathbb{M}_{-}^2$  satisfy

$$L = \frac{E - P_3}{-r}, \quad M = \frac{F}{-r}, \quad N = \frac{G}{-r}$$

and

$$EG - F^2 = r^2 P_3^2, \quad LN - M^2 = r P_3 Q_3, \quad (25)$$

where

$$\begin{aligned} P_3 &= rr'' - r\kappa \sinh \varphi \cos \theta + \sinh^2 \varphi = rQ_3 + \sinh^2 \varphi, \\ Q_3 &= r'' - \kappa \sinh \varphi \cos \theta. \end{aligned} \quad (26)$$

**Remark 8.** Due to regularity, we see  $P_3 \neq 0$  everywhere by Equation (25).

By Proposition 3, Gaussian curvature  $K$  and mean curvature  $H$  of  $\mathbb{M}_{-}^2$  are given by, respectively,

$$K = -\frac{LN - M^2}{EG - F^2} = -\frac{Q_3}{rP_3}, \quad (27)$$

$$H = -\frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{2P_3 - \sinh^2 \varphi}{2rP_3}. \quad (28)$$

**Theorem 5.** Gaussian curvature  $K$  and mean curvature  $H$  of canal surface  $\mathbb{M}_{-}^2$  are related by

$$H = -\frac{1}{2}\left(Kr - \frac{1}{r}\right). \quad (29)$$

Next, we study canal surface  $\mathbb{M}_{-}^2$  whose Gaussian curvature and mean curvature satisfy some particular conditions. We omitted the proofs for  $\mathbb{M}_{-}^2$  since they are similar to  $\mathbb{M}_{+}^2$ ,  $\mathbb{M}_{-}^{12}$ .

**Theorem 6.** Let  $\mathbb{M}_{-}^2$  be a linear Weingarten canal surface; then, it is a surface of revolution, such as

$$x(s, \theta) = (r(s) \sinh \varphi(s) \sin \theta, r(s) \sinh \varphi(s) \cos \theta, r(s) \cosh \varphi(s) + s), \quad (30)$$

where  $r(s)$  is given by

$$s = c_2 \pm \int \sqrt{\frac{r^2 - 2ar + b}{r^2 - 2ar + b + c_1}} dr, \quad (c_1 > -r^2 + 2ar - b, c_2 \in \mathbb{R}).$$

**Corollary 3.** Let  $\mathbb{M}_{-}^2$  be a canal surface with nonzero constant Gaussian curvature. Then, it is a surface of revolution with positive constant Gaussian curvature, such as

$$x(s, \theta) = (r(s) \sinh \varphi(s) \sin \theta, r(s) \sinh \varphi(s) \cos \theta, r(s) \cosh \varphi(s) + s),$$

where  $r(s)$  is given by

$$s = c_2 \pm \int \sqrt{\frac{r^2 + b}{r^2 + b + c_1}} dr, \quad (c_1 > -r^2 - b, c_2 \in \mathbb{R}).$$

**Corollary 4.** Canal surface  $\mathbb{M}_{-}^2$  with nonzero constant mean curvature does not exist.

**Theorem 7.** A canal surface  $\mathbb{M}_{-}^2$  is developable iff it is a circular cone.

**Proof of Theorem 7.**  $\mathbb{M}_{-}^2$  is developable iff  $K \equiv 0$ . By Equation (27), we have  $Q_3 \equiv 0$ . Then, we get

$$r'' - \kappa \sinh \varphi \cos \theta = 0.$$

It follows that  $r'' = 0$  and  $\kappa = 0$  (if  $\sinh \varphi = 0$ , by Equation (25),  $\mathbb{M}_{-}^2$  is degenerate). Then,  $r(s) = c_1 s + c_2$ , where  $c_1, c_2$  are constants, and  $|c_1| > 1$ . If  $|c_1| \leq 1$ , by (23), it is a contradiction. Therefore,  $\mathbb{M}_{-}^2$  is a circular cone ( $|c_1| > 1$ ) in  $\mathbb{E}_1^3$ . The converse is obvious.  $\square$

**Theorem 8.** A canal surface  $\mathbb{M}_{-}^2$  is minimal iff it is a part of a surface of revolution such as

$$x(s, \theta) = (r(s) \sinh \varphi(s) \sin \theta, r(s) \sinh \varphi(s) \cos \theta, r(s) \cosh \varphi(s) + s),$$

where  $r(s)$  satisfies

$$s = c_2 \pm \int \sqrt{\frac{r}{r + c_1}} dr, \quad (c_1 > -r, c_2 \in \mathbb{R}).$$

### 3.3. Canal Surfaces of Type $\mathbb{M}_{-}^{13}$ and $\mathbb{M}_{-}^3$

Let  $\mathbb{M}$  be a canal surface formed by the movement of  $\mathbb{H}_0^2$  along a null type spacelike curve  $c(s)$ , i.e.,  $\mathbb{M}_{-}^{13}$ . By the definition of  $\mathbb{M}_{-}^{13}$  and Frenet equations, we obtain

$$\begin{cases} \lambda(s) = r(s)r'(s), \\ 2\mu(s, \theta)\omega(s, \theta) = -r^2(s)(1 + r'^2(s)) \end{cases} \quad (31)$$

in Equation (1). Then,  $\mathbb{M}_{-}^{13}$  can be parameterized by

$$x(s, \theta) = c(s) + r(s)r'(s)T + \mu(s, \theta)N + \omega(s, \theta)B, \quad (32)$$

where  $c(s)$  is parameterized by arc length  $s$ .

Initially, we have

$$x_s = U_1(s, \theta)T + V_1(s, \theta)N + W_1(s, \theta)B, \quad x_\theta = \mu_\theta N + \omega_\theta B,$$

where

$$U_1(s, \theta) = 1 + r'^2 + rr'' - \omega, \quad V_1(s, \theta) = rr' + \mu + \mu\kappa, \quad W_1(s, \theta) = \omega_s - \omega\kappa.$$

From Equation (31), we can get

$$\mu_\theta = -\frac{\mu\omega_\theta}{\omega}, \quad \mu_s = \frac{-rr'(U_1 + \omega) - \mu\omega_s}{\omega}.$$

Then, the quantities of the first fundamental form are given by

$$E = U_1^2 + 2V_1W_1, \quad F = \mu_\theta W_1 + \omega_\theta V_1, \quad G = 2\mu_\theta\omega_\theta. \quad (33)$$

$$EG - F^2 = \frac{r^2\omega_\theta^2 U_1^2}{\omega^2}.$$

Unit normal vector field  $n$  to  $\mathbb{M}_-^{13}$  is given by

$$n = \frac{x_s \times x_\theta}{\|x_s \times x_\theta\|} = -\frac{1}{r}(rr'T + \mu N + \omega B), \quad (34)$$

which point canal surface  $\mathbb{M}_-^{13}$  and  $\langle n, n \rangle = -1$  outwards.

Furthermore, by Equation (34) we have

$$n_s = \frac{1}{r^2}\{(rr'^2 - rU_1 + r)T + (r'\mu - rV_1)N + (r'\omega - rW_1)B\},$$

$$n_\theta = -\frac{1}{r}(\mu_\theta N + \omega_\theta B).$$

Then, the quantities of the second fundamental form are obtained by

$$L = \frac{1}{r}(U_1^2 - U_1 + 2V_1W_1), \quad M = \frac{\omega_\theta}{\omega r}(\omega V_1 - \mu W_1), \quad N = \frac{2\mu_\theta\omega_\theta}{r}. \quad (35)$$

From Equations (34) and (35), we have

**Proposition 4.** The quantities of the first and second fundamental forms of canal surfaces  $\mathbb{M}_-^{13}$  satisfy

$$L = \frac{E - U_1}{r}, \quad M = \frac{F}{r}, \quad N = \frac{G}{r}$$

and

$$EG - F^2 = \frac{r^2\omega_\theta^2 U_1^2}{\omega^2}, \quad LN - M^2 = -\frac{\omega_\theta^2 U_1 Y_1}{\omega^2}, \quad (36)$$

where  $Y_1 = 1 + r'^2 - U_1 = \omega - rr''$ .

**Remark 9.** Due to regularity, we see  $U_1 \neq 0$  everywhere by Equation (36).

By Proposition 4, Gaussian curvature  $K$  and mean curvature  $H$  of  $\mathbb{M}_-^{13}$  are given by, respectively,

$$K = -\frac{LN - M^2}{EG - F^2} = \frac{Y_1}{r^2 U_1}, \quad (37)$$

$$H = -\frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{Y_1 - U_1}{2rU_1}. \quad (38)$$

Second, we study canal surface  $\mathbb{M}_-^3$ . By the definition of  $\mathbb{M}_-^3$  and the Frenet equations, we obtain

$$\begin{cases} \omega(s) = r(s)r'(s), \\ 2\lambda(s, \theta)r(s)r'(s) + \mu^2(s, \theta) = -r^2(s) \end{cases} \quad (39)$$

in Equation (1). Then,  $\mathbb{M}_-^3$  can be parameterized by

$$x(s, \theta) = c(s) + \lambda(s, \theta)T + \mu(s, \theta)N + r(s)r'(s)B \quad (40)$$

where  $c(s)$  is parameterized by null arc length  $s$ .

**Remark 10.** According to Equation (39), tube  $\mathbb{M}_-^3$  does not exist.

**Proposition 5.** The quantities of the first and second fundamental forms of canal surfaces  $\mathbb{M}_-^3$  satisfy

$$L = \frac{E - W_2}{r}, \quad M = \frac{F}{r}, \quad N = \frac{G}{r}$$

and

$$EG - F^2 = \frac{r^2\lambda_\theta^2 W_2^2}{\mu^2}, \quad LN - M^2 = \frac{\lambda_\theta^2 W_2 Y_2}{\mu^2}, \quad (41)$$

where  $W_2 = rr'' - r'^2 - \mu$ ,  $Y_2 = W_2 - r'^2 = rr'' - \mu$ .

By Proposition 5, Gaussian curvature  $K$  and mean curvature  $H$  of  $\mathbb{M}_-^3$  are given by, respectively,

$$K = -\frac{LN - M^2}{EG - F^2} = -\frac{Y_2}{r^2 W_2}, \quad (42)$$

$$H = -\frac{EN - 2FM + GL}{2(EG - F^2)} = -\frac{W_2 + Y_2}{2rW_2}. \quad (43)$$

Based on the Gaussian curvature and mean curvature of  $\mathbb{M}_-^{13}$  and  $\mathbb{M}_-^3$ , it is easy to get the following results.

**Theorem 9.** Gaussian curvature  $K$  and mean curvature  $H$  of canal surface  $\mathbb{M}_-^{13}$  ( $\mathbb{M}_-^3$ ) can be related by

$$H = \frac{1}{2}\left(Kr - \frac{1}{r}\right). \quad (44)$$

**Proof of Theorem 9.** For  $\mathbb{M}_-^{13}$ , from Equations (37) and (38), we can easily obtain the conclusion. For  $\mathbb{M}_-^3$ , we can refer to Equations (42) and (43).  $\square$

Next, we study canal surface  $\mathbb{M}_-^{13}$  ( $\mathbb{M}_-^3$ ) whose Gaussian curvature and mean curvature satisfy some particular conditions.

**Remark 11.** In the following, we just prove the results for  $\mathbb{M}_-^{13}$  and omit the proofs for  $\mathbb{M}_-^3$ , since they can be similarly done to those of  $\mathbb{M}_-^{13}$  and the results are similar.

**Theorem 10.** Let  $\mathbb{M}_-^{13}$  be a linear Weingarten canal surface; then, it is a tube with radius  $r = -a$  ( $a < 0$ ).

**Proof of Theorem 10.** From Equation (2) with  $c = 1$  and Equation (44), we obtain

$$(ar^2 + br)K = r + a.$$

By Equation (37), we get

$$\frac{(ar^2 + br)(\omega - rr'')}{r^2(1 + r'^2 + rr'' - \omega)} = r + a, \quad (45)$$

i.e.,

$$\omega(r^2 + 2ar + b) - rr''(r^2 + 2ar + b) - (r^2 + ar)(1 + r'^2) = 0.$$

Therefore, we get

$$\omega(r^2 + 2ar + b) = 0 \quad \text{and} \quad rr''(r^2 + 2ar + b) + (r^2 + ar)(1 + r'^2) = 0.$$

Assume  $r^2 + 2ar + b \neq 0$ , then  $\omega = 0$ . By (36),  $\mathbb{M}^{13}_-$  is degenerate. Thus,  $r^2 - 2ar + b = 0$ . Hence,  $r = -a$  ( $a < 0$ ) is a nonzero constant.  $\mathbb{M}^{13}_-$  is a tube and  $a, b$  satisfy  $a^2 - b = 0$ .  $\square$

**Theorem 11.** Linear Weingarten canal surface  $\mathbb{M}^3_-$  does not exist.

**Proof of Theorem 11.** Similar to the proof of Theorem 10, through calculation, we obtain that  $r = -a$  ( $a < 0$ ) is a nonzero constant. This contradicts the result of Remark 10. Thus it is completed.  $\square$

**Corollary 5.** Canal surface  $\mathbb{M}^{13}_-$  ( $\mathbb{M}^3_-$ ) with nonzero constant Gaussian curvature or nonzero constant mean curvature does not exist.

**Proof of Corollary 5.** If  $\mathbb{M}^{13}_-$  has nonzero constant Gaussian curvature or nonzero constant mean curvature, by Equations (37) and (38), the functions  $\omega = \omega(s)$  and  $\mu = \mu(s)$ , obviously. It is impossible. The proof is completed.  $\square$

Similar to Corollary 5, when the Gaussian curvature or mean curvature equal to zero, by (37) and (38), the functions  $\omega = \omega(s)$  and  $\mu = \mu(s)$ , obviously. Then, we have

**Theorem 12.** Canal surface  $\mathbb{M}^{13}_-$  ( $\mathbb{M}^3_-$ ) is nondevellopable and nonminimal.

From the calculations above, we have the following common conclusions.

**Theorem 13.** Umbilical canal surface  $\mathbb{M}_-$  does not exist.

**Proof of Theorem 13.** Canal surface  $\mathbb{M}_-$  is umbilical; this means

$$E : F : G = L : M : N,$$

from Propositions 1–5, we obtain  $P_1 = P_2 = P_3 = U_1 = W_2 = 0$ . It is impossible by the regularity of those canal surfaces.  $\square$

**Theorem 14.** Canal surfaces  $\mathbb{M}_-$  are spacelike surfaces in  $\mathbb{E}^3_1$ .

**Proof of Theorem 14.** The normal vector of  $\mathbb{M}_-$  satisfies  $\langle n, n \rangle = -1$ ; it is obtained easily.  $\square$

**Remark 12.** The canal surfaces obtained by pseudo spheres  $\mathbb{S}^2_1$  along a space curve, i.e.,  $\mathbb{M}_+$  are discussed in [7]. The canal surfaces foliated by lightcones  $\mathbb{Q}^2$  along a space curve, i.e.,  $\mathbb{M}_0$  are degenerate surfaces by simple calculation. Here, the proof is omitted.

#### 4. Examples

Canal surfaces are very popular in CAGD. In this section, we want to show a method to characterize  $\mathbb{M}_-$  geometrically via Mathematica Programme.

**Example 1.** Let the timelike curve  $c(s) = (\sin \frac{s}{2}, \cos \frac{s}{2}, \frac{\sqrt{5}s}{2})$ , then the Frenet frame are

$$\begin{cases} T(s) = (\frac{1}{2} \cos \frac{s}{2}, -\frac{1}{2} \sin \frac{s}{2}, \frac{\sqrt{5}}{2}), \\ N(s) = (-\sin \frac{s}{2}, -\cos \frac{s}{2}, 0), \\ B(s) = (\frac{\sqrt{5}}{2} \cos \frac{s}{2}, -\frac{\sqrt{5}}{2} \sin \frac{s}{2}, \frac{1}{2}). \end{cases}$$

Denoting radius function  $r(s) = 2s$ , then the canal surface of type  $\mathbb{M}_-^2$  (see Figure 1) can be written by

$$\begin{aligned} x(s, \theta) = & (\sin \frac{s}{2} - 2s \cos \frac{s}{2} - 2\sqrt{3}s \sin \frac{s}{2} \cos \theta + \sqrt{15}s \cos \frac{s}{2} \sin \theta, \\ & \cos \frac{s}{2} + 2s \sin \frac{s}{2} - 2\sqrt{3}s \cos \frac{s}{2} \cos \theta - \sqrt{15}s \sin \frac{s}{2} \sin \theta, -\frac{3\sqrt{5}s}{2} + \sqrt{3}s \sin \theta). \end{aligned}$$

**Example 2.** Let null curve  $c(s) = (\cos s, \sin s, s)$ ; then, the Frenet frame can be given by

$$\begin{cases} T(s) = (-\sin s, \cos s, 1), \\ N(s) = (-\cos s, -\sin s, 0), \\ B(s) = (-\frac{1}{2} \sin s, \frac{1}{2} \cos s, -\frac{1}{2}). \end{cases}$$

Here, we denote radius function  $r(s) = s^2$  and  $\lambda(s, \theta) = -e^\theta$  in (39), the canal surface of type  $\mathbb{M}_-^3$  (see Figure 2) as

$$\begin{aligned} x(s, \theta) = & (\cos s + e^\theta \sin s - s\sqrt{(4e^\theta - s)s \cos s - s^3 \sin s}, \\ & \sin s - e^\theta \cos s - s\sqrt{(4e^\theta - s)s \sin s + s^3 \cos s}, s - e^\theta - s^3). \end{aligned}$$

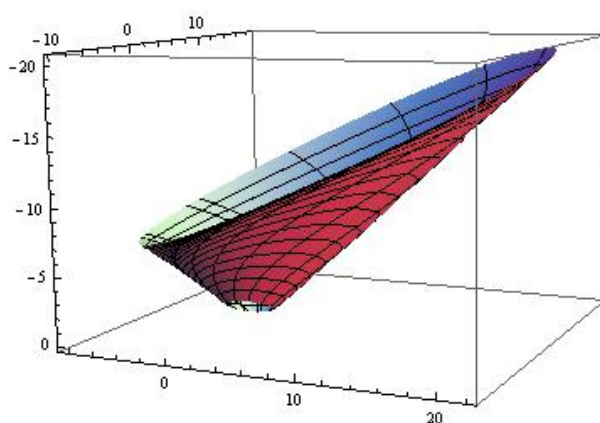


Figure 1.  $\mathbb{M}_-^2$  with  $r(s) = 2s$ .

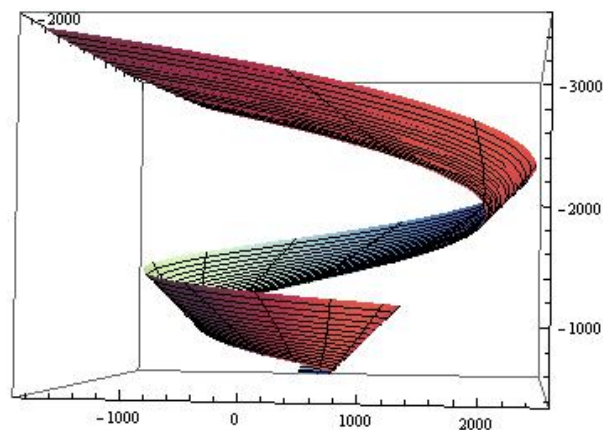


Figure 2.  $\mathbb{M}^3_-$  with  $r(s) = s^2$ .

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## References

1. Cho, H.C.; Choi, H.I.; Kwon, S.H.; Lee, D.S.; Wee, N.S. Clifford algebra, Lorentzian geometry, and rational parametrization of canal surfaces. *Comput. Aided Geom. D* **2004**, *21*, 327–339. [\[CrossRef\]](#)
2. Farouki, R.T.; Sverrisor, R. Approximation of rolling-ball blends for free-form parametric surfaces. *Comput. Aided Des.* **1996**, *28*, 871–878. [\[CrossRef\]](#)
3. Kim, Y.H.; Yoon, D.W. On non-developable ruled surface in Lorentz-Minkowski 3-spaces. *Taiwan J. Math.* **2007**, *11*, 197–214. [\[CrossRef\]](#)
4. Kim, Y.H.; Liu, H.; Qian, J. Some characterizations of canal surfaces. *Bull. Korean Math. Soc.* **2016**, *53*, 461–477. [\[CrossRef\]](#)
5. Xu, Z.Q.; Feng, R.Z.; Sun, J.G. Analytic and algebraic properties of canal surfaces. *Appl. Math. Comput.* **2016**, *195*, 220–228. [\[CrossRef\]](#)
6. Ucum, A.; Ilarslan, K. New Types of Canal Surfaces in Minkowski 3-Space. *Adv. Appl. Clifford Algebr.* **2016**, *26*, 449–468. [\[CrossRef\]](#)
7. Fu, X.; Jung, S.D.; Qian, J.; Su, M. Geometric characterizations of canal surfaces in Minkowski 3-space II. *Bull. Korean Math. Soc.* **2019**, *56*, 867–883.
8. Brunt, B.; Grant, K. Potential applications of Weingarten surfaces in CAGD, Part I: Weingarten surfaces and surface shape investigation. *Comput. Aided Geom. D* **1996**, *13*, 569–582. [\[CrossRef\]](#)
9. Karacan, M.K.; Yoon, D.W.; Tuncer, Y. Tubular surface of Weingarten types in Minkowski 3-space. *Gen. Math. Notes* **2014**, *22*, 44–56.
10. Lopez, R. Rotational linear Weingarten surfaces of hyperbolic type. *Isr. J. Math.* **2008**, *167*, 283–302. [\[CrossRef\]](#)



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