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Geodesic Mappings of $V_n(K)$-Spaces and Concircular Vector Fields

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Abstract: In the present paper, we study geodesic mappings of special pseudo-Riemannian manifolds called $V_n(K)$-spaces. We prove that the set of solutions of the system of equations of geodesic mappings on $V_n(K)$-spaces forms a special Jordan algebra and the set of solutions generated by concircular fields is an ideal of this algebra. We show that pseudo-Riemannian manifolds admitting a concircular field of the basic type form the class of manifolds closed with respect to the geodesic mappings.

Keywords: pseudo-Riemannian manifold; Jordan algebra; concircular vector field; geodesic mapping

1. Introduction

The problem of geodesic mappings of the pseudo-Riemannian manifold was first studied by Levi-Civita [1]. There exist many monographs and papers devoted to the theory of geodesic mappings and transformations [1–37]. Geodesic mappings play an important role in the general theory of relativity [8,26].

Let $A_n = (M_n, \nabla)$ be an n-dimensional manifold M_n with an affine connection ∇ without torsion. We denote the ring of smooth functions on M_n by $f(M_n)$, the Lie algebra of smooth vector fields on M_n by $X(M_n)$ and arbitrary smooth vector fields on M_n by X, Y, Z.

A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a *geodesic mapping* of A_n onto $\bar{A}_n$ if f maps any geodesic curve on A_n onto a geodesic curve on $\bar{A}_n$ [6,24–26,33].

A manifold A_n admits a geodesic mapping onto $\bar{A}_n$ if and only if the equation [6,24–26,33]

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X$$

holds for any vector fields X, Y and where ψ is a differential form on $M_n (= \bar{M}_n)$.

If $\psi = 0$ then geodesic mapping is called *trivial* and *nontrivial* if $\psi \neq 0$.

Let $V_n = (M_n, g)$ be an n-dimensional pseudo-Riemannian manifold with a metric tensor g and ∇ be a *Levi-Civita connection*.

A pseudo-Riemannian manifold V_n admits a geodesic mapping onto a pseudo-Riemannian manifold $\bar{V}_n$ if and only if there exists a differential form ψ on V_n such that the *Levi-Civita equation* [6,24,26,33]

$$(\nabla_Z \bar{g})(X, Y) = 2\psi(Z)\bar{g}(X, Y) + \psi(X)\bar{g}(Y, Z) + \psi(Y)\bar{g}(X, Z) \tag{1}$$

holds for any vector field X, Y, Z.

Or in the coordinate form

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \tag{2}$$

where $\psi_i = \nabla_i \Psi$, Ψ is a scalar field, $\bar{g}_{ij}$ are components of the metric $\bar{g}$ and comma "," denotes a covariant derivative with respect to ∇.

The Levi-Civita Equation (1) is not linear so it is not convenient for investigations. Sinyukov [24,33] proved that a pseudo-Riemannian manifold V_n admits a geodesic mapping if and only if there exist a differential form λ and a regular symmetric bilinear form a on V_n such that the equation

$$(\nabla_Z a)(X,Y) = \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z) \tag{3}$$

holds for any vector field X, Y, Z. Or in the coordinate form

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{4}$$

where a_{ij} and λ_i are components of a and λ, respectively. Note that $\lambda_i = \nabla_i \Lambda$, Λ is a scalar field.

Solutions of (2) and solutions of (4) are related by the equalities

$$a_{ij} = \exp(2\Psi(x)) \cdot \bar{g}^{\alpha\beta} g_{i\alpha} g_{j\beta} \quad \text{and} \quad \lambda_i = -\exp(2\Psi(x)) \cdot \bar{g}^{\alpha\beta} g_{i\alpha} \psi_\beta$$

where g_{ij} are components of the metric g, $(g^{ij}) = (g_{ij})^{-1}$ and $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$.

If V_n $(n > 2)$ admits two linearly independent solutions not proportional to the metric tensor g then [24]

$$(\nabla_Y \lambda)(X) = K\, a(X,Y) + \mu\, g(X,Y) \quad \text{and} \quad \nabla_X \mu = 2K\, \lambda(X), \tag{5}$$

where K is a constant and μ is a scalar field on $V_n$or in the coordinate form

$$\nabla_j \lambda_i = K\, a_{ij} + \mu\, g_{ij} \quad \text{and} \quad \nabla_k \mu = 2K\, \lambda_k. \tag{6}$$

A pseudo-Riemannian manifold satisfying the Equations (3) and (5) is called a $V_n(K)$-*space*.

These spaces for Riemannian manifolds were introduced by Solodovnikov [34] as $V(K)$-space and in another problem for pseudo-Riemannian manifolds were introduced by Mikeš [14,24] as $V_n(B)$-space (in this case $B = -K$).

A vector field φ on a pseudo-Riemannian manifold V_n is called *concircular* if

$$(\nabla_Y \varphi)X = \varrho\, g(X,Y), \tag{7}$$

where ϱ is a scalar field on V_n, see Reference [24] (p. 247), Reference [33] (p. 83) and Yano [38].

If $\varrho \neq 0$ a concircular field belongs to the *basic type* otherwise it belongs to the *exceptional type*.

A pseudo-Riemannian manifold V_n admitting a concircular field is called an *equidistant space* [24,33]. The equidistant space belongs to the *basic type* if it admits a concircular field of the basic type and it belongs to the *exceptional type* if it admits concircular fields only of the exceptional type [33].

Concircular fields play an important role in the theories of conformal and geodesic mappings and transformations. They were studied by a number of geometers: Brinkmann [39], Fialkow [40], Yano [38], Sinyukov [33], Aminova [3], Mikeš [13–16,24], Shandra [28–31] and so forth.

Let us denote the linear space of all concircular fields on V_n by $\mathrm{Con}(V_n)$. If $\overset{1}{\varphi}, \ldots, \overset{m}{\varphi}$ is a basis in $\mathrm{Con}(V_n)$ then the tensor field

$$a = \sum_{\alpha,\beta=1}^{m} \underset{\alpha\beta}{C} \, (\overset{\alpha}{\varphi} \otimes \overset{\beta}{\varphi})$$

is a solution of the system (3), where $\underset{\alpha\beta}{C}$ $(= \underset{\beta\alpha}{C})$ are some constants. So V_n admits the geodesic mapping.

Pseudo-Riemannian manifolds admitting concircular fields form the class of manifolds which is closed with respect to the geodesic mappings [24,33]. Let a pseudo-Riemannian manifold V_n admit a geodesic mapping onto a pseudo-Riemannian manifold $\bar{V}_n$, if there exists a concircular field φ on V_n then there exists a concircular field $\bar{\varphi}$ on $\bar{V}_n$ such that

$$\bar{\varrho} = \exp(\Psi)\,(\varrho + g^{ij}\varphi_i\psi_j). \tag{8}$$

A concircular field φ is said to be *special* if

$$Z(\varrho) = K\,g(Z,\varphi), \tag{9}$$

where K is a constant and it is said to be *convergent* if ϱ is a constant. A pseudo-Riemannian manifold V_n admitting a convergent field is called a *Shirokov space*, see References [24,31–33].

If there exist two linearly independent concircular fields on V_n then all concircular fields on V_n are special with the same constant K, see Reference [24]. A pseudo-Riemannian manifold V_n admitting a special concircular field is a $V_n(K)$-space. On a $V_n(K)$-space any concircular field is special.

2. Shirokov Spaces and $V_n(K)$ Spaces $(K \neq 0)$

Lemma 1. *Let a pseudo-Riemannian manifold $V_{n+1} = (M_{n+1}, G)$ admit convergent fields $\tilde{\varphi}$ such that*

$$a)\quad \|\tilde{\varphi}\| < 0 \quad and \quad b)\quad (\tilde{\nabla}_{\tilde{Y}}\tilde{\varphi})\tilde{X} = K\cdot G(\tilde{X},\tilde{Y}), \tag{10}$$

for any vector field $\tilde{X}, \tilde{Y}$ on M_{n+1}, where K $(\neq 0)$ is a constant. Then there exists the adapted coordinate system $(x^I) = (x^0, x^i)$ in which the components G_{IJ} of the metric G are reduced to the form

$$G_{IJ} = \exp(2\,K\,x^0)\cdot\begin{pmatrix} -1 & 0 \\ 0 & \dfrac{g_{ij}(x^k)}{K}\end{pmatrix} \tag{11}$$

where $g_{ij}(x^k)$ are the components of the metric of some $V_n{=}(M_n, g)$, $I, J, \ldots = 1, \ldots, n{+}1$, $i, j, \ldots = 1, \ldots, n$.

Proof. Let $\tilde{\varphi}^I$ be the components of the vector fields $\tilde{\varphi}$ g-conjugate with a convergent fields $\tilde{\varphi}$ in a coordinate system (x^I) on $V_{n+1} = (M_{n+1}, G)$. Then due to (10b) they satisfy

$$\tilde{\nabla}_J\tilde{\varphi}^I = K\,\delta^I_J. \tag{12}$$

Let D be the linear space of all vector fields on V_{n+1} which are orthogonal to $\overset{*}{\varphi}$. It is easy to check that D is involutive. So if we use as a natural basis of $X(M_{n+1})$ the basis $\{e_I\} = \{\overset{*}{\varphi}, e_i\}$, where $\{e_i\}$, is the basis in D, we get the coordinate system $(x^I) = (x^0, x^i)$ in which

$$\text{a)}\quad \tilde{\varphi}^I = \delta^I_0;\quad \text{b)}\quad G_{i0} = 0. \tag{13}$$

In these coordinates the Equations (12) are equivalent to

$$\tilde{\Gamma}^I_{0J} = K\,\delta^I_J, \tag{14}$$

where $\tilde{\Gamma}^I_{JK}$ are the components of the Levi-Civita connection of the metric G.

Let us consider the conditions (14). If $I = 0, J = j$ we have

$$\partial_j G_{00} = 0. \tag{15}$$

If $I = 0, J = 0$ we get

$$\partial_j G_{00} = 2K\,G_{00}. \tag{16}$$

It follows from (15) and (16) that $G_{00} = C \cdot \exp(2K\, x^0)$, where C is a constant. Due to (10a) it holds $C < 0$. We can choose it such that $C = -1$. So

$$G_{00} = -\exp(2K\, x^0). \tag{17}$$

If $I = i, J = j$ we obtain $\partial_0 G_{ij} = 2K\, G_{ij}$. So

$$G_{ij} = \exp(2K\, x^0)\, \frac{g_{ij}(x^k)}{K}. \tag{18}$$

It follows from (13b), (17) and (18) that in the coordinate system $(x^I) = (x^0, x^i)$ the components G_{IJ} reduce to the form (11).

Conversely, if the components G_{IJ} of the metric G in the coordinate system $(x^I) = (x^0, x^i)$ reduce to the form (11) then the components $\tilde{\Gamma}^I_{JK}$ of the Levi-Civita connection reduce to the form:

$$\tilde{\Gamma}^0_{00} = K,\ \ \tilde{\Gamma}^0_{0j} = 0,\ \ \tilde{\Gamma}^i_{0j} = \delta^i_j,\ \ \tilde{\Gamma}^0_{ij} = g_{ij},\ \ \tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}, \tag{19}$$

where Γ^k_{ij} are the components of the Levi-Civita connection of the metric g. Using direct calculations it is easy to verify that a vector field with components $\tilde{\varphi}^I_0 = \delta^I_0$ by virtue (19) satisfies the conditions (10a) and (12). □

Remark 1. *The components G^{IJ} of the inverse metric G in the adapted coordinate system $(x^I) = (x^0, x^i)$ reduce to the form*

$$G^{IJ} = \exp(-2\, K\, x^0) \begin{pmatrix} -1 & 0 \\ 0 & K\, g^{ij}(x^k) \end{pmatrix}. \tag{20}$$

Lemma 2. *The pseudo-Riemannian manifold $V_{n+1} = (M_{n+1}, G)$ with the metric defined by the conditions (11) admits an absolutely parallel covector field $\tilde{\varphi}$ if and only if its components in the adapted coordinate system $(x^I) = (x^0, x^i)$ reduce to the form*

$$\tilde{\varphi}_I = \exp(Kx^0) \left(\varrho(x^k), \varphi_i(x^k)\right), \tag{21}$$

where $\varrho(x^k)$ and $\varphi_i(x^k)$ satisfy the following equations on $V_n = (M_n, g)$:

$$\nabla_j \varphi_i = \varrho\, g_{ij}, \tag{22}$$

$$\nabla_j \varrho = K\, \varphi_j. \tag{23}$$

Proof. Let $\tilde{\varphi}_I$ be the components of an absolutely parallel covector field $\tilde{\varphi}$ in the adapted coordinate system $(x^I) = (x^0, x^i)$ on $V_{n+1} = (M_{n+1}, G)$. So

$$\tilde{\nabla}_J \tilde{\varphi}_I = 0 \tag{24}$$

If $I = 0, J = 0$ we get from (24) by virtue (19): $\partial_0 \tilde{\varphi}_0 - K\, \tilde{\varphi}_0 = 0$.
Thus

$$\tilde{\varphi}_0 = \exp(Kx^0)\, \varrho(x^k). \tag{25}$$

If $I = i, J = 0$: $\partial_0 \tilde{\varphi}_i - K\, \tilde{\varphi}_i = 0$. Hence,

$$\tilde{\varphi}_i = \exp(Kx^0)\, \tilde{\varphi}_i(x^k). \tag{26}$$

If $I = 0, J = j$: $\partial_j \tilde{\varphi}_0 - K\, \tilde{\varphi}_j = 0$. Due to (25) and (26) we have (23) and if $I = i, J = j$: $\partial_j \tilde{\varphi}_i - g_{ij} \tilde{\varphi}_0 - \Gamma^a_{ij} \tilde{\varphi}_a = 0$. Thus, we obtain (22).

Conversely, using direct calculations it is easy to check that if the covector field $\tilde{\varphi}$ has components $\tilde{\varphi}_i = \exp(Kx^0)\,(\varrho(x^k), \varphi_i(x^k))$ in the adapted coordinate system $(x^I) = (x^0, x^i)$ on $V_{n+1} = (M_{n+1}, G)$ with metric (11), where $\varrho(x^k)$ and $\varphi_i(x^k)$ satisfy the Equations (22) and (23) on $V_n = (M_n, g)$, then $\tilde{\varphi}$ due to (19) it is absolutely parallel. $\square$

Remark 2. *The Equations* (22) *and* (23) *are the coordinate forms of the Equations* (7) *and* (9) *defining a special concircular field. So the conditions* (21) *establish a one-to-one correspondence between absolutely parallel covector fields on the Shirokov space $V_{n+1} = (M_{n+1}, G)$ and special concircular fields on the $V_n(K)$-space $K \neq 0$.*

In a similar way, it is possible to prove the following statement.

Lemma 3. *The pseudo-Riemannian manifold $V_{n+1} = (M_{n+1}, G)$ with the metric defined by the conditions* (11) *admits an absolutely parallel symmetric bilinear form $\tilde{a}$ if and only if its components in the adapted coordinate system $(x^I) = (x^0, x^i)$ reduce to the form*

$$\tilde{a}_{IJ} = \exp(2\,K\,x^0) \begin{pmatrix} \mu(x^k) & \lambda_i(x^k) \\ \lambda_j(x^k) & a_{ij}(x^k) \end{pmatrix} \tag{27}$$

where $a_{ij}(x^k)$, $\lambda_i(x^k)$ and $\mu(x^k)$ satisfy the Equations (4) *and* (6) *on $V_n = (M_n, g)$.*

Remark 3. *The Equations* (4) *and* (6) *define a $V_n(K)$-space. So the conditions* (27) *establish a one-to-one correspondence between absolutely parallel symmetric bilinear forms on the Shirokov space $V_{n+1} = (M_{n+1}, G)$ and solutions of the system* (4) *and* (6) *defining geodesic mappings of the $V_n(K)$-space $(K \neq 0)$.*

Remark 4. *The set of absolutely parallel symmetric bilinear forms on $V_n = (M_n, g)$ is a special Jordan algebra J_0 with the operation of multiplication $\overset{1}{A} * \overset{2}{A} = \{\overset{1}{A}; \overset{2}{A}\}$, where A is the linear operator g-conjugate with a bilinear form a, defined by $g(AX, Y) = a(X, Y)$ and $\{\overset{1}{A}; \overset{2}{A}\}$ are Jordan brackets*

$$\{\overset{1}{A}; \overset{2}{A}\} = \frac{1}{2}\left(\overset{1}{A}\,\overset{2}{A} + \overset{2}{A}\,\overset{1}{A}\right). \tag{28}$$

The condition (28) *can be rewritten in the vector form as*

$$2\,\{\overset{1}{a}; \overset{2}{a}\}(X, Y) = \overset{1}{a}\left(\overset{2}{A}X, Y\right) + \overset{1}{a}\left(\overset{2}{A}Y, X\right) \tag{29}$$

or in the coordinate form

$$2\,\{\overset{1}{a}; \overset{2}{a}\}_{ij} = g^{ab}\left(\overset{1}{a}_{ai}\overset{2}{a}_{bj} + \overset{1}{a}_{aj}\overset{2}{a}_{bi}\right). \tag{30}$$

This statement follows from the Lemma 2.

Theorem 1. *The set of solutions of the system* (4) *and* (6) *on a $V_n(K)$-space $(K \neq 0)$ forms a special Jordan algebra J with the operation of multiplication $\left\{(\overset{1}{a}, \overset{1}{\lambda}, \overset{1}{\mu}); (\overset{2}{a}, \overset{2}{\lambda}, \overset{2}{\mu})\right\} = (\overset{3}{a}, \overset{3}{\lambda}, \overset{3}{\mu})$, where*

$$2\,\overset{3}{a}(X, Y) = K\left(\overset{1}{a}(\overset{2}{A}X, Y) + \overset{1}{a}(\overset{2}{A}Y, X)\right) - \left(\overset{1}{\lambda} \otimes \overset{2}{\lambda} + \overset{2}{\lambda} \otimes \overset{1}{\lambda}\right)(X, Y), \tag{31}$$

$$2\overset{3}{\lambda}(X) = K\left(\overset{1}{\lambda}(\overset{2}{A}X) + \overset{2}{\lambda}(\overset{1}{A}X)\right) - \left(\overset{1}{\mu}\overset{2}{\lambda}(X) + \overset{2}{\mu}\overset{1}{\lambda}(X)\right), \tag{32}$$

$$\overset{3}{\mu} = K g^{-1}\left(\overset{1}{\lambda}\,\overset{2}{\lambda}\right) - \overset{1}{\mu}\,\overset{2}{\mu}. \tag{33}$$

The algebra J is isomorphic to the special Jordan algebra J_0 of absolutely parallel symmetric bilinear forms on the Shirokov space $V_{n+1} = (M_{n+1}, G)$ with the metric (11).

Proof of the theorem follows immediately from the Lemma 2 and (20), (27) and (30).

Remark 5. *Due to* (29) *the unit of the algebra J_0 is G so the unit of the algebra J is $\left(\frac{g}{K}, 0, -1\right)$.*

Remark 6. *If there exists a convergent field $\tilde{\varphi}$ on $V_{n+1} = (M_{n+1}, G)$ such that $\|\tilde{\varphi}\| > 0$, then there exists the adapted coordinate system $(x^I) = (x^0, x^i)$ in which the components G_{IJ} of the metric G reduce to the form*

$$G_{IJ} = \exp(2Kx^0)\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{-g_{ij}(x^k)}{K} \end{pmatrix},$$

where $g_{ij}(x^k)$ are the components of the metric of some $V_n = (M_n, g)$. Using this metric and (29) *we can define a new operation of multiplication $\{\cdot,\cdot\}_2$. It is obvious that $\{\overset{1}{A};\overset{2}{A}\} = -\{\overset{1}{A};\overset{2}{A}\}_2$.*

Corollary 1. *Let $V_n = (M_n, g)$ be a $V_n(K)$-space $(K \neq 0)$ then there exists the solution (a, λ, μ) of the system* (4) *and* (6) *satisfying the following conditions:*

$$K\,a(AX, Y) - (\lambda \otimes \lambda)(X, Y) = \frac{e\,g(X, Y)}{K}, \tag{34}$$

$$K\,\lambda(AX) - \mu\,\lambda(X) = 0, \tag{35}$$

$$K\,g^{-1}(\lambda, \lambda) - \mu^2 = -e, \tag{36}$$

where e takes values $\pm 1, 0$.

Proof. Let $\tilde{b}$ be an absolutely parallel symmetric bilinear form on the Shirokov space $V_{n+1} = (M_{n+1}, G)$ with the metric (11). Then as it has been shown in Reference [11] there exists the absolutely parallel symmetric bilinear form $\tilde{a}$ on $V_{n+1} = (M_{n+1}, G)$ such that $\tilde{A}^2 = e$ or in the equivalent form

$$\tilde{a}(\tilde{A}\tilde{X}, \tilde{Y}) = e\,G(\tilde{X}, \tilde{Y}). \tag{37}$$

The Equation (37) means that $\{\tilde{a}, \tilde{a}\} = e\,G$. Hence if (a, λ, μ) is the corresponding solution of the system (4) and (6) on the $V_n(K)$-space $(K \neq 0)$ then taking into account (31)–(33) we get (34)–(36). □

As mentioned above concircular fields generate a solution of the Equation (2). Denote this set of solutions by J_c.

Theorem 2. *J_c is an ideal of J.*

Proof. To prove that J_c is an ideal of J on $V_n = (M_n, g)$ it is equivalent to prove that J_{0c} is an ideal of J_0 on $V_{n+1} = (M_{n+1}, G)$, where J_{0c} is the set of absolutely parallel symmetric bilinear forms generated by absolutely parallel covector fields.

Let $\overset{1}{\varphi}, \dots, \overset{m}{\varphi}$ be a basis of the linear space $\mathrm{Conv}(V_{n+1})$ of absolutely parallel covector fields on $V_{n+1} = (M_{n+1}, G)$. Then any absolutely parallel symmetric bilinear form generated by absolutely parallel covector fields has the components

$$\tilde{b}_{IJ} = \sum_{\alpha,\beta=1}^{m} \underset{\alpha\beta}{C} (\overset{\alpha}{\varphi}_I \overset{\beta}{\varphi}_J),$$

where $\underset{\alpha\beta}{C}$ $(= \underset{\beta\alpha}{C})$ are some constants. Let $\tilde{a}_{IJ}$ be the components of the arbitrary absolutely parallel symmetric bilinear form $\tilde{a}$. We should prove that $\{\tilde{a}, \tilde{b}\} \in J_{0c}$. We have

$$2\{\tilde{a}, \tilde{b}\} = G^{DT} \sum_{\alpha,\beta=1}^{m} \underset{\alpha\beta}{C} (\overset{\alpha}{\varphi}_I \overset{\beta}{\varphi}_D \tilde{a}_{TJ} + \overset{\alpha}{\varphi}_J \overset{\beta}{\varphi}_D \tilde{a}_{TI}) = \sum_{\alpha,\beta=1}^{m} \underset{\alpha\beta}{C} (\overset{\alpha}{\varphi}_I \overset{\beta}{\Phi}_J + \overset{\alpha}{\varphi}_J \overset{\beta}{\Phi}_I), \tag{38}$$

where $\overset{\beta}{\Phi}_I = \overset{\beta}{\varphi}_D \tilde{a}_{TI} G^{DT}$ is an absolutely parallel covector field. Therefore,

$$\overset{\beta}{\Phi}_I = \sum_{\gamma=1}^{m} F^{\beta}_{\gamma} \overset{\gamma}{\varphi}_I \tag{39}$$

where F^{β}_{γ} are some constants. It follows from (38) and (39) that

$$2\{\tilde{a}, \tilde{b}\}_{IJ} = \sum_{\alpha,\beta,\gamma=1}^{m} \left(F^{\gamma}_{\beta} \underset{\alpha\gamma}{C} + F^{\gamma}_{\alpha} \underset{\beta\gamma}{C} \right) \overset{\alpha}{\varphi}_I \overset{\beta}{\varphi}_J.$$

Thus, $\{\tilde{a}, \tilde{b}\} \in J_{0c}$. □

3. $V_n(0)$-Spaces

Let $(M_n.g)$ be a $V_n(0)$-space, then there exists a solution of the system

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{40}$$

$$\nabla_k \lambda_i = \mu\, g_{ik}, \tag{41}$$

where μ is a constant and $\lambda_i = \nabla_i \Lambda$. Thus, a $V_n(0)$-space is a Shirokov space.

Lemma 4. *If the $V_n(0)$-space does not admit any convergent field of the basic type and φ is an absolutely parallel covector field on it, then there exists the sequence of absolutely parallel covector fields $\left\{\overset{\alpha}{\varphi}\right\}$ $(\alpha \in \mathbb{N})$ such that*

$$a)\ \overset{\alpha+1}{\varphi}(X) = \overset{\alpha}{\varphi}(AX) - \overset{\alpha}{f} \lambda(X), \quad b)\ \overset{\alpha}{\varphi}(\lambda^*) = 0, \ \forall \alpha \in \mathbb{N}, \tag{42}$$

where $\overset{1}{\varphi} = \varphi$, $d\overset{\alpha}{f} = \overset{\alpha}{\varphi}$, λ^* is the vector field g-conjugate with λ.

Proof. Taking into account that the $V_n(0)$ does not admit any convergent fields of the basic type we obtain from (41) that

$$\nabla_k \lambda_i = 0. \tag{43}$$

Let φ_i be the components of an absolutely parallel covector field φ on a $V_n(0)$. Denote $\overset{1}{\varphi} = \varphi$. Consider the covector field

$$\overset{2}{\varphi}_i = a^t_i \overset{1}{\varphi} - \overset{1}{f} \lambda_i, \tag{44}$$

where a_i^t are components of the linear operator A ($a_i^j = g^{jl} a_{il}$). It follows from (44) due to (40) and (43)

$$\nabla_k \overset{2}{\varphi}_i = \overset{1}{\varphi}_t \lambda^t g_{ik}, \tag{45}$$

where $\lambda^t = g^{ti}\lambda_i$. According to our assumption it follows from (45) that

$$\overset{1}{\varphi}_t \lambda^t = 0 \quad \text{and} \quad \nabla_k \overset{2}{\varphi}_i = 0.$$

Applying now similar argumentation to the covector $\overset{2}{\varphi}_i$ and continuing the process in this way, we obtain the desired sequence. □

Remark 7. *The Equation (42b) due to (42a) can be rewritten as*

$$\varphi(\overset{\alpha-1}{\lambda}{}^*) = 0, \;\; \forall \alpha \in \mathbb{N}, \tag{46}$$

where $\overset{\alpha}{A}$ is the α-s power of the linear operator A.

Theorem 3. *Let a pseudo-Riemannian manifold V_n be a $V_n(0)$-space. Then there exists a convergent field of the basic type on V_n or there exists the sequence of linearly independent absolutely parallel covector fields $\{\overset{\alpha}{\lambda}\}$, $(\alpha = 1, 2 \ldots, p \leq n-1)$ such that*

$$\overset{\alpha+1}{\lambda}(X) = \overset{\alpha}{\lambda}(AX) - \overset{\alpha}{\Lambda}\, \alpha(X), \quad \lambda(\overset{\alpha-1}{A}\lambda^*) = 0, \;\; \forall \alpha \in A, \tag{47}$$

$$\overset{p}{\lambda}(AX) = \overset{p}{\Lambda}\,\lambda(X), \tag{48}$$

where $\overset{1}{\lambda} = \lambda$, λ^* is the vector field g-conjugate with λ.

Proof. (1) It follows from (41) that if $\mu \neq 0$ then λ is a convergent field of the basic type on $V_n(0)$.

(2) Let $\mu = 0$, then $\nabla\lambda = 0$. According to the Lemma 4 and the Remark 7 we can construct the sequence of absolutely parallel covector fields $\{\overset{\alpha}{\lambda}\}$ $(\alpha \in \mathbb{N})$ such that

$$\overset{\alpha+1}{\lambda}(X) = \overset{\alpha}{\lambda}(AX) - \overset{\alpha}{\Lambda}\,\lambda(X), \quad \lambda(\overset{\alpha-1}{A}\lambda^*) = 0, \;\; \forall \alpha \in \mathbb{N}.$$

This sequence contains no more than p $(\leq n-1)$ linearly independent covectors. Otherwise, $V_n(0)$ will be locally flat and so it will admit a convergent field of the basic type. Thus,

$$\overset{p+1}{\lambda} = \sum_{\alpha=1}^{p} C_\alpha \overset{\alpha}{\lambda},$$

where C_α are constants and $\overset{1}{\lambda}, \ldots, \overset{p}{\lambda}$ are linearly independent. Changing $\overset{\alpha}{\Lambda}$ (defined to a constant) we can make $\overset{p+1}{\lambda} = 0$. So we get (48). □

Corollary 2. *If the $V_n(0)$-space does not admit any converging fields of the basic type and φ is an absolutely parallel covector field on it, then*

$$\overset{\alpha-1}{\lambda}(\varphi^*) = 0, \;\; \forall \alpha \in \mathbb{N} \tag{49}$$

where φ^* is the vector field g-conjugate with φ.

Proof. We get from (46): $(\overset{\alpha-1}{A}\lambda^*) = \overset{\alpha-1}{A}\lambda(\varphi^*) = \overset{\alpha-1}{\lambda}(\varphi^*) = 0.$ $\square$

The following statement holds.

Theorem 4. *Let a pseudo-Riemannian manifold V_n admit a geodesic mapping onto a pseudo-Riemannian manifold $\bar{V}_n$ if there exists a concircular field of the basic type on $\bar{V}_n$, then there exists a concircular field of the basic type on V_n.*

Proof. Let $\bar{\varphi}$ be a concircular field of the basic type on $\bar{V}_n$ ($\bar{\varrho} \neq 0$), then there exists a concircular field φ on V_n. Let us suppose the contrary, namely that V_n does not admit concircular fields of the basic type. It means that $\varrho = 0$. So φ is an absolutely parallel covector field and, therefore, V_n is a $V_n(0)$-space [30]. So according to Theorem 3 there exists a V_n on the sequence of linearly independent absolutely parallel covector fields $\{\overset{\alpha}{\lambda}\}$ $(\alpha = 1, 2, \ldots, p \leq n-1)$ satisfying (47) and (48). The Equation (48) in the coordinate form can be written as

$$a_i^t \overset{p}{\lambda}_t = \overset{p}{\Lambda} \lambda_i. \tag{50}$$

Contracting (50) with $\bar{a}_j^i$ (the inverse operator to a_j^i) by i and taking into account that $\lambda_i = -a_i^t \psi_t$ we get

$$\overset{p}{\lambda}_j = -\overset{p}{\Lambda} \psi_j. \tag{51}$$

The condition (49) means that $\varphi^t \overset{p}{\lambda}_t = 0$. Hence, due to $\overset{p}{\Lambda} \neq 0$ it follows from (51) that $\varphi^t \psi_t = 0$. On the other hand since $\bar{\varrho} \neq 0$ and $\varrho = 0$ the Equation (8) gives us $\varphi^t \psi_t \neq 0$. This contradiction proves the theorem. $\square$

Remark 8. *The Theorem 4 shows that pseudo-Riemannian manifolds admitting a concircular field of the basic type (i.e., equidistant spaces of the basic type) form the class of manifolds closed with respect to the geodesic mappings. The same properties have spaces of constant curvature [24,33], Einstein spaces [17,24] and $V_n(K)$-spaces [24].*

Corollary 3. *Let an equidistant space of the exeptional type V_n admit a geodesic mapping onto a pseudo-Riemannian manifold $\bar{V}_n$, then $\bar{V}_n$ is an equidistant space of the exeptional type.*

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