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A Study of Multivalent q -starlike Functions Connected with Circular Domain

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Abstract: Starlike functions have gained popularity both in literature and in usage over the past decade. In this paper, our aim is to examine some useful problems dealing with q -starlike functions. These include the convolution problem, sufficiency criteria, coefficient estimates, and Fekete–Szegő type inequalities for a new subfamily of analytic and multivalent functions associated with circular domain. In addition, we also define and study a Bernardi integral operator in its q -extension for multivalent functions. Furthermore, we will show that the class defined in this paper, along with the obtained results, generalizes many known works available in the literature.

Keywords: multivalent functions; q -Ruschweyh differential operator; q -starlike functions; circular domain; q -Bernardi integral operator

1. Introduction

The study of q -extension of calculus and q -analysis has attracted and motivated many researchers because of its applications in different parts of mathematical sciences. Jackson was one of the main contributors among all mathematicians who initiated and established the theory of q -calculus [1,2]. As an interesting sequel to [3], in which the q -derivative operator was used for the first time for studying the geometry of q -starlike functions, a firm footing of the usage of the q -calculus in the context of Geometric Function Theory was provided and the basic (or q -) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [4] (pp. 347 et seq.)). The theory of q -starlike functions was later extended to various families of q -starlike functions by Agrawal and Sahoo in [5] (see also the recent investigations on this subject by Srivastava et al. [6–11]). Motivated by these q -developments in Geometric Function Theory, many authors added their contributions in this direction which has made this research area much more attractive in works like [4,12].

In 2014, Kanas and Răducanu [13] used the familiar Hadamad products to define a q -extension of the Ruschweyh operator and discussed important applications of this operator in detail. Moreover, the extensive study of this q -Ruschweyh operator was further made by Mohammad and Darus [14] and Mahmood and Sokół in [15]. Recently, a new idea was presented by Darus [16] that introduced a new differential operator called a generalized q -differential operator, with the help of q -hypergeometric functions where they studied some useful applications of this operator. For the recent extensions of different operators in q -analogue, see the work in [17–19]. The operator defined in [13] was extended further for multivalent functions by Arif et al. in [20] where they investigated its important applications.

The aim of this paper is to define a family of multivalent q -starlike functions associated with circular domains and to study some of its useful properties.

Background

Let \mathfrak{A}_p ($p \in \mathbb{N} = \{0, 1, 2, \dots\}$) contain all multivalent functions say f that are holomorphic or analytic in a subset $\mathbb{D} = \{z : |z| < 1\}$ of a complex plane \mathbb{C} and having the series form:

$$f(z) = z^p + \sum_{l=1}^{\infty} a_{l+p} z^{l+p}, \quad (z \in \mathbb{D}). \tag{1}$$

For two analytic functions f and g in \mathbb{D} , then f is subordinate to g , symbolically presented as $f \prec g$ or $f(z) \prec g(z)$, if we can find an analytic function w with the properties $w(0) = 0$ & $|w(z)| < 1$ such that $f(z) = g(w(z))$. Also, if g is univalent in \mathbb{D} , then we have

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

For given $q \in (0, 1)$, the derivative in q -analogue of f is given by

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (z \neq 0, q \neq 1). \tag{2}$$

Making (1) and (2), we easily get that for $n \in \mathbb{N}$ and $z \in \mathbb{D}$:

$$\mathcal{D}_q \left\{ \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \right\} = \sum_{n=1}^{\infty} [n+p]_q a_{n+p} z^{n+p-1}, \tag{3}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n-1} q^l, \quad [0, q] = 0.$$

For $n \in \mathbb{Z}^* := \mathbb{Z} \setminus \{-1, -2, \dots\}$, the q -number shift factorial is given as

$$[n]_q! = \begin{cases} 1, & n = 0, \\ [1]_q [2]_q \cdots [n]_q, & n \in \mathbb{N}. \end{cases}$$

Also, with $x > 0$, the q -analogue of the Pochhammer symbol has the form

$$[x, q]_{qn} = \begin{cases} 1, & n = 0, \\ [x, q][x+1, q] \cdots [x+n-1, q], & n \in \mathbb{N}, \end{cases}$$

and, for $x > 0$, the Gamma function in q -analogue is presented as

$$\Gamma_q(x+1) = [x, q] \Gamma_q(x) \quad \text{and} \quad \Gamma_q(1) = 1.$$

We now consider a function

$$\Phi_p(q, \mu + 1; z) = z^p + \sum_{n=2}^{\infty} \Lambda_{n+p} z^{n+p}, \quad (\mu > -1, z \in \mathbb{D}), \tag{4}$$

with

$$\Lambda_{n+p} = \frac{[\mu + 1, q]_{n+p}}{[n+p]_q!}. \tag{5}$$

The series defined in (4) converges absolutely in \mathbb{D} . Using $\Phi_p(q, \mu; z)$ with $\mu > -1$ and idea of convolution, Arif et al. [20] established a differential operator $\mathcal{L}_q^{\mu+p-1} : \mathfrak{A}_p \rightarrow \mathfrak{A}_p$ by

$$\mathcal{L}_q^{\mu+p-1} f(z) = \Phi_p(q, \mu; z) * f(z) = z^p + \sum_{n=2}^{\infty} \Lambda_{n+p} a_{n+p} z^{n+p}, \quad (z \in \mathbb{D}). \tag{6}$$

We also note that

$$\lim_{q \rightarrow 1^-} \Phi_p(q, \mu; z) = \frac{z^p}{(1-z)^{\mu+1}} \quad \text{and} \quad \lim_{q \rightarrow 1^-} \mathcal{L}_q^{\mu+p-1} f(z) = f(z) * \frac{z^p}{(1-z)^{\mu+1}}.$$

Now, when $q \rightarrow 1^-$, the operator defined in (6) becomes the familiar differential operator investigated in [21] and further, setting $p = 1$, we get the most familiar operator known as Ruscheweyh operator [12] (see also [22,23]). Also, for different types of operators in q -analogue, see the works [16,17,19,24–26].

Motivated from the work studied in [3,18,27–29], we establish a family $\mathcal{S}_p^*(q, \mu, A, B)$ using the operator $\mathcal{L}_q^{\mu+p-1}$ as follows:

Definition 1. Suppose that $q \in (0, 1)$ and $-1 \leq B < A \leq 1$. Then, $f \in \mathfrak{A}_p$ belongs to the set $\mathcal{S}_p^*(q, \mu, A, B)$, if it satisfies

$$\frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{7}$$

where the function $\frac{1+Az}{1+Bz}$ is known as Janowski function studied in [30].

Alternatively,

$$f \in \mathcal{S}_p^*(q, \mu, A, B) \Leftrightarrow \left| \frac{\frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)} - 1}{A - B \frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)}} \right| < 1. \tag{8}$$

Note: We will assume throughout our discussion, unless otherwise stated,

$$-1 \leq B < A \leq 1, \quad q \in (0, 1), \quad p \in \mathbb{N}, \quad \text{and} \quad \mu > -1.$$

2. A Set of Lemmas

Lemma 1. [31] Let $h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \prec K(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$ in \mathbb{D} . If $K(z)$ is convex univalent in \mathbb{D} , then,

$$|d_n| \leq |k_1|, \quad \text{for } n \geq 1.$$

Lemma 2. Let \mathcal{W} contain all functions w that are analytic in \mathbb{D} , which satisfies $w(0) = 0$ & $|w(z)| < 1$ if the function $w \in \mathcal{W}$, given by

$$w(z) = \sum_{k=1}^{\infty} w_k z^k \quad (z \in \mathbb{D}).$$

Then, for $\lambda \in \mathbb{C}$, we have

$$\left| w_2 - \lambda w_1^2 \right| \leq \max \{1; |\lambda|\}, \tag{9}$$

and

$$\left| w_3 + \frac{1}{4} w_1 w_2 + \frac{1}{16} w_1^3 \right| \leq 1. \tag{10}$$

These results are the best possible.

For the first and second part, see references [32,33], respectively.

3. Main Results and Their Consequences

Theorem 1. Let $f \in \mathfrak{A}_p$ have the series form (1) and satisfy the inequality given by

$$\sum_{n=1}^{\infty} \wedge_{n+p} ([n+p, q] (1-B) - [p, q] (1-A)) |a_{n+p}| \leq [p, q] (A-B). \tag{11}$$

Then, $f \in \mathcal{S}_p^*(q, \mu, A, B)$.

Proof. To show $f \in \mathcal{S}_p^*(q, \mu, A, B)$, we just need to show the relation (8). For this, we consider

$$\left| \frac{\frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)} - 1}{A - B \frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)}} \right| = \left| \frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z) - [p, q] \mathcal{L}_q^{\mu+p-1} f(z)}{A [p, q] \mathcal{L}_q^{\mu+p-1} f(z) - B zD_q \mathcal{L}_q^{\mu+p-1} f(z)} \right|.$$

Using (6), and with the help of (11) and (3), we have

$$\begin{aligned} &= \left| \frac{[p, q] z^p + \sum_{n=1}^{\infty} \wedge_{n+p} a_{n+p} [n+p, q] z^{n+p} - [p, q] (z^p + \sum_{n=1}^{\infty} \wedge_{n+p} a_{n+p} z^{n+p})}{A [p, q] (z^p + \sum_{n=1}^{\infty} \wedge_{n+p} a_{n+p} z^{n+p}) - B ([p, q] z^p + \sum_{n=1}^{\infty} \wedge_{n+p} a_{n+p} [n+p, q] z^{n+p})} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} \wedge_{n+p} a_{n+p} ([n+p, q] - [p, q]) z^{n+p}}{(A-B) [p, q] z^p + \sum_{n=1}^{\infty} \wedge_{n+p} a_{n+p} (A [p, q] - B [n+p, q]) z^{n+p}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} \wedge_{n+p} |a_{n+p}| ([n+p, q] - [p, q]) |z|^{n+p}}{(A-B) [p, q] |z|^p - \sum_{n=1}^{\infty} \wedge_{n+p} |a_{n+p}| (A [p, q] - B [n+p, q]) |z|^{n+p}} \\ &\leq \frac{\sum_{n=1}^{\infty} \wedge_{n+p} |a_{n+p}| ([n+p, q] - [p, q])}{(A-B) [p, q] - \sum_{n=1}^{\infty} \wedge_{n+p} |a_{n+p}| (A [p, q] - B [n+p, q])} < 1, \end{aligned}$$

where we have used the inequality (11) and this completes the proof. \square

Varying the parameters $\mu, b, A,$ and B in the last Theorem, we get the following known results discussed earlier in [34].

Corollary 1. Let $f \in \mathfrak{A}$ be given by (1) and satisfy the inequality

$$\sum_{n=2}^{\infty} ([n, q] (1-B) - 1 + A) |a_n| \leq A - B.$$

Then, the function $f \in \mathcal{S}_q^*[A, B]$.

By choosing $q \rightarrow 1^-$ in the last corollary, we get the known result proved by Ahuja [22] and, furthermore, for $A = 1 - \alpha$ and $B = -1$, we obtain the result for the family $\mathcal{S}^*(\xi)$ which was proved by Silverman [35].

Theorem 2. Let $f \in \mathcal{S}_p^*(q, \mu, A, B)$ be of the form (1). Then,

$$|a_{p+1}| \leq \frac{\psi_1 (A - B)}{\wedge_{1+p}}, \tag{12}$$

and for $n \geq 2$,

$$|a_{n+p}| \leq \frac{(A - B) \psi_n}{\wedge_{n+p}} \prod_{t=1}^{n-1} \left(1 + \frac{[p, q] (A - B)}{([p+t, q] - [p, q])} \right), \tag{13}$$

where

$$\psi_n := \psi_n(p, q) = \frac{[p, q]}{([n + p, q] - [p, q])}. \tag{14}$$

Proof. If $f \in \mathcal{S}_p^*(q, \mu, A, B)$, then by definition we have

$$\frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{15}$$

Let us put

$$p(z) = 1 + \sum_{n=1}^{\infty} d_n z^n = \frac{1 + Aw(z)}{1 + Bw(z)}.$$

Then, by Lemma 1, we get

$$|d_n| \leq A - B. \tag{16}$$

Now, from (15) and (6), we can write

$$z^p + \sum_{n=1}^{\infty} \frac{[n+p, q]}{[p, q]} \Lambda_{n+p} a_{n+p} z^{n+p} = \left(1 + \sum_{n=1}^{\infty} d_n z^n \right) \left(z^p + \sum_{n=1}^{\infty} \Lambda_{n+p} a_{n+p} z^{n+p} \right). \tag{17}$$

Equating coefficients of z^{n+p} on both sides,

$$\Lambda_{n+p} ([n + p, q] - [p, q]) a_{n+p} = [p, q] \Lambda_{n+p-1} a_{n+p-1} d_1 + \dots + [p, q] \Lambda_{1+p} a_{1+p} d_{n-1}.$$

Taking absolute on both sides and then using (16), we have

$$\Lambda_{n+p} ([n + p, q] - [p, q]) |a_{n+p}| \leq [p, q] (A - B) \left(1 + \sum_{k=1}^{n-1} \Lambda_{k+p} |a_{k+p}| \right),$$

and this further implies

$$|a_{n+p}| \leq \frac{(A - B) \psi_n}{\Lambda_{n+p}} \left(1 + \sum_{k=1}^{n-1} \Lambda_{k+p} |a_{k+p}| \right), \tag{18}$$

where ψ_n is given by (14). So, for $n = 1$, we have from (18)

$$|a_{p+1}| \leq \frac{(A - B) \psi_1}{\Lambda_{1+p}},$$

and this shows that (12) holds for $n = 1$. To prove (13), we apply mathematical induction. Therefore, for $n = 2$, we have from (12):

$$|a_{p+2}| \leq \frac{(A - B) \psi_2}{\Lambda_{2+p}} (1 + \Lambda_{1+p} |a_{1+p}|),$$

using (12), we have

$$|a_{p+2}| \leq \frac{(A - B) \psi_2}{\Lambda_{2+p}} (1 + (A - B) \psi_1),$$

which clearly shows that (13) holds for $n = 2$. Let us assume that (13) is true for $n \leq m - 1$, that is,

$$|a_{m-1+p}| \leq \frac{(A - B) \psi_{m-1}}{\Lambda_{m+p-1}} \prod_{t=1}^{m-2} (1 + (A - B) \psi_t).$$

Consider

$$\begin{aligned}
 |a_{m+p}| &\leq \frac{(A - B) \psi_m}{\Lambda_{m+p}} \left(1 + \sum_{k=1}^{m-1} \Lambda_{k+p} |a_{k+p}| \right) \\
 &= \frac{(A - B) \psi_m}{\Lambda_{m+p}} \left\{ 1 + (A - B) \psi_1 + \dots + (A - B) \psi_{m-1} \prod_{t=1}^{m-2} (1 + (A - B) \psi_t) \right\} \\
 &= \frac{(A - B) \psi_m}{\Lambda_{m+p}} \prod_{t=1}^{m-1} \left(1 + \frac{[p, q] (A - B)}{([p + t, q] - [p, q])} \right),
 \end{aligned}$$

this implies that the given result is true for $n = m$. Hence, using mathematical induction, we achieve the inequality (13). □

Theorem 3. Let $f \in S_p^*(q, \mu, A, B)$, and be given by (1). Then, for $\lambda \in \mathbb{C}$

$$\left| a_{p+2} - \lambda a_{p+1}^2 \right| \leq \frac{(A - B) \psi_2}{\Lambda_{p+2}} \{1; |v|\},$$

where v is given by

$$v = (B - (A - B) \psi_1) + \frac{\Lambda_{p+2} \psi_1^2}{\Lambda_{p+1}^2 \psi_2} (A - B) \lambda. \tag{19}$$

Proof. Let $f \in S_p^*(q, \mu, A, B)$, and consider the right-hand side of (15), we have

$$\frac{1 + Aw(z)}{1 + Bw(z)} = \left(1 + A \sum_{k=1}^{\infty} w_k z^k \right) \left(1 + B \sum_{k=1}^{\infty} w_k z^k \right)^{-1},$$

where

$$w(z) = \sum_{k=1}^{\infty} w_k z^k,$$

and after simple computations, we can rewrite

$$\frac{1 + Aw(z)}{1 + Bw(z)} = 1 + (A - B) w_1 z + (A - B) \left\{ w_2 - B w_1^2 \right\} z^2 + \dots \tag{20}$$

Now, for the left hand side of (15), we have

$$\begin{aligned}
 \frac{z D_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)} &= \left(1 + \sum_{n=1}^{\infty} \frac{[n + p, q]}{[p, q]} \Lambda_{n+p} a_{n+p} z^n \right) \left(1 + \sum_{n=1}^{\infty} \Lambda_{n+p} a_{n+p} z^n \right)^{-1} \\
 &= 1 + \frac{\Lambda_{1+p}}{\psi_1} a_{1+p} z + \left(\frac{\Lambda_{2+p} a_{2+p}}{\psi_2} - \frac{\Lambda_{1+p}^2 a_{1+p}^2}{\psi_1} \right) z^2 + \dots
 \end{aligned} \tag{21}$$

From (20) and (21), we have

$$a_{p+1} = \frac{\psi_1}{\Lambda_{p+1}} (A - B) w_1, \tag{22}$$

$$a_{p+2} = \frac{(A - B) \psi_2}{\Lambda_{p+2}} \left\{ w_2 + ((A - B) \psi_1 - B) w_1^2 \right\}. \tag{23}$$

Now, consider

$$\begin{aligned} \left| a_{p+2} - \lambda a_{p+1}^2 \right| &= \left| \frac{(A - B)\psi_2}{\Lambda_{p+2}} \left\{ w_2 + ((A - B)\psi_1 - B) w_1^2 \right\} - \lambda \frac{\psi_1^2}{\Lambda_{p+1}^2} (A - B)^2 w_1^2 \right| \\ &= \frac{(A - B)\psi_2}{\Lambda_{p+2}} \left| w_2 - \left\{ (B - (A - B)\psi_1) + \frac{\Lambda_{p+2}\psi_1^2}{\Lambda_{p+1}^2\psi_2} (A - B)\lambda \right\} w_1^2 \right|, \end{aligned}$$

using Lemma 2, we have

$$\left| a_{p+2} - \lambda a_{p+1}^2 \right| \leq \frac{(A - B)\psi_2}{\Lambda_{p+2}} \{1; |v|\},$$

where v is given by

$$v = (B - (A - B)\psi_1) + \frac{\Lambda_{p+2}\psi_1^2}{\Lambda_{p+1}^2\psi_2} (A - B)\lambda.$$

This completes the proof. \square

Theorem 4. Let $f \in \mathcal{S}_p^*(q, \mu, A, B)$ and be given by (1). Then,

$$\left| a_{p+3} - \frac{q+2}{q^2+q+1} \frac{\Lambda_{1+p}\Lambda_{2+p}}{\Lambda_{3+p}} a_{p+2}a_{p+1} + \frac{1}{[3, q]} \frac{\Lambda_{1+p}^3}{\Lambda_{3+p}} a_{p+1}^3 \right| \leq (A - B) \left\{ \frac{4(2B - 1)^2 + 1}{8\Lambda_{3+p}} \right\} \psi_3,$$

where ψ_n and Λ_{n+p} are defined by (14) and (5), respectively.

Proof. From the relations (20) and (21), we have

$$\left(a_{p+3} - \frac{q+2}{q^2+q+1} \frac{\Lambda_{1+p}\Lambda_{2+p}}{\Lambda_{3+p}} a_{p+2}a_{p+1} + \frac{1}{[3, q]} \frac{\Lambda_{1+p}^3}{\Lambda_{3+p}} a_{p+1}^3 \right) = \frac{(A - B)\psi_3}{\Lambda_{3+p}} \left\{ w_3 - 2Bw_1w_2 + B^2w_1^3 \right\},$$

equivalently, we have

$$\begin{aligned} &\left| \left(a_{p+3} - \frac{q+2}{q^2+q+1} \frac{\Lambda_{1+p}\Lambda_{2+p}}{\Lambda_{3+p}} a_{p+2}a_{p+1} + \frac{1}{[3, q]} \frac{\Lambda_{1+p}^3}{\Lambda_{3+p}} a_{p+1}^3 \right) \right| \\ &= \frac{(A - B)\psi_3}{\Lambda_{3+p}} \left| \left(w_3 + \frac{1}{4}w_1w_2 + \frac{1}{16}w_1^3 \right) - \frac{16B^2 - 1}{16} (w_2 - w_1^2) + \frac{16B^2 - 32B - 5}{16} w_2 \right| \\ &\leq \frac{(A - B)\psi_3}{\Lambda_{3+p}} \left\{ 1 + \frac{16B^2 - 1}{16} + \frac{16B^2 - 32B - 5}{16} \right\} \\ &\leq \frac{(A - B)\psi_3}{\Lambda_{3+p}} \left\{ \frac{16B^2 - 16B + 5}{8} \right\}, \end{aligned}$$

where we have used (9) and (10). This completes the proof. \square

Theorem 5. Let $f \in \mathfrak{A}_p$ be given by (1). Then, the function f is in the class $\mathcal{S}_p^*(q, \mu, A, B)$, if and only if

$$\frac{e^{i\theta} (B - [p, q] A)}{z} \left[\mathcal{L}_q^{\mu+p-1} f(z) * \left(\frac{(N + 1) z^p - qLz^{p+1}}{(1 - z)(1 - qz)} \right) \right] \neq 0, \tag{24}$$

for all

$$\begin{aligned}
 N &= N_\theta = \frac{([p, q] - 1) e^{-i\theta}}{([p, q] A - B)}, \\
 L &= L_\theta = \frac{(e^{-i\theta} + [p, q] A)}{([p, q] A - B)},
 \end{aligned}
 \tag{25}$$

and also for $N = 0, L = 1$.

Proof. Since the function $f \in \mathcal{S}_p^*(q, \mu, A, B)$ is analytic in \mathbb{D} , it implies that $\mathcal{L}_q^{\mu+p-1} f(z) \neq 0$ for all $z \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ —that is

$$\frac{e^{i\theta} (B - [p, q] A)}{z} \mathcal{L}_q^{\mu+p-1} f(z) \neq 0 \quad (z \in \mathbb{D}),$$

and this is equivalent to (24) for $N = 0$ and $L = 1$. From (7), according to the definition of the subordination, there exists an analytic function w with the property that $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z D_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathbb{D}),$$

which is equivalent for $z \in \mathbb{D}, 0 \leq \theta < 2\pi$

$$\frac{z D_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p, q] \mathcal{L}_q^{\mu+p-1} f(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}},
 \tag{26}$$

and further written in a more simplified form

$$(1 + Be^{i\theta}) z D_q \mathcal{L}_q^{\mu+p-1} f(z) - [p, q] (1 + Ae^{i\theta}) \mathcal{L}_q^{\mu+p-1} f(z) \neq 0.
 \tag{27}$$

Now, using the following convolution properties in (27)

$$\mathcal{L}_q^{\mu+p-1} f(z) * \frac{z^p}{(1-z)} = \mathcal{L}_q^{\mu+p-1} f(z) \quad \text{and} \quad \mathcal{L}_q^{\mu+p-1} f(z) * \frac{z^p}{(1-z)(1-qz)} = z D_q \mathcal{L}_q^{\mu+p-1} f(z),$$

then, simple computation gives

$$\frac{1}{z} \left[\mathcal{L}_q^{\mu+p-1} f(z) * \left(\frac{(1 + Be^{i\theta}) z^p}{(1-z)(1-qz)} - \frac{[p, q] (1 + Ae^{i\theta}) z^p}{(1-z)} \right) \right] \neq 0,$$

or equivalently

$$\frac{(B - [p, q] A) e^{i\theta}}{z} \left[\mathcal{L}_q^{\mu+p-1} f(z) * \left(\frac{(N + 1) z^p - Lqz^{p+1}}{(1-z)(1-qz)} \right) \right] \neq 0,$$

which is the required direct part.

Assume that (11) holds true for $L_\theta - 1 = N_\theta = 0$, it follows that

$$\frac{e^{i\theta} (B - [p, q] A)}{z} \mathcal{L}_q^{\mu+p-1} f(z) \neq 0, \text{ for all } z \in \mathbb{D}.$$

Thus, the function $h(z) = \frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p,q] \mathcal{L}_q^{\mu+p-1} f(z)}$ is analytic in \mathbb{D} and $h(0) = 1$. Since we have shown that (27) and (11) are equivalent, therefore we have

$$\frac{zD_q \mathcal{L}_q^{\mu+p-1} f(z)}{[p,q] \mathcal{L}_q^{\mu+p-1} f(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{D}). \tag{28}$$

Suppose that

$$H(z) = \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D}.$$

Now, from relation (28) it is clear that $H(\partial\mathbb{D}) \cap h(\mathbb{D}) = \emptyset$. Therefore, the simply connected domain $h(\mathbb{D})$ is contained in a connected component of $\mathbb{C} \setminus H(\partial\mathbb{D})$. The univalence of the function h , together with the fact that $H(0) = h(0) = 1$, shows that $h \prec H$, which shows that $f \in \mathcal{S}_p^*(q, \mu, A, B)$. \square

We now define an integral operator for the function $f \in \mathfrak{A}_p$ as follows:

Definition 2. Let $f \in \mathfrak{A}_p$. Then, $\mathcal{L} : \mathfrak{A}_p \rightarrow \mathfrak{A}_p$ is called the q -analogue of Bernardi integral operator for multivalent functions defined by $\mathcal{L}(f) = F_{\eta,p}$ with $\eta > -p$, where $F_{\eta,p}$ is given by

$$F_{\eta,p}(z) = \frac{[\eta + p, q]}{z^\eta} \int_0^z t^{\eta-1} f(t) d_q t, \tag{29}$$

$$= z^p + \sum_{n=1}^{\infty} \frac{[\eta + p, q]}{[\eta + p + n, q]} a_{n+p} z^{n+p}, \quad (z \in \mathbb{D}). \tag{30}$$

We easily obtain that the series defined in (30) converges absolutely in \mathbb{D} . Now, if $q \rightarrow 1$, then the operator $F_{\eta,p}$ reduces to the integral operator studied in [29] and further by taking $p = 1$, we obtain the q -Bernardi integral operator introduced in [36]. If $q \rightarrow 1$ and $p = 1$, we obtain the familiar Bernardi integral operator [37].

Theorem 6. If f is of the form (1), it belongs to the family $\mathcal{S}_p^*(q, \mu, A, B)$ and

$$F_{\eta,p}(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}, \tag{31}$$

where $F_{\eta,p}$ is the integral operator given by (29), then

$$|b_{p+1}| \leq \frac{[\eta + p, q]}{[\eta + p + 1, q]} \frac{\psi_1(A - B)}{\wedge_{1+p}},$$

and for $n \geq 2$

$$|b_{p+n}| \leq \frac{[\eta + p, q]}{[\eta + p + n, q]} \frac{(A - B) \psi_n}{\wedge_{n+p}} \prod_{t=1}^{n-1} \left(1 + \frac{[p, q](A - B)}{([p + t, q] - [p, q])} \right),$$

where ψ_n and \wedge_{n+p} are defined by (14) and (5), respectively.

Proof. The proof follows easily by using (30) and Theorem 2. \square

Theorem 7. Let $f \in \mathcal{S}_p^*(q, \mu, A, B)$ and be given by (1). In addition, if $F_{\eta,p}$ is the integral operator is defined by (29) and is of the form (31), then for $\sigma \in \mathbb{C}$

$$\left| b_{p+2} - \sigma b_{p+1}^2 \right| \leq \frac{[\eta + p, q]}{[\eta + p + 2, q]} \frac{(A - B) \psi_2}{\wedge_{p+2}} \{1; |v|\},$$

where

$$v = (B - (A - B)\psi_1) + \frac{\Lambda_{p+2}\psi_1^2}{\Lambda_{p+1}^2\psi_2} (A - B) \frac{[\eta + p, q][\eta + p + 2, q]}{[\eta + p + 1, q]^2} \sigma. \tag{32}$$

Proof. From (30) and (31), we easily have

$$\begin{aligned} b_{p+1} &= \frac{[\eta + p, q]}{[\eta + p + 1, q]} a_{p+1}, \\ b_{p+2} &= \frac{[\eta + p, q]}{[\eta + p + 2, q]} a_{p+2}. \end{aligned}$$

Now,

$$\left| b_{p+2} - \sigma b_{p+1}^2 \right| = \frac{[\eta + p, q]}{[\eta + p + 2, q]} \left| a_{p+2} - \sigma \frac{[\eta + p, q][\eta + p + 2, q]}{([\eta + p + 1, q])^2} a_{p+1}^2 \right|.$$

By using (22) and (23), we have

$$\left| b_{p+2} - \sigma b_{p+1}^2 \right| = \frac{[\eta + p, q]}{[\eta + p + 2, q]} \frac{(A - B)}{\Lambda_{p+2}} \left| w_2 - v w_1^2 \right|,$$

where v is given by (32). Applying (9), we get

$$\left| b_{p+2} - \sigma b_{p+1}^2 \right| \leq \frac{[\eta + p, q]}{[\eta + p + 2, q]} \frac{(A - B)}{\Lambda_{p+2}} \{1, |v|\}.$$

Hence, we have the required result. \square

4. Future Work

The idea presented in this paper can easily be implemented to define some more subfamilies of analytic and univalent functions connected with different image domains [38–40].

5. Conclusions

In this article, we have defined a new class of multivalent q -starlike functions by using multivalent q -Ruscheweyh differential operator. We studied some interesting problems, which are helpful to study the geometry of the image domain, and also used some of the achieved results to find the growth of Hankel determinant. The idea of this determinant is applied in the theory of singularities [39] and in the study of power series with integral coefficients. For deep insight, the reader is invited to read [38–44]. Further, we have generalized the Bernardi integral operator and defined the multivalent q -Bernardi integral operator. Some useful properties of this class of multivalent functions have been studied.

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