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Dynamic Parallel Mining Algorithm of Association Rules Based on Interval Concept Lattice

Yafeng Yang [1,2,3], Ru Zhang [4] and Baoxiang Liu [1,2,*]

1 College of Science, North China University of Science and Technology, 21 Bohai Road, Tangshan 063210, China
2 Hebei Key Laboratory of Data Science and Application, 21 Bohai Road, Tangshan 063210, China
3 Tangshan Key Laboratory of Engineering Computing, 21 Bohai Road, Tangshan 063210, China
4 Department of mathematics and information sciences, Tangshan Normal University, No. 156 Jianshe North Road, Tangshan 063009, China
* Correspondence: www1673@163.com

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Abstract: An interval concept lattice is an expansion form of a classical concept lattice and a rough concept lattice. It is a conceptual hierarchy consisting of a set of objects with a certain number or proportion of intent attributes. Interval concept lattices refine the proportion of intent containing extent to get a certain degree of object set, and then mine association rules, so as to achieve minimal cost and maximal return. Faced with massive data, the structure of an interval concept lattice is more complex. Even if the lattice structures have been united first, the time complexity of mining interval association rules is higher. In this paper, the principle of mining association rules with parameters is studied, and the principle of a vertical union algorithm of interval association rules is proposed. On this basis, a dynamic mining algorithm of interval association rules is designed to achieve rule aggregation and maintain the diversity of interval association rules. Finally, the rationality and efficiency of the algorithm are verified by a case study.

Keywords: interval concept lattice; association rules; mining algorithm; vertical union

1. Introduction

A concept lattice [1] is a conceptual hierarchy constructed according to the binary relationship between objects and attributes in data sets. As an effective tool for knowledge representation, a concept lattice is widely used in knowledge discovery, rule mining, information retrieval, and other fields because of its accuracy and completeness [2–4].

Concept lattice theory mainly focuses on the following aspects: A concept lattice extent model [5–7], concept lattice construction and rule extraction [8–10], concept lattice merging [11–15], concept lattice reduction [16,17], concept lattice modification [18], etc.

In a classic concept lattice, the concept extents have all the attributes or only one attribute, sometimes. Hence, the support and confidence degree of the extracted association rules would be reduced greatly. To solve this problem, the authors have put forward a new concept lattice structure: An interval concept lattice, and the construction methods, compression, and maintenance of lattice structure were studied [19–22].

From the perspective of a concept lattice, the relationship between intents is association rules, while the relationship between extents is its embodiment. A concept lattice is the unity of intent and extent, and the relationship between its nodes also reflects the relationship between the generalization and specialization of concepts. Therefore, a concept lattice is suitable for the application of a basic data structure in association rules mining. Many scholars have conducted in-depth research on rule mining based on a concept lattice [23–28].

Previously, we studied the structure characteristics of an interval concept lattice and gave two measurement standards of the uncertainty rule—precision and uncertainty. Then, a mining model of interval association rule with parameters was constructed [29]. The algorithm can mine and optimize rules according to the adjustment of parameters, which is of great significance to the mining of rules with uncertainties. Next, the complex relationship between interval parameters and association rules and the optimization algorithm of the rule base were given [30]. By adjusting the parameters, the purpose of controlling and optimizing rules was achieved.

In the era of data explosion, people's demand for data processing is getting higher and higher. The real-time updating of data requires the efficient processing of dynamic data. For example, in the supermarket shopping system, the massive transaction information generated every day can only mine local association rules, but cannot provide a timely and accurate decision-making plan for decision-makers as a whole. However, the time and space complexity of the process will increase rapidly with the increase of the amount of data, and the mining association rules will be missing. Therefore, it is necessary to study the dynamic mining of interval association rules in order to grasp uncertain rules in real time.

The authors have studied the consistency of interval concept lattices, discussed the decision theorem of the concept of consistent intent, and designed a vertical union algorithm of interval concept lattices based on the breadth-first principle [31]. Furthermore, the sequential traversal method was used to scan the lattice structure, and a union algorithm of interval concept lattices was proposed from a transverse point of view [32]. Based on the research results of the union algorithm of interval concept lattice, this paper carries out the dynamic mining of interval association rules.

2. Concepts and Methods

2.1. Interval Concept Lattice

Definition 1 ([33]). *For the formal context (U, A, R), where $U = \{x_1, x_2, \cdots, x_3\}$ is the object sets and each $x_i (i \le n)$ denotes an object; $A = \{a_1, a_2, \cdots, a_m\}$ is the attribute set, and each $a_j (j \le m)$ denotes an attribute; R is the binary relationship between U and A. $R \subseteq U \times a$. If $(x, a) \in R$, then we record that x has the attribute a, and write as xRa.*

Definition 2 ([33]). *For the formal context (U, A, R), operators f, g are defined as follows:*

$\forall x \in U, f(x) = \{y \big| \forall y \in A, xRy\}$, i.e., f is the mapping between x and its attributes;
$\forall y \in a, g(y) = \{x \big| \forall x \in U, xRy\}$, i.e., g is the mapping between y and its objects.

Definition 3 ([33]). *For the formal context (U, A, R), if $f(X) = Y, g(Y) = X$ for $X \subseteq U, Y \subseteq A$, then the sequence $< X, Y >$ is called a formal concept, or concept for short. X is the extent and Y is the intent.*

Rough concept lattice $RL(U, A, R)$ based on rough set theory was studied in Reference [7], where the upper approximation extent and lower approximation extent refer to the maximal concept set and the minimal concept set respectively which have all the attributes in $Y \subseteq A$.

Definition 4 ([33]). *For the formal context (U, A, R) and its rough concept lattice $RL(U, A, R)$, (M, N, Y) is the rough concept. Set an interval $[\alpha, \beta]$ $(0 \le \alpha \le \beta \le 1)$, then α upper bound extent M^α and β lower bound extent M^β are:*

$$M^\alpha = \left\{x \middle| x \in M, \left|f(x) \cap Y\right| / |Y| \ge \alpha, 0 \le \alpha \le 1\right\} \tag{1}$$

$$M^\beta = \left\{x \middle| x \in M, \left|f(x) \cap Y\right| / |Y| \ge \beta, 0 \le \alpha \le \beta \le 1\right\} \tag{2}$$

Y is the concept intent and $|Y|$ is the number of elements in Y, that is base number. M^{α} refers to the objects which may be covered by $\alpha \times |Y|$ attributes or more in Y. M^{β} refers to the objects which may be covered by $\beta \times |Y|$ attributes or more in Y.

Definition 5 ([33]). *Let (U, A, R) be a formal context and $(M^{\alpha}, M^{\beta}, Y)$ be an interval concept. Then, Y is the intent; M^{α} is the α upper bound extent and M^{β} is the β lower bound extent.*

Definition 6. *Suppose that (U, A, R) has two interval concepts, $(M_1^{\alpha}, M_1^{\beta}, Y_1)$ and $(M_2^{\alpha}, M_2^{\beta}, Y_2)$. If the two meet $Y_1 \subseteq Y_2$, $|Y_2| - |Y_1| = 1$, $M_1^{\alpha} = M_2^{\alpha}$ and $M_1^{\beta} = M_2^{\beta}$, then $(M_1^{\alpha}, M_1^{\beta}, Y_1)$ is called the redundant concept.*

Definition 7. *Suppose that (U, A, R) has an interval concept, $(M^{\alpha}, M^{\beta}, Y)$. If it meets $M^{\alpha} = M^{\beta} = \varnothing$, then $(M^{\alpha}, M^{\beta}, Y)$ is called the empty concept.*

Definition 8. *Suppose (U, A, R) has an interval concept, $C = (M^{\alpha}, M^{\beta}, Y)$. If C is neither the redundant concept nor the empty concept, then C is called the existence concept. $L_{\alpha}^{\beta}(U, A, R)$ is a collection of all the existence concepts.*

Definition 9. *$\overline{L_{\alpha}^{\beta}(U, A, R)}$ refers to all the $[\alpha, \beta]$ interval concepts, which include: Existence concepts, redundant concepts, and empty concepts, that is:*

$$(M_1{}^{\alpha}, M_1{}^{\beta}, Y_1) \leq (M_2{}^{\alpha}, M_2{}^{\beta}, Y_2) \Leftrightarrow Y_1 \supseteq Y_2, \tag{3}$$

Then "$\leq$"is called the partial order relationship of $\overline{L_{\alpha}^{\beta}(U, A, R)}$.

Definition 10. *If all the concepts in $\overline{L_{\alpha}^{\beta}(U, A, R)}$ meet "$\leq$", then $L_{\alpha}^{\beta}(U, A, R)$ is called interval concept lattice on the formal context (U, A, R).*

Definition 11. *In the interval concept lattice $L_{\alpha}^{\beta}(U, A, R)$, if $C = (M^{\alpha}, M^{\beta}, Y) \in L_{\alpha}^{\beta}(U, A, R)$, then the layer of the Lattice Structure is $|A| + 1$ and node C is at Layer $|Y|$. In particular, when $Y = \varnothing$, C was recorded on the zeroth layer.*

2.2. Interval Association Rules

Formal context (U, A, R) can describe a database, where U represents an object set; A represents an attribute set. For $x \in U$, $a \in A$, xRa represents the item-set where a belongs to x.

Definition 12. *Given the minimal support threshold θ, for any interval concept node C, if the number of objects in the upper bound extent is not less than $|U| \times \theta$, then C is called the α-upper bound frequent node, and the corresponding Y is called the α-upper bound frequent item-set; if the number of objects in the lower bound extent is not less than $|U| \times \theta$, then C is called the β-lower bound frequent node, the corresponding Y is called the β-lower bound frequent item-set.*

The father and son concept in the interval concept lattice does not have a specific relationship in frequency, which is different from the classical concept lattice.

If the association rule $A \Rightarrow B$ corresponds to the interval concept node $(C1, C2)(C1 = (M_1^{\alpha}, M_1^{\beta}, Y_1)$, $C2 = (M_2^{\alpha}, M_2^{\beta}, Y_2))$ and $C1 \geq C2$, then rule $A \Rightarrow B$ is generated by node binary $(C1, C2)$.

The α-upper bound association rules and the β-lower bound association rules can be extracted by two lower bound extents of the interval concept. The calculation methods of confidence and support are as follows:

The α-upper bound rule $A \Rightarrow B$:

$$Conf(A \Rightarrow B) = \left|M_2^{\alpha}\right| / \left|M_1^{\alpha}\right| \tag{4}$$

$$Support(A \Rightarrow B) = \left|M_2^{\alpha}\right| / |U| \tag{5}$$

The β-lower bound $A \Rightarrow B$:

$$Conf(A \Rightarrow B) = \left|M_2^{\beta}\right| / \left|M_1^{\beta}\right| \tag{6}$$

$$Support(A \Rightarrow B) = \left|M_2^{\beta}\right| / |U| \tag{7}$$

Definition 13. *Given the minimal support threshold θ and the minimal confidence threshold c. Node binary $(C1, C2)$ is called α-upper bound candidate binary which consists of two frequent concept nodes $C1 = (M_1^{\alpha}, M_1^{\beta}, Y_1)$ and $C2 = (M_2^{\alpha}, M_2^{\beta}, Y_2)$, where $M_2^{\alpha} \subseteq M_1^{\alpha}$and $\left|M_2^{\alpha}\right| / \left|M_1^{\alpha}\right| \geq c$. When $(C1, C2)$ meets $M_2^{\beta} \subseteq M_1^{\beta}$ and $\left|M_2^{\beta}\right| / \left|M_1^{\beta}\right| \geq c$, it is called β-lower bound candidate binary.*

Definition 14. *For interval association rule $A \Rightarrow B$, if $A \cup B$ is frequent item-sets, $Support(A \Rightarrow B) \geq \theta$ and $Conf(A \Rightarrow B) \geq c$, i.e., $\left|P(A \cup B)\right| / \left|P(A)\right| \geq c$, then it is called the strong association rule.*

Definition 15. *If $A \Rightarrow B$ is the strong association rule, then $C \Rightarrow D$ must be the strong association rule, then we say "$A \Rightarrow B$ can derive $C \Rightarrow D$".*

Theorem 1. *If $C \subset D$, then $A \Rightarrow B$ can derive $A \Rightarrow C$.*

Proof. $C \subset B \Rightarrow |C| < |B| \Rightarrow |A \cup C| < |A \cup B| \Rightarrow Support(A \Rightarrow C) = |C| < |U| \quad < Support(A \Rightarrow B) = |B|/|U|$ and $Conf(A \Rightarrow C) = |A \cup C|/|A| < Conf(A \Rightarrow B) = |A \cup B|/|A|$. □

Theorem 2. *In the interval concept lattice, if $(C1, C2)$ and $(C1, C3)$ are candidate binaries and $C3 > C2$, then all the rules in $Rules(C1, C3)$ can be derived from $Rules(C1, C2)$.*

Definition 16. *Suppose $A \Rightarrow B$ is α-upper bound association rule derived from $(C1, C2)$. The upper bound extent of $C1$ is $M_1^{\alpha} = \{x_1, x_2, \ldots, x_m,\}$. The intent of $C1$ is Y_1.The upper bound extent of $C2$ is $M_2^{\alpha} = \{o_1, o_2, \ldots, o_m,\}$. The intent of $C2$ is Y_2. The precision of $A \Rightarrow B$ is*

$$PD_{A \Rightarrow B} = \min\left\{\min_{i=1}^{m} \frac{|x_i.Y \cap Y_1|}{|Y_1|}, \min_{i=1}^{n} \frac{|o_i.Y \cap Y_2|}{|Y_2|}\right\} \tag{8}$$

The uncertainty of $A \Rightarrow B$ is $UD_{A \Rightarrow B} = 1 - PD_{A \Rightarrow B}$.

Definition 17. *Let $\Omega = \{Rule1, Rule2, \cdots, Rulek\}$ denotes to α -rules set, the uncertainty of $Rulei$ is $UD_{\alpha-Ri}$, then the uncertainty of α -rules sets is*

$$UD_{\alpha-Rluesset} = \max_{i=1}^{k}(UD_{\alpha-Ri}) \tag{9}$$

Let $\Omega = \{Rule1, Rule2, \cdots, Rulem\}$ denote to β -rules set, the uncertainty of $Rulej$ is $UD_{\beta-Ri}$, then the uncertainty of β -rules sets is

$$UD_{\beta-Rluesset} = \max_{j=1}^{m}(UD_{\beta-Ri}) \tag{10}$$

and the uncertainty of interval association rules is $UD = \max(UD_{\alpha-Rluesset}, UD_{\beta-Rluesset})$.

Theorem 3. *Suppose the minimal support threshold θ and the minimal confidence threshold c . In the same context, when one of the interval parameters α and β is unchanged and the other is larger, the process of extracting association rules will change as follows:*

(1) The number of frequent nodes generated does not increase;
(2) The node with the largest intent in frequent nodes does not increase in intent cardinality.
(3) The number of candidate binary arrays generated does not increase;
(4) The number of generated association rules does not increase.

3. Algorithm and Results

In the mining method of interval association rules, the lattice structure is united with the vertical union algorithm first, and then the association rules are extracted by using the parametric association rules mining algorithm. Because the structure of the interval concept lattice is complex, the memory space required is large, and the time complexity of mining interval association rules after vertical merging is high, this method does not meet the requirements of the current era of large data. In this section, the principle of a vertical union algorithm of interval association rules is proposed, and on this basis, a dynamic mining algorithm of interval association rules is designed.

3.1. Vertical Union Principle of Interval Association Rules

If the interval concept lattices $\overline{L_\alpha^\beta(U_1, A_1, R_1)}$ and $\overline{L_\alpha^\beta(U_2, A_2, R_2)}$ are consistent and $A_1 = A_2 = A$. $U_1 \cap U_2 = \phi$, then $\overline{L_\alpha^\beta(U, A, R)}$ can be gotten through the union of the two. Suppose the minimal support threshold θ and the minimal confidence threshold c, we can extract the association rules of $\overline{L_\alpha^\beta(U_1, A, R_1)}$ and $\overline{L_\alpha^\beta(U_2, A, R_2)}$. Here, $\alpha - Rluesset^*$ is the upper bound association rules set derived from $\overline{L_\alpha^\beta(U_1, A, R_1)}$. $\alpha - Rluesset^{**}$ is the upper bound association rules set derived from $\overline{L_\alpha^\beta(U_2, A, R_2)}$; $\alpha - Rluesset$ is the upper bound association rules set derived from $\overline{L_\alpha^\beta(U, A, R)}$.

$$C_1^* = (M_1^{\alpha *}, M_1^{\beta *}, Y_1),\ C_2^* = (M_2^{\alpha *}, M_2^{\beta *}, Y_2) \text{ and } C_1^*, C_2^* \in \overline{L_\alpha^\beta(U_1, A, R_1)};$$

$$C_1^{**} = (M_1^{\alpha **}, M_1^{\beta **}, Y_1),\ C_2^{**} = (M_2^{\alpha **}, M_2^{\beta **}, Y_2) \text{ and } C_1^{**}, C_2^{**} \in \overline{L_\alpha^\beta(U_2, A, R_2)};$$

$$C_1 = (M_1^{\alpha}, M_1^{\beta}, Y_1),\ C_2 = (M_2^{\alpha}, M_2^{\beta}, Y_2) \text{ and } C_1, C_2 \in \overline{L_\alpha^\beta(U, A, R)}.$$

Now, taking the upper bound extent as an example (the union principle of lower bound extent and upper bound extent), the vertical union of interval association rules $\alpha - Rluesset^*$ and $\alpha - Rluesset^{**}$ is carried out as follows:

Theorem 4. *If C_1^*, C_2^* constitutes the association rule, and $Rules(C_1^*, C_2^*) \in \alpha - Rluesset^*$; C_1^{**}, C_2^{**} the constitutes association rule, and $Rules(C_1^{**}, C_2^{**}) \in \alpha - Rluesset^{**}$, then $Rules(C_1, C_2) \in \alpha - Rluesset$.*

Proof. $Rules(C_1^*, C_2^*) \in \alpha - Rluesset^*$ and $Rules(C_1^{**}, C_2^{**}) \in \alpha - Rluesset^{**}$, so:
Relation 1: $M_2^{\alpha *} \subseteq M_1^{\alpha *}$ and $M_2^{\alpha **} \subseteq M_1^{\alpha **}$, then $M_2^{\alpha *} \cup M_2^{\alpha **} \subseteq M_1^{\alpha *} \cup M_1^{\alpha **}$, i.e., $M_2^{\alpha} \subseteq M_1^{\alpha}$.
Relation 2: $\frac{|M_2^{\alpha *}|}{|M_1^{\alpha *}|} \geq c$ and $\frac{|M_2^{\alpha **}|}{|M_1^{\alpha **}|} \geq c$, then

$$\left|M_2^{\alpha *}\right| + \left|M_2^{\alpha **}\right| \geq c\left(\left|M_1^{\alpha *}\right| + \left|M_1^{\alpha **}\right|\right), \frac{\left|M_2^{\alpha *}\right| + \left|M_2^{\alpha **}\right|}{\left|M_1^{\alpha *}\right| + \left|M_1^{\alpha **}\right|} \geq c, \text{i.e.,} \frac{\left|M_2^{\alpha}\right|}{\left|M_1^{\alpha}\right|} \geq c.$$

Then $Rules(C_1, C_2) \in \alpha - Rluesset$ can be derived from two relations. □

Theorem 5. *If at least one of the two association rules* $Rules(C_1^*, C_2^*)$ *and* $Rules(C_1^{**}, C_2^{**})$ *in* $\overline{L_\alpha^\beta(U_1, A, R_1)}$ *and* $\overline{L_\alpha^\beta(U_2, A, R_2)}$ *does not exist, then there must be no* $Rules(C_1, C_2)$ *in* $\alpha - Rluesset$.

Because the vertical union of interval association rules is based on the objects set with certain proportion attributes, it is necessary to introduce support, confidence, accuracy, and uncertainty to measure accurately.

The upper bound frequencies of C_1^*, C_2^* are θ_1^* and θ_2^*, respectively.

The upper bound frequencies of C_1^{**}, C_2^{**} are θ_1^{**} and θ_2^{**}, respectively.

The upper bound frequencies of C_1, C_2 are θ_1 and θ_2, respectively.

The support degree, confidence degree, accuracy, and uncertainty of $Rules(C_1, C_2)$ obtained by vertical union of association rules $Rules(C_1^*, C_2^*)$ and $Rules(C_1^{**}, C_2^{**})$ can be deduced from frequency to its solution formula, as follows.

Theorem 6. *The support degree of* $Rules(C_1, C_2)$ *is:*

$$Support(C_1 \Rightarrow C_2) = \frac{|U_1|\theta_2^* + |U_2|\theta_2^{**}}{|U_1| + |U_2|} \tag{11}$$

Theorem 7. *The confidence degree of* $Rules(C_1, C_2)$ *is*

$$Conf(C_1 \Rightarrow C_2) = \frac{|U_1|\theta_2^* + |U_2|\theta_2^{**}}{|U_1|\theta_1^* + |U_2|\theta_1^{**}} \tag{12}$$

Theorem 8. $Rules(C_1, C_2) = Min\left\{Rules(C_1^*, C_2^*), Rules(C_1^{**}, C_2^{**})\right\}$.

3.2. Dynamic Mining Algorithms for Interval Association Rules

3.2.1. Algorithm Design

According to Theorems 4 and 5, the result of the vertical union of interval association rules only occurs in the data with the same interval association rules and adjacent data. In order to ensure that the mined interval association rules are not lost, the algorithm retains the non-united rules on the basis of a vertical union of the same rules, which facilitates the mining of interval association rules later, and measures the occurrence frequency and frequency of the same rules with frequency. The basic idea of the algorithm is that whenever an interval association rule is generated, it is transformed into an array representation, and it is mined with the set of interval rules that have been united and retain the non-united rules, so that the interval rules can be aggregated again and again.

In order to distinguish different rules, association rules mined from interval concept lattices are stored in the form of arrays $RS[i] = \left\{Rule, FN_1, FN_2, U, Support, Conf, PD, UD, Flag, Num\right\}$.

$RS[i]$ represents the ith association rule in the interval association rules set RS;

$Rule$ represents the interval association rules $Rule(C_x, C_y)$;

FN_1, FN_2 represents the frequency degree of frequency nodes C_x and C_y in $Rule(C_x, C_y)$;

U represents the number of objects in the context;

$Support$, $Conf$, PD and UD represents the support, confidence, accuracy, and uncertainty degree of $Rule(C_x, C_y)$;

$Flag$ marked according to whether rules have been merged or not. $Flag = 0$ denotes that rules are not united vertically; $Flag = 1$ indicates that the rules have been vertically united; Num denotes the number of occurrences of rule $Rule(C_x, C_y)$ in previously united rule sets.

Based on the above algorithm principle and analysis, we design a dynamic extraction and union algorithm of interval association rules, DMA (Dynamic Mining Algorithm). See Algorithm 1.

Algorithm 1. DMA (Dynamic Mining Algorithm)

Input: Association rule sets $RS_1, RS_2, \ldots RS_k \ldots$
Output: Association rule set RS
Step1 $RS = RS_1$;
Step2 The interval association rules in the rule set RS_k is stored in the form of arrays. Set RS^* and initialize. Comparing the *Rule* of rule set RS and RS_k, we unit interval association rules vertically according to Theorems 4 and 5. For the rules that have been united in RS and RS_k, let $Flag = 1$ and put the united rule number in RS^*. According to Theorems 6–8, calculate the frequency,$Support$, $Conf$, PD, and UD of C_x and C_y in RS^*. $Flag$=1, Num = Num + 1. Delete the rules of $Flag = 1$ in RS and RS_k, and renumbering. Let rules number in RS^* add the renumbering number in RS. Putting the remain rules of RS into RS^*; then renumbering in RS_k, and putting the remain rules of RS_k into RS^*.
Rule Vertical Union (RS, RS_k)

{
$g = 0$;
$RS[g]^* = \{Rule, FN_1, FN_2, U, Support, Conf, \quad PD, UD, Flag, Num\}$
{ $Rule = \varnothing$;
$FN_1 = FN_2 = 0$;
$U = RS[1].U + RS_k[1].U$;
$Support = Conf = PD = UD = Flag = 0$;
$Num = 1$;
}
For (i=1;|RS|; i++)
 {For (j=1;|RS_k|; j++)
  If($RS[i].Flag = RS_k[j].Flag = 0 \| \quad RS[i].Rule = RS_k[j].Rule$)
  { $g + 1$;
   $Num + 1$;
   $RS[i].Flag = 1$;
   $RS_k[j].Flag = 1$;
   $RS[g]^*.FN_1 = \frac{RS[i].U*RS[i].FN_1 + RS_k[j].U*RS_k[j].FN_1}{RS[i].U + RS_k[j].U}$
   $RS[g]^*.FN_2 = \frac{RS[i].U*RS[i].FN_2 + RS_k[j].U*RS_k[j].FN_2}{RS[i].U + RS_k[j].U}$
  $RS[g]^*.Support = RS[g]^*.FN_2$
  $RS[g]^*.Conf = \frac{RS[i].U*RS[i].FN_2 + RS_k[j].U*RS_k[j].FN_2}{RS[i].U*RS[i].FN_1 + RS_k[j].U*RS_k[j].FN_1}$
  $RS[g]^*.PD = \min\{RS[i].PD, RS_k[j].PD\}$;
   $RS[g]^*.UD = 1 - RS[g]^*.PD$;
    }
  }
  For each $RS[i]$ in RS;
  {If $RS[i].Flag = 0$;
   $g + 1$;
   $RS^*[g] = RS[i]$;}
  For each $RS_k[j]$ in RS_k;
  {If $RS_k[j].Flag = 0$;
   $g + 1$;
  $RS^*[g] = RS_k[j]$;}
 }

Step3 $RS = RS^*$

3.2.2. Algorithm Analysis

The main content of DMA is embodied in the function Rule Vertical Union (RS, RS_k). Lines 1 to 9 implement the initialization of parallel rule set RS^*. Lines 10–12 nested for loop implements searching for interval association rules in RS and RS_k that correspond to the same rules and are not united. Lines 13 to 16 number the interval association rules found in RS^*. Num+1 is used to count the rule, and the corresponding rule *Flag* in united RS, RS_k is recorded as 1. Lines 17 to 25 implement the assignment of interval association rule $RS^*[g]$. In lines 26–34, the unconsolidated rules in RS, RS_k are put into RS^* to maintain the diversity of the united rule set.

Compared with Step 2, the algorithm achieves the step requirements, and the assignment of Step 1 and Step 3 make the DMA algorithm dynamic, so that the algorithm has integrity and correctness. Compared with the general method, the algorithm realizes the merging of rules to rules, eliminating the process of vertical merging of interval concept lattices, thus greatly reducing the time complexity and space complexity of the implementation process.

If the number of association rules in two sub lattices is n and m respectively, the time complexity of the algorithm is less than $O(n \times m) + O(n) + O(m)$. Compared with the interval association rule mining algorithm with parameters, the algorithm is more efficient.

3.3. Example Study

Set the formal contexts as shown in Tables 1 and 2:

Table 1. Formal context FC_1.

	a	**b**	**c**	**d**	**e**
1	1	1	1	0	0
2	0	0	0	1	0
3	1	1	0	1	0
4	1	0	1	0	1

Table 2. Formal context FC_2.

	a	**b**	**c**	**d**	**e**
(1)	1	0	1	1	0
(2)	1	1	0	1	0
(3)	0	1	0	1	1

Set $\alpha = 0.6, \beta = 0.7, \theta = 50\%, c = 60\%$.

The DMA algorithm is used to mine interval association rules in parallel. Take the upper bound interval association rules as an example (the mining of lower bound association rules is similar).

Among them, the upper bound frequent nodes from FC_1 and FC_2 are shown in Tables 3 and 4 respectively, and the corresponding upper bound association rules are shown in Tables 5 and 6 respectively.

Table 3. The frequent nodes of FC_1.

Frequent Node	Frequent Degree	Frequent Node	Frequent Degree
a	75%	acd	100%
b	50%	ace	50%
c	75%	ade	50%
d	50%	bcd	75%
ab	50%	bce	50%
ac	50%	cde	50%
abc	75%	abcd	50%
abd	50%	abce	50%
abe	75%	abcde	75%

Table 4. The frequent nodes of FC_2.

Frequent Node	Frequent Degree	Frequent Node	Frequent Degree
a	67%	acd	67%
b	67%	ade	100%
d	100%	bcd	100%
ad	67%	bde	67%
bd	67%	cde	67%
abc	67%	abcd	67%
abd	100%	abde	67%
abe	67%	abcde	100%

Table 5. The previous rules and its measurement results on FC_1.

Rule	Support	Confidence	Accuracy	Uncertainty	Frequent Number
$a \Rightarrow bcde$	75%	100%	60%	40%	1
$abc \Rightarrow de$	75%	100%	60%	40%	1
$abe \Rightarrow cd$	75%	100%	60%	40%	1
$acd \Rightarrow be$	75%	75%	60%	40%	1
$a \Rightarrow bce$	50%	67%	75%	25%	1
$c \Rightarrow abe$	50%	67%	75%	25%	1
$ac \Rightarrow be$	50%	100%	75%	25%	1
$abc \Rightarrow e$	50%	67%	67%	33%	1
$abe \Rightarrow c$	50%	67%	67%	33%	1
$ace \Rightarrow b$	50%	100%	67%	33%	1
$a \Rightarrow bcd$	50%	67%	75%	25%	1
$b \Rightarrow acd$	50%	100%	75%	25%	1
$ab \Rightarrow cd$	50%	100%	75%	25%	1
$abc \Rightarrow d$	50%	67%	67%	33%	1
$abd \Rightarrow c$	50%	100%	67%	33%	1
$bcd \Rightarrow a$	50%	67%	67%	33%	1
$c \Rightarrow de$	50%	67%	67%	33%	1
$c \Rightarrow be$	50%	67%	67%	33%	1
$a \Rightarrow de$	50%	67%	67%	33%	1
$a \Rightarrow ce$	50%	67%	67%	33%	1
$c \Rightarrow ae$	50%	67%	67%	33%	1
$ac \Rightarrow e$	50%	100%	67%	33%	1
$a \Rightarrow be$	75%	100%	67%	33%	1
$a \Rightarrow bd$	50%	67%	67%	33%	1
$b \Rightarrow ae$	50%	100%	67%	33%	1
$ab \Rightarrow e$	50%	100%	67%	33%	1
$a \Rightarrow bc$	75%	100%	67%	33%	1
$a \Rightarrow c$	50%	67%	100%	0%	1
$c \Rightarrow a$	50%	67%	100%	0%	1
$a \Rightarrow b$	50%	67%	100%	0%	1
$b \Rightarrow a$	50%	100%	100%	0%	1

The DMA algorithm is used to mine the previous association rules of FC_1 and FC_2 in parallel. The parallel results of the different association rules are the same part of the combined association rules deleted in Tables 5 and 6, respectively.

The parallel results of the same association rules are shown in Table 7.

Table 6. The previous rules and its measurement results on FC_2.

Rule	Support	Confidence	Accuracy	Uncertainty	Frequent Number
$d \Rightarrow abce$	100%	100%	60%	40%	1
$abd \Rightarrow ce$	100%	100%	60%	40%	1
$ade \Rightarrow bc$	100%	100%	60%	40%	1
$bcd \Rightarrow ae$	100%	100%	60%	40%	1
$b \Rightarrow ade$	67%	100%	75%	25%	1
$d \Rightarrow abe$	67%	67%	75%	25%	1
$bd \Rightarrow ae$	67%	100%	75%	25%	1
$abd \Rightarrow e$	67%	67%	67%	33%	1
$abe \Rightarrow d$	67%	100%	67%	33%	1
$ade \Rightarrow b$	67%	67%	67%	33%	1
$bde \Rightarrow a$	67%	100%	67%	33%	1
$a \Rightarrow bcd$	67%	100%	75%	25%	1
$d \Rightarrow abc$	67%	67%	75%	25%	1
$ad \Rightarrow bc$	67%	100%	75%	25%	1
$abc \Rightarrow d$	67%	100%	67%	33%	1
$abd \Rightarrow c$	67%	67%	67%	33%	1
$acd \Rightarrow b$	67%	100%	67%	33%	1
$bcd \Rightarrow a$	67%	67%	67%	33%	1
$d \Rightarrow ce$	50%	67%	67%	33%	1
$b \Rightarrow de$	67%	100%	67%	33%	1
$d \Rightarrow be$	67%	67%	67%	33%	1
$bd \Rightarrow e$	67%	100%	67%	33%	1
$d \Rightarrow bc$	100%	100%	67%	33%	1
$d \Rightarrow ae$	100%	100%	67%	33%	1
$a \Rightarrow cd$	67%	100%	67%	33%	1
$d \Rightarrow ac$	67%	67%	67%	33%	1
$ad \Rightarrow c$	67%	100%	67%	33%	1
$b \Rightarrow ae$	67%	100%	67%	33%	1
$d \Rightarrow ab$	100%	100%	67%	33%	1
$a \Rightarrow bc$	67%	100%	67%	33%	1

Table 7. The previous rules set and its measurement results after vertical union.

Rule	Support	Confidence	Accuracy	Uncertainty	Frequent Number
$a \Rightarrow bcd$	57%	80%	75%	25%	2
$abc \Rightarrow d$	57%	80%	67%	33%	2
$abd \Rightarrow c$	57%	80%	67%	33%	2
$bcd \Rightarrow a$	57%	67%	67%	33%	2
$a \Rightarrow bc$	71%	100%	67%	33%	2

4. Discussion

Considering the characteristics of interval concept lattices, the vertical union method of interval concept lattices and the principle of mining association rules with parameters were studied in this paper. Considering the intrinsic relationship between interval association rules, this paper presents some metrics, such as the confidence and credibility of united rules, and proposes a dynamic mining algorithm for interval association rules, which realizes rule aggregation and keeps the diversity of interval association rules. The rationality and efficiency of the algorithm are proved by algorithm analysis and case study, which provides timely, effective and abundant decision information for decision makers in data analysis. The next step is to optimize the algorithm for incomplete information systems in the context of large data. In the mining process, the stability and timing choice of the rule union is another problem to be studied in depth.

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