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Dependence on the Initial Data for the Continuous Thermostatted Framework

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Abstract: The paper deals with the problem of continuous dependence on initial data of solutions to the equation describing the evolution of a complex system in the presence of an external force acting on the system and of a thermostat, simply identified with the condition that the second order moment of the activity variable (see Section 1) is a constant. We are able to prove that these solutions are stable with respect to the initial conditions in the Hadamard's sense. In this connection, two remarks spontaneously arise and must be carefully considered: first, one could complain the lack of information about the "distance" between solutions at any time $t \in [0, +\infty)$; next, one cannot expect any more complete information without taking into account the possible distribution of the transition probabiliy densities and the interaction rates (see Section 1 again). This work must be viewed as a first step of a research which will require many more steps to give a sufficiently complete picture of the relations between solutions (see Section 5).

Keywords: kinetic theory; integro-differential equations; complex systems; stability; evolution equations

1. Introduction

The aim of the present paper is to contribute a first result about the stability (and, consequently, the uniqueness) of the solutions to the equation describing the evolution of a thermostatted complex system. As is well known, a complex system is a set of a very large number of objects that, in connection with the physical origin of the notion, can be called "particles", but—in view of its present applications—should perhaps be identified by the more general name of "individuals". These objects, of course, enjoy a number of empirical properties, that define their "state": in a purely mechanical framework, these properties are position and velocity. In principle, the state of the whole system should be considered as completely known when the states of all of its particles are known. But, since these states are modified by the mutual interactions of the particles, and these interactions in turn depend on the states of the particles, also in view of the extremely large number of interactions to be taken into account, it is readily seen that a complete knowledge of the state of the system is impossible [1] at any time. This gave rise to statistical mechanics and to Boltzmann's Kinetic Theory of Gases, that aimed to describe the states of any such system and their evolution in terms of average states of the particles and of probability of their interactions.

But the "complexity" of a system is more than the large number of its particles. In the mechanical framework, each interaction between two particles p and q is assumed to be independent of the presence of the remaining particles of the system, so that the result of simultaneous interactions of p with $q_1, q_2, \ldots, q_n$ is simply the sum of the results of single interaction. When this assumption is given up, and each particle is allowed to interact—under suitable conditions—not only with other particles but also with their interactions, then the system is "complex" in the full meaning of the term. So, not only the states of a complex systems and their evolution can be described only by using a statistical (or probabilistic) language and statistical (or probabilistic) methods, but also the interactions between its particles. Accordingly, all what we may assume to be able to know about complex systems at each instant are (a) a probability (or relative frequency) distribution f on the set D_u of all possible states of all of its particles; (b) for any pair (u_*, u^*) of possible states, an average frequency $\eta(u_*, u^*)$ of interactions between particles that are in the state u_* and particles that are in the state u^*; (c) for any pair (u_*, u^*) of possible states, a probability (or relative frequency) distribution on the possible effects of interactions (see Section 2 below).

This is a generalized form of Boltzmann's framework [1,2]. As a matter of fact, the introduction of the notion of "complex systems", in addition to the methodological choice to work on probability distributions in the same way as in the Kinetic Theory of Gases, corresponds to the need, hence to the attempt, to go beyond the limits of purely mechanical applications of the farmework, in order to give some contributions to a formal statement, and if possible to the solution, of many problems arising in the life of large sets of individuals of many different kinds (cell populations [3], with particular respect to the interaction between cancer cells and cells of immune systems [4], animal populations, social and economic communities like human nations [5,6], problems about vehicular traffic [7] and about the behaviour of swarms [8], crowds [9,10] and pedestrians [11], opinion formation [12], etc.). Of course, this attempt started about thirty years ago, was quite succesful and gave origin to a wide literature, investigating both the mathematical problems posed by the framework and its possible new applications [5,6,13]. The mathematical scheme provided by the equations proposed in this framework (see Section 2 below) could perhaps be considered as the most versatile among the ones from time to time constructed to formulate and solve problems arising in medicine, natural sciences, economics and even in behavioural sciences.

In this perspective, in order to widen the "world" of applications of the model, the notion of thermostat was recently introduced [14–17] to describe e.g. a crowd in a place hall compelled to go out by an emergency (the external force) but calmed down and organized to move in a suitable order by a security service (the thermostat) [18,19]. Of course, this extended model presents the same mathematical problems of all the models of the same kind (existence and uniqueness of solutions and their continuous dependence on initial and boundary conditions). Existence and uniqueness results were given in [18,20] (for the case in which the variable u representing the state of any particle is continuous) and [21,22] (for the discrete case). This paper offers a result about contiunous dependence for the continuous model.

The matter of the paper is distributed as follows: in Section 2 a short description of the problem is given, with the explicit statement of the equation governing the evolution of a general thermostatted complex system; the statement of our result about the continuous dependence of solutions to the evolution equation on the initial data is given in Section 3, and its proof in Section 4; finally, Section 5 is devoted to point out the reasons why the result stated and proved in the previous sections must be just considered as the first step of a further research, as well as to draw the perspectives of such reasearch.

2. The Framework

Let $\mathcal{C}$ be a complex system [13,23–25] composed by a large number of objects, called active particles [13], which can interact with each other. The system is assumed to be homogeneous with respect to mechanical variables (position and velocity). The state of each of its particles will be then described, at any time $t \geq 0$, by only one scalar variable $u \in D_u \subseteq \mathbb{R}$, called activity variable, which will be denoted by u. According to the statistical (or stochastic) scheme adopted to describe complex systems (see Introduction), the states of single particles cannot be known at any time, and only a probability (or relative frequency) distribution on the set D_u of all possible states (values of u) can be considered. This distribution function is

$$f(t, u) \in [0, +\infty[\times D_u \to \mathbb{R}^+.$$

Interactions between particles are described by the following two functions:

- a function $\eta(u_*, u^*) : D_u \times D_u \to \mathbb{R}^+$, the interaction rate between particles in the state u_* and particles in the state u^*;
- a function $\mathcal{A}(u_*, u^*, u) : D_u \times D_u \times D_u \to \mathbb{R}^+$, the transition probability density that a particle in the state u_* falls into the state u after interacting with a particle in the state u^*, such that for all $u_*,\ u^* \in D_u$:

$$\int_{D_u} \mathcal{A}(u_*, u^*, u)\, du = 1.$$

Let now $p \in \mathbb{N}$. The p-th order moment of the system, a time $t > 0$, is:

$$\mathbb{E}_p[f](t) := \int_{D_u} u^p f(t,u)\, du.$$

We assume that the system $\mathcal{C}$ is subject to an external force field acting on it, described by a function $F(u) : D_u \to \mathbb{R}^+$. Moreover, a thermostat is applied on $\mathcal{C}$, in order to keep the second order moment

$$\mathbb{E}_2[f](t) = \int_{D_u} u^2 f(t,u)\, du$$

constant [18].

Accordingly, the evolution equation for the thermostatted framework of the system takes the form

$$\partial_t f(t,u) + \partial_u \left(F(u) \left(1 - u\mathbb{E}_1[f](t)\right) f(t,u) \right) = J[f,f](t,u), \tag{1}$$

where the operator $J[f,f](t,u)$ can be split as follows:

$$\begin{aligned} J[f,f](t,u) &= G[f,f](t,u) - L[f,f](t,u) \\ &= \int_{D_u \times D_u} \eta(u_*, u^*)\, \mathcal{A}(u_*, u^*, u)\, f(t,u_*)\, f(t,u^*)\, du_* du^* \\ &\quad - f(t,u) \int_{D_u} \eta(u,u^*)\, f(t,u^*)\, du^*. \end{aligned}$$

The term $G[f,f,](t,u)$ is called the gain operator, and the term $L[f,f](t,u)$ is the loss operator.

Given an initial datum $f^0(u)$, from Equation (1) we obtain the Cauchy problem related to the thermostatted:

$$\begin{cases} \partial_t f(t,u) + \partial_u \left(F(u) \left(1 - u\mathbb{E}_1[f](t)\right) f(t,u) \right) = J[f,f](t,u) & [0,+\infty[\times D_u \\ \\ f(0,u) = f^0(u) & D_u \end{cases}. \tag{2}$$

Let now

$$\mathcal{K}(D_u) := \left\{ f(t,u) \in [0,+\infty[\times D_u \to \mathbb{R}^+ : \mathbb{E}_0[f](t) = \mathbb{E}_2[f](t) = 1 \right\}$$

and assume that

Assumption 1 (H1): for all $u_*, u^* \in D_u$:

$$\int_{D_u} \mathcal{A}(u_*, u^*, u)\, u^2\, du = u_*^2;$$

Assumption 2 (H2): the interaction function is constant, i.e., there exists an $\eta > 0$ such that $\eta(u_*, u^*) = \eta$, for all $u_*,\ u^* \in D_u$;

Assumption 3 (H3): the distribution function vanishes on the boundary of D_u, i.e. $f(t,u) = 0$ for $u \in \partial D_u$;

Assumption 4 (H4): there exists $F > 0$ such that $F(t) = F$, for all $t > 0$;

Assumption 5 (H5): the initial data verifies the conditions $\mathbb{E}_0[f^0] = \mathbb{E}_2[f^0] = 1$.

If assumptions H1–H5 are satisfied, then there exists a unique solution $f(t,u) \in C\left((0,+\infty); L^1(D_u)\right) \cap \mathcal{K}(D_u)$ to the Cauchy problem (2) [18].

Lemma 1. *[18] Under assumptions H1–H5, the relations*

1. $$G[f,f] - G[g,g] = G[f-g,f] + G[g,f-g]\,,$$
2. $$\int_{D_u} G[f,f](t,u)\,du = \eta\,,$$
3. $$\int_{D_u} u\,G[f,f](t,u)\,du = 0\,,$$
4. $$\int_{D_u} u^2\,G[f,f](t,u)\,du = \eta$$

are true.

Now, the evolution equation of $\mathbb{E}_1[f](t)$ reads, [18]:

$$\mathbb{E}_1'[f](t) = F(1 - (\mathbb{E}_1[f](t))^2) - \eta\mathbb{E}_1[f](t).$$

Lemma 2. [18] *Under the assumptions H1–H5:*

$$\mathbb{E}_1[f](t) = \frac{\mathbb{E}_1^+(\mathbb{E}_1^- - \mathbb{E}_1^0) - \mathbb{E}_1^-(\mathbb{E}_1^+ - \mathbb{E}_1^0)e^{-\frac{\sqrt{\eta^2+4F^2}}{F}t}}{(\mathbb{E}_1^- - \mathbb{E}_1^0) - (\mathbb{E}_1^+ - \mathbb{E}_1^0)e^{-\frac{\sqrt{\eta^2+4F^2}}{F}t}},$$

where $\mathbb{E}_1^0 := \mathbb{E}_1[f^0]$ *and* $\mathbb{E}_1^\pm := \frac{-\eta\pm\sqrt{\eta^2+4F^2}}{2F}$.

Furthermore, the following result holds.

Lemma 3. *If assumptions H1–H5 are satisfied, then*

$$\mathbb{E}_1[f](t) \to \mathbb{E}_1^+, \text{ as } t \to +\infty,$$

and

$$\left|\mathbb{E}_1[f](t) - \mathbb{E}_1^+\right| \leq Ce^{-\frac{\sqrt{\eta^2+4F^2}}{F}t},$$

where $C = C(\eta, F)$ *is a constant depending on the system.*

Remark 1. *Assumptions H2 and H4 can be relaxed as follows: there exist* $\eta > 0$ *and* $F > 0$ *such that*

$$\eta(u_*, u^*) \leq \eta,$$

for all $u_*, u^* \in D_u$*, and*

$$F(t) \leq F,$$

for all $t > 0$.

3. Dependence on the Initial Data

The main aim of this paper is to prove a result about the dependence on the initial data of the continuous thermostatted framework (1) that defines the related Cauchy problem (2).

Let $f_1(t,u)$ and $f_2(t,u)$ be two solutions to the Cauchy problem (2) corresponding to the initial data $f_1^0(u)$ and $f_2^0(u)$ respectively. If assumptions H1–H5 hold true, then $f_1(t,u)$, $f_2(t,u) \in C\left((0,+\infty); L^1(D_u)\right) \cap \mathcal{K}(D_u)$. Suppose that $f_1^0(u)$ and $f_2^0(u)$ belong to $L^1(D_u)$.

Theorem 1 obtains an estimate of the norm

$$\|f_1(t,u) - f_2(t,u)\|_{C\left((0,+\infty);L^1(D_u)\right)\cap\mathcal{K}(D_u)} = \max_{t\in[0,T]} \left(\int_{D_u} |f_1(t,u) - f_2(t,u)|\, du \right),$$

when

$$\left\|f_1^0(u) - f_2^0(u)\right\|_{L^1(D_u)} = \int_{D_u} \left|f_1^0(u) - f_2^0(u)\right| du \le \delta,$$

where $\delta > 0$. Precisely,

Theorem 1. *Assume that conditions **H1–H5** are verified. If*

$$\left\|f_1^0(u) - f_2^0(u)\right\|_{L^1(D_u)} \le \delta,$$

for $\delta > 0$, then, for all $T > 0$

$$\|f_1(t,u) - f_2(t,u)\|_{C\left([0,T];L^1(D_u)\right)\cap\mathcal{K}(D_u)} \le \delta\, e^{\bar{C}\,T},$$

where $\bar{C} := \left(3\,\eta + 2\,F\left(C(\eta,F) + \mathbb{E}_1^+\right)\right)$, and $C(\eta,F)$ is the constant of Lemma 1.

This result ensures that the dependence of solution to the continuous thermostatted framework (2) on the initial data is continuous, so that any solution to the Cauchy problem (2) is stable in the Hadamard sense.

4. Proof of the Result

The aim of this section is to give the proof of Theorem 1.

Proof of Theorem 1. Let $f_1(t,u)$ and $f_2(t,u)$ be two solutions to the Cauchy problem (2) belonging to the space $C\left([0,T]; L^1(D_u)\right) \cap \mathcal{K}(D_u)$, corresponding to the initial data $f_1^0(u)$ and $f_2^0(u)$ in the space $L^1(D_u)$, respectively.

Then, Equation (1) can be written:

$$\partial_t f(t,u) = J[f,f](t,u) - F\partial_u \left((1 - u\mathbb{E}_1[f](t))f(t,u)\right). \tag{3}$$

Integrating Equation (3) between 0 and t, for $t > 0$, one has:

$$\int_0^t \partial_t f(\tau,u)\, d\tau = \int_0^t J[f,f](\tau,u)\, d\tau - \int_0^t F\partial_u \left((1 - u\mathbb{E}_1[f](\tau))f(\tau,u)\right) d\tau,$$

and

$$\begin{aligned} f(t,u) = f^0(u) &+ \int_0^t J[f,f](\tau,u)\,d\tau \\ &- F\int_0^t \partial_u\left((1-u\mathbb{E}_1[f](\tau))f(\tau,u)\right)\,d\tau. \end{aligned} \tag{4}$$

Bearing in mind the initial data $f_1^0(u)$ and $f_2^0(u)$ and using relation (4), we obtain

$$\begin{aligned} f_1(t,u) = f_1^0(u) &+ \int_0^t J[f_1,f_1](\tau,u)\,d\tau \\ &- F\int_0^t \partial_u\left((1-u\mathbb{E}_1[f_1](\tau))f_1(\tau,u)\right)\,d\tau, \end{aligned} \tag{5}$$

and

$$\begin{aligned} f_2(t,u) = f_2^0(u) &+ \int_0^t J[f_2,f_2](\tau,u)\,d\tau \\ &- F\int_0^t \partial_u\left((1-u\mathbb{E}_1[f_2](\tau))f_2(\tau,u)\right)\,d\tau, \end{aligned} \tag{6}$$

so that, subtracting the (5) and (6), we find

$$\begin{aligned} f_1(t,u) - f_2(t,u) = &\left(f_1^0(u) - f_2^0(u)\right) \\ &+ \int_0^t \left(J[f_1,f_1](\tau,u) - J[f_2,f_2](\tau,u)\right)\,d\tau \\ &+ F\int_0^t \partial_u\left((1-u\mathbb{E}_1[f_2](\tau))f_2(\tau,u)\right)\,d\tau \\ &- F\int_0^t \partial_u\left((1-u\mathbb{E}_1[f_1](\tau))f_1(\tau,u)\right),d\tau, \end{aligned} \tag{7}$$

In virtue of Lemma 1:

$$\begin{aligned} J[f_1,f_1](\tau,u) - J[f_2,f_2](\tau,u) &= G[f_1,f_1](\tau,u) - \eta f_1(\tau,u) \\ &\quad - G[f_2,f_2](\tau,u) + \eta f_2(\tau,u) \\ &= G[f_1-f_2,f_1](\tau,u) \\ &\quad + G[f_2,f_1-f_2](\tau,u) \\ &\quad + \eta\left(f_2(\tau,u) - f_1(\tau,u)\right). \end{aligned} \tag{8}$$

Furthermore:

$$\begin{aligned} &\partial_u\Big((1-u\mathbb{E}_1[f_2](\tau))f_2(\tau) - (1-u\mathbb{E}_1[f_1](\tau))f_1(\tau)\Big) \\ &= \partial_u\Big((f_2(\tau,u) - f_1(\tau,u)) - u\mathbb{E}_1[f_2](\tau)f_2(\tau,u) + u\mathbb{E}_1[f_1](\tau)f_1(\tau,u)\Big) \\ &= \partial_u\big(f_2(\tau,u) - f_1(\tau,u)\big) - \partial_u\big(u(\mathbb{E}_1[f_2](\tau)f_2(\tau,u) - \mathbb{E}_1[f_1](\tau)f_1(\tau,u))\big). \end{aligned} \tag{9}$$

In virtue of relations (8) and (9), Equation (7) may be rewritten in the form

$$\begin{aligned} f_1(t,u) - f_2(t,u) = & \left(f_1^0(u) - f_2^0(u)\right) \\ & + \int_0^t \left(G[f_1 - f_2, f_1](\tau,u) + G[f_2, f_1 - f_2](\tau,u)\right) d\tau \\ & + \eta \int_0^t \left(f_2(\tau,u) - f_1(\tau,u)\right) d\tau + F \int_0^t \partial_u \left(f_2(\tau,u) - f_1(\tau,u)\right) d\tau \\ & - F \int_0^t \partial_u \Big(u(\mathbb{E}_1[f_2](\tau) f_2(\tau,u) - \mathbb{E}_1[f_1](\tau) f_1(\tau,u))\Big)\, d\tau, \end{aligned} \tag{10}$$

and the application of the triangle inequality to Equation (10) and an integration on D_u lead to

$$\begin{aligned} & \int_{D_u} |f_1(t,u) - f_2(t,u)|\, du \le \int_{D_u} \left|f_1^0(u) - f_2^0(u)\right| du \\ & + \int_{D_u} \left| \int_0^t G[f_1 - f_2, f_1](\tau,u) + G[f_2, f_1 - f_2](\tau,u)\, d\tau \right| du \\ & + \eta \int_0^t \left(\int_{D_u} |f_2(\tau,u) - f_1(\tau,u)|\, du \right) d\tau \\ & + F \int_{D_u} \left| \int_0^t \partial_u \left(f_2(\tau,u) - f_1(\tau,u)\right) d\tau \right| du \\ & + F \int_{D_u} \left| \int_0^t \partial_u \big(u(\mathbb{E}_1[f_2](\tau) f_2(\tau,u) - \mathbb{E}_1[f_1](\tau) f_1(\tau,u))\big)\, d\tau \right| du. \end{aligned} \tag{11}$$

To conclude the proof, we only need now to estimate the five terms at the right hand side of Equation (11).

First of all, by the assumptions of the theorem,

$$\int_{D_u} \left|f_1^0(u) - f_2^0(u)\right| du \le \delta\,, \tag{12}$$

so that, by straightforward calculations, we obtain for the second term at the right hand side of inequality (11) the following estimate:

$$\begin{aligned} & \int_{D_u} \left| \int_0^t G[f_1 - f_2, f_1](\tau,u) + G[f_2, f_1 - f_2](\tau,u)\, d\tau \right| du \\ & \le \int_{D_u} \int_0^t \int_{D_u \times D_u} \left|\eta \mathcal{A}(u_*, u^*, u) \left(f_1(\tau,u_*) - f_2(\tau,u_*)\right) f_1(\tau,u^*)\right| du_*\, du^*\, du\, d\tau \\ & + \int_{D_u} \int_0^t \int_{D_u \times D_u} \left|\eta \mathcal{A}(u_*, u^*, u) \left(f_1(\tau,u^*) - f_2(\tau,u^*)\right) f_2(\tau,u_*)\right| du_*\, du^*\, du\, d\tau \\ & \le 2\eta \int_{D_u} \left(\int_0^t |f_1(\tau,u) - f_2(\tau,u)|\, d\tau \right) du \\ & \le 2\eta \int_0^t \left(\int_{D_u} |f_2(\tau,u) - f_1(\tau,u)|\, du \right) d\tau. \end{aligned} \tag{13}$$

Next, since

$$\begin{aligned} & F \int_{D_u} \left| \int_0^t \partial_u \left(f_2(\tau,u) - f_1(\tau,u)\right) d\tau \right| du \\ & \le F \int_0^t \left(\int_{D_u} |\partial_u (f_2(\tau,u) - f_1(\tau,u))|\, du \right) d\tau, \end{aligned}$$

and $f(t,u) = 0$ for $u \in \partial D_u$, the fourth term at the right hand side of inequality (11) vanishes.

Furthermore, the fifth term at the right hand side of relation (11) is easily estimated as follows:

$$\begin{aligned}
&F\int_{D_u}\left|\int_0^t \partial_u\big(u(\mathbb{E}_1[f_2](\tau)f_2(\tau,u)-\mathbb{E}_1[f_1](\tau)f_1(\tau,u))\big)\,d\tau\right|\,du\\
&\le F\int_{D_u}\int_0^t |\mathbb{E}_1[f_2](\tau)f_2(\tau,u)-\mathbb{E}_1[f_1](\tau)f_1(\tau,u)|\,d\tau\,du\\
&+F\int_{D_u}\int_0^t |u\,\partial_u\,(\mathbb{E}_1[f_2](\tau)f_2(\tau,u)-\mathbb{E}_1[f_1](\tau)f_1(\tau,u))|\,d\tau\,du.
\end{aligned} \tag{14}$$

Now, integrating by parts the second term of the right hand side of relation (14) and bearing in mind that $f(t,u)=0$ for $u\in\partial D_u$, one has

$$\begin{aligned}
&F\int_{D_u}\left|\int_0^t \partial_u\big(u(\mathbb{E}_1[f_2](\tau)f_2(\tau,u)-\mathbb{E}_1[f_1](\tau)f_1(\tau,u))\big)\,d\tau\right|\,du\\
&\le 2F\int_{D_u}\int_0^t |\mathbb{E}_1[f_2](\tau)f_2(\tau,u)-\mathbb{E}_1[f_1](\tau)f_1(\tau,u)|\,d\tau\,du.
\end{aligned} \tag{15}$$

By adding and subtracting $\mathbb{E}_1^+\,f_2(\tau,u)$ and $\mathbb{E}_1^+\,f_1(\tau,u)$ at the right hand side of relation (15) and using Lemma (2), it follows that

$$\begin{aligned}
&F\int_{D_u}\left|\int_0^t \partial_u\big(u(\mathbb{E}_1[f_2](\tau)f_2(\tau,u)-\mathbb{E}_1[f_1](\tau)f_1(\tau,u))\big)\,d\tau\right|\,du\\
&\le 2F\left[\int_{D_u}\int_0^t \Big|f_2(\tau,u)\,(\mathbb{E}_1[f_2](\tau)-\mathbb{E}_1^+)\right.\\
&-f_1(\tau,u)\,(\mathbb{E}_1[f_1](\tau)-\mathbb{E}_1^+)\Big|\,d\tau\,du\\
&\left.+\mathbb{E}_1^+\int_0^t\left(\int_{D_u}|f_2(\tau,u)-f_1(\tau,u)|\,du\right)d\tau\right]\\
&\le 2\,F\left[C(\eta,F)\int_{D_u}\int_0^t e^{-\frac{\sqrt{\eta^2+4F^2}}{F}t}\,|f_2(\tau,u)-f_1(\tau,u)|\,d\tau\,du\right.\\
&\left.+\mathbb{E}_1^+\int_0^t\left(\int_{D_u}|f_2(\tau,u)-f_1(\tau,u)|\,du\right)d\tau\right]\\
&\le 2F\,(C(\eta,F)+\mathbb{E}_1^+)\int_0^t\left(\int_{D_u}|f_2(\tau,u)-f_1(\tau,u)|\,du\right)d\tau.
\end{aligned} \tag{16}$$

By relations (12), (13), (16) and the condition $f(t,u)=0$ for $u\in\partial D_u$, inequality (11) becomes

$$\begin{aligned}
&\int_{D_u}|f_1(t,u)-f_2(t,u)|\,du\le\delta\\
&+2\,\eta\int_0^t\left(\int_{D_u}|f_2(\tau,u)-f_1(\tau,u)|\,du\right)d\tau\\
&+\eta\int_0^t\left(\int_{D_u}|f_2(\tau,u)-f_1(\tau,u)|\,du\right)d\tau\\
&+2F\,(C(\eta,F)+\mathbb{E}_1^+)\int_0^t\left(\int_{D_u}|f_2(\tau,u)-f_1(\tau,u)|\,du\right)d\tau,
\end{aligned} \tag{17}$$

and, by reordering the terms of this last relation and using the constant

$$\bar{C}:=\left(3\,\eta+2\,F\,(C(\eta,F)+\mathbb{E}_1^+)\right),$$

we get

$$\begin{aligned}&\int_{D_u} |f_1(t,u) - f_2(t,u)| \, du \\ &\leq \delta + \bar{C} \int_0^t \left(\int_{D_u} |f_2(\tau,u) - f_1(\tau,u)| \, du \right) d\tau.\end{aligned} \tag{18}$$

Finally, the Gronwall inequality [26] and inequality (15) yield

$$\int_{D_u} |f_1(t,u) - f_2(t,u)| \, du \leq \delta \, e^{\bar{C}\, t}, \tag{19}$$

for all $t > 0$, so that, for $T > 0$,

$$\max_{t \in [0,T]} \left(\int_{D_u} |f_1(t,u) - f_2(t,u)| \, du \right) \leq \delta \, e^{\bar{C}\, t},$$

and our claim is proved. □

5. Concluding Remarks and Research Perspectives

As already pointed out at the end of Section 3, the result proved in the foregoing section can be expressed by the statement that the solutions to the Cauchy problem (2) are stable in the Hadamard sense. As it stands, this result could seem to be not quite satisfactory, and the question spontaneously arises on whether these solutions can be proved to be stable or unstable in the whole interval $(0, +\infty)$, and whether the answer to such a question could be find by some suitable technical refinements of the method of proof used in the previous section. Several observations witness for a negative answer to this last question, but a simple heuristic reasoning—based on the acknowledged versatility of the scheme expressed by problem (2)—will be sufficient. As said in the Introduction, and shown in the references quoted there, the scheme applies to many different kinds of systems, in particular to social and economic systems. Our experience shows that slightly different distributions of wealth can evolve in time in very similar or very different ways depending on the behaviour of the particles of each system, i.e. on the terms in $J[f,f](t,u)$ describing their interaction rates and the probable effects of such interactions. The relations between these terms should play an important role in producing stability or instability: for instance, frequent interactions could produce instability when joined with strongly asymmetric transition probability distributions, but even uniform stability in the whole interval $(0, +\infty)$ when the transition probabilities are symmetric for any pair of initial states of interacting individuals. Thus, one of the first goals of subsequent researches about the problem considered in the present paper should be to check whether different assumptions on the forms of $\eta(u_*, u^*)$ and $\mathcal{A}(u_*, u^*, u)$ give stronger but different results about stability or instability.

In this connection, a very important step of subsequent research will be the numerical formulation and some numerical simulations of its solutions. In particular, problem (2) should be formulated in particular contexts, like the one of socio–economic problems, and the interaction rate $\eta(u_*, u^*)$ and the probability distribution $\mathcal{A}(u_*, u^*, u)$ on the effects of interactions should be assigned according to the results of previous statistical researches and the laws of micro-economy.

Furthermore, the choice of the distance

$$d(f_1, f_2) = \int_{D_u} |f_1(t,u) - f_2(t,u)| \, du$$

in the space of solutions of the Cauchy problem (2) can be improved, as it is not so significant from a statistical viewpoint. For instance, if f_1 is a uniform probability density, it tells us that the deviations of f_2 from uniformity are, in some sense, "small", but says nothing about the symmetry (or asymmetry) of these deviations. From a statistical viewpoint, we are interested to know whether such deviations are "large" but concentrated in a small subset of D_u, or "small" and spread over D_u. So, the analysis must be concentrated on a comparison of the derivatives $\partial_u f_1$ and $\partial_u f_2$.

Finally, it is of the greatest relevance to study continuous dependence on initial data in the discrete case, mainly in order to check it also by means of numerical simulations. This must be a preliminary step to the statistical studies proposed above.

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