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$\mathcal{F}$-Metric, F-Contraction and Common Fixed-Point Theorems with Applications

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Abstract: In this paper, we noticed that the existence of fixed points of F-contractions, in $\mathcal{F}$-metric space, can be ensured without the third condition (F3) imposed on the Wardowski function $F : (0, \infty) \to \mathbb{R}$. We obtain fixed points as well as common fixed-point results for Reich-type F-contractions for both single and set-valued mappings in $\mathcal{F}$-metric spaces. To show the usability of our results, we present two examples. Also, an application to functional equations is presented. The application shows the role of fixed-point theorems in dynamic programming, which is widely used in computer programming and optimization. Our results extend and generalize the previous results in the existing literature.

Keywords: $\mathcal{F}$-metric; Reich-type F-contraction; Kannan-type F-contraction; dynamic programming

MSC: 47H09; 47H10

1. Introduction

In recent years, many authors have presented interesting generalizations of metric spaces (see for example [1–11]). Among them, Jleli et al. [12] presented the idea of $\mathcal{F}$-metric space and, while comparing it with metric spaces, they proved that every metric space is an $\mathcal{F}$-metric space but the converse is not true, confirming that $\mathcal{F}$-metric space is more general than the metric space. With the help of concrete examples, they obtained a similar result for s-relaxed metric space. They discussed a relation between b-metric and $\mathcal{F}$-metric spaces, defined a natural topology on these spaces and proved that after imposing a sufficient condition, the closed ball is closed with respect to the given topology. Finally, they established a fixed-point theorem of the Banach contraction in the frame of $\mathcal{F}$-metric spaces.

Wardowski [13] extended the Banach Contraction Principle by introducing F-contraction and established fixed-point theorems in metric spaces. Klim et al. [9] discussed F-contractions with respect to dynamic process and proved fixed-point theorems involving F-contractions. Nazam et al. [14] proved fixed-point theorems for Kannan-type F-contractions on closed balls in complete partial metric spaces. Fixed points and common fixed points of F-contractions under various structures (set-valued mappings, partial order, α-admissibility, graphic structure, b-metric, partial metric, G-metric etc.) have been investigated, (see the references [14–22] and references therein).

In this article, we relax the restrictions on Wardowski's mapping [13] by eliminating the third condition and prove common fixed-point results of Reich-type F-contractions for both single and set-valued mappings in $\mathcal{F}$-metric spaces.

This article is organized into seven sections. This first section contains a short history of the literature providing a motivation for this article. Section 2 contains some basic definitions which help readers to understand our results. In Section 3, we established theorems of fixed points and common fixed points of single-valued F-contractions in $\mathcal{F}$-metric spaces. An example is provided to explain our results. Section 4 deals with fixed-point theorems of F-contractions with respect to closed balls in $\mathcal{F}$-metric spaces along with an example. In Section 5, the common fixed-point theorems of multi-valued modified F-contractions are proved in $\mathcal{F}$-metric spaces. Section 6 is concerned with an application of the mentioned results to the functional equations in dynamic programming. Section 7 consists of conclusions.

2. Basic Relevant Notions

Wardowski [13] considered a nonlinear function $F : (0, \infty) \to \mathbb{R}$ with the following characteristics:

(F1) F is strictly increasing.

(F2) for any sequence $\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(t_n) = -\infty.$$

(F3) there exists $l \in (0, 1)$ such that $\lim_{t\to 0^+} t^l F(t) = 0$.

Wardowski [13] called the mapping $T : X \to X$, defined on a metric space $X = (X, d)$, an F-contraction if there exist $\tau > 0$ and F satisfying (F1)-(F3) such that

$$\tau + F\big(d(Tx, Ty)\big) \le F\big(d(x, y)\big) \text{ for all } x, y \in X.$$

In what follows, we let

$$\mathcal{B} = \{F : (0, \infty) \to \mathbb{R} \,|\, F \text{ satisfies } (F1) - (F2)\}.$$

Definition 1. *[12] Suppose A is a non-empty set and $(g, \alpha) \in \mathcal{B} \times [0, \infty)$. Let the function $d : A \times A \to [0, \infty)$ be such that*

(d1) for all $(a, b) \in A \times A,\ d(a, b) = 0 \Longleftrightarrow a = b$.

(d2) for all $(a, b) \in A \times A,\ d(a, b) = d(b, a)$.

(d3) for every $(a, b) \in A \times A$, for each $N' \in \mathbb{N}, N' \ge 2$ and for every $\{t_n\}_{i=1}^{n} \subset A$ with $(t_1, t_{N'}) = (a, b)$, we have $d(a, b) > 0 \Longrightarrow g(d(a, b)) \le g\left(\sum_{i=1}^{N'-1} d(t_i, t_{i+1})\right) + \alpha$.

Then d is known as an $\mathcal{F}$-metric on A and the pair (A, d) is called an $\mathcal{F}$-metric space.

Example 1. *[12] Let $A = \mathbb{N}$ (the set of natural numbers) and $d : A \times A \to (0, \infty)$ be defined by*

$$d(a, b) = \begin{cases} (a-b)^2 & \text{if } (a, b) \in [0, 3] \times [0, 3] \\ |a-b| & \text{if } (a, b) \notin [0, 3] \times [0, 3] \end{cases}$$

for all $(a, b) \in A \times A$. Then d is an $\mathcal{F}$-metric on A.

Example 2. *[12] Let $A = \mathbb{N}$ and $d : A \times A \to (0, \infty)$ be defined by*

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ e^{|a-b|} & \text{if } a \ne b \end{cases}$$

for all $(a, b) \in A \times A$. Then d is an $\mathcal{F}$-metric on A.

Definition 2. *[12] Suppose $\{a_n\}$ is a sequence in A. Then*

(i) *$\{a_n\}$ is $\mathcal{F}$-convergent to a point $a \in A$ if $\lim_{n\to\infty} d(a_n, a) = 0$.*
(ii) *$\{a_n\}$ is an $\mathcal{F}$-Cauchy sequence if $\lim_{n,m\to\infty} d(a_n, a_m) = 0$.*
(iii) *The space (A, d) is $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence $\{a_n\}$ is $\mathcal{F}$-convergent to a point $a \in A$.*

Definition 3. *[12] Let (A, d) be an $\mathcal{F}$-metric space. A subset O of A is said to be $\mathcal{F}$-open if for every $a \in O$, there is some $r > 0$ such that $B(a, r) \subset O$, where*

$$B(a,r) = \{b \in A \mid d(a,b) < r\}.$$

We say that a subset $\mathcal{C}$ of A is $\mathcal{F}$-closed if $A \setminus \mathcal{C}$ is $\mathcal{F}$-open.

Definition 4. *[12] Let (A, d) be an $\mathcal{F}$-metric space and B be a non-empty subset of A. Then the following statements are equivalent:*

(i) *B is $\mathcal{F}$-closed.*
(ii) *For any sequence $\{a_n\} \subset B$, we have*

$$\lim_{n\to\infty} d(a_n, a) = 0,\ a \in A \Longrightarrow a \in B.$$

Theorem 1. *[12] Suppose $(f, \alpha) \in \mathcal{B} \times [0, \infty)$ and (A, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $g : A \to A$ be a given mapping. Suppose that there exists $k \in (0, 1)$ such that*

$$d(g(a), g(b)) \leq kd(a,b),\ (a,b) \in A \times A.$$

Then g has a unique fixed point $a^* \in A$. Moreover, for any $a_0 \in A$, the sequence defined by $a_{n+1} = g(a_n)$, $n \in \mathbb{N}$ is $\mathcal{F}$-convergent to a^*.

Theorem 2. *[11] Suppose A is a complete metric space with metric d, and let $g : A \to A$ be a function such that*

$$d(g(a), g(b)) \leq \alpha d(a,b) + \beta d(a, g(a)) + \gamma d(b, g(b))$$

for all $a, b \in A$, where α, β, γ are non-negative and satisfy $\alpha + \beta + \gamma < 1$. Then g has a unique fixed point.

Remark 1. *[1] If F is right continuous and satisfies (F1), then*

$$F(\inf A) = \inf F(A) \text{ for all } A \subset (0,\infty) \text{ with } \inf A > 0.$$

3. Fixed Points of Reich-Type F-Contractions in $\mathcal{F}$-Metric Spaces

In this section, we construct fixed-point and common fixed-point results for single-valued Reich-type and Kannan-type F-contractions in the setting of $\mathcal{F}$-metric spaces.

Theorem 3. *Suppose $(f, \alpha) \in \mathcal{B} \times [0, \infty)$ and (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $S, T : X \to X$ be self-mappings. Suppose there exist $F \in \mathcal{B}$ and $\tau > 0$ such that*

$$\tau + F(d(Sx, Ty)) \leq F\big(ad(x,y) + bd(x, Sx) + cd(y, Ty)\big) \tag{1}$$

for $a, b, c \in [0, \infty)$ such that $a + b + c < 1$ with $\min\{d(Sx, Ty), d(x,y), d(x, Sx), d(y, Ty)\} > 0$, for all $(x, y) \in X \times X$. Then S and T have at most one common fixed point in X.

Proof. Suppose x_0 is an arbitrary point and define a sequence (x_n) by

$$Sx_{2j} = x_{2j+1} \text{ and } Tx_{2j+1} = x_{2j+2};\ j = 0, 1, 2, \ldots \tag{2}$$

Using (1) and (2) we can write

$$\begin{aligned} &\tau + F\big(d(x_{2j+1}, x_{2j+2})\big) \\ &= \tau + F\big(d(Sx_{2j}, Tx_{2j+1})\big) \\ &\leq F\big(ad(x_{2j}, x_{2j+1}) + bd(x_{2j}, Sx_{2j}) + c(x_{2j+1}, Tx_{2j+1})\big) \\ &= F\big(ad(x_{2j}, x_{2j+1}) + bd(x_{2j}, x_{2j+1}) + cd(x_{2j+1}, x_{2j+2})\big). \end{aligned}$$

Using (F1) and letting $\lambda = \frac{a+b}{1-c}$, we have

$$d(x_{2j+1}, x_{2j+2}) < \tfrac{a+b}{1-c}\, d(x_{2j}, x_{2j+1}) = \lambda d(x_{2j}, x_{2j+1}).$$

Similarly

$$d(x_{2j+2}, x_{2j+3}) < \tfrac{a+b}{1-c}\, d(x_{2j+1}, x_{2j+2}) = \lambda d(x_{2j+1}, x_{2j+2}).$$

Hence

$$d(x_n, x_{n+1}) < \lambda d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N},$$

which yields

$$d(x_n, x_{n+1}) < \lambda d(x_{n-1}, x_n) < \lambda^2 d(x_{n-2}, x_{n-1}) < \ldots < \lambda^n d(x_0, x_1), n \in \mathbb{N}. \tag{3}$$

Using (3), we can write

$$\begin{aligned} \sum_{k=n}^{m-1} d(x_k, x_{k+1}) &< \lambda^n \left(1 + \lambda + \lambda^2 + \ldots + \lambda^{m-n-1}\right) d(x_0, x_1) \\ &\leq \tfrac{\lambda^n}{1-\lambda}\, d(x_0, x_1),\ m > n. \end{aligned}$$

Since $\lim_{n\to\infty} \frac{\lambda^n}{1-\lambda} d(x_0, x_1) = 0$, for any $\delta > 0$ there exists some $n' \in \mathbb{N}$ such that

$$0 < \tfrac{\lambda^n}{1-\lambda}\, d(x_0, x_1) < \delta,\ n \geq n'. \tag{4}$$

Furthermore, let $\epsilon > 0$ be fixed. Since $(f, \alpha) \in \mathcal{B} \times [0, \infty)$ satisfies (d3), by (F2) it follows that there is some $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \alpha. \tag{5}$$

By (4) and (5), we write

$$f\left(\sum_{k=n}^{m-1} d(x_k, x_{k+1})\right) \leq f\left(\tfrac{\lambda^n}{1-\lambda}\, d(x_0, x_1)\right) < f(\epsilon) - \alpha,\ m > n \geq n'.$$

By (d3) and the above inequality, we obtain

$$d(x_n, x_m) > 0,\ m > n > n' \Longrightarrow f(d(x_n, x_m)) < f(\epsilon).$$

This shows

$$d(x_n, x_m) < \epsilon,\ m > n \geq n'.$$

Hence, we showed that (x_n) is an $\mathcal{F}$-Cauchy sequence in X. Since (X, d) is $\mathcal{F}$-complete, there exists $z^* \in X$ such that (x_n) is $\mathcal{F}$-convergent to z^*, i.e.,

$$\lim_{n\to\infty} d(x_n, z^*) = 0.$$

To prove that z^* is the fixed point of S, assume $d(Sz^*, z^*) > 0$. Then

$$\tau + F\big(d(Sz^*, x_{2j+2})\big) \leq F\big(ad(z^*, x_{2j+1}) + bd(z^*, Sz^*) + cd(x_{2j+1}, x_{2j+2})\big).$$

By (F1) and letting $j \to \infty$, we have

$$(1-b)d(Sz^*, z^*) < 0,$$

which is a contradiction. Hence $d(Sz^*, z^*) = 0$, i.e., $Sz^* = z^*$. Following the same steps, we get $Tz^* = z^*$. Hence $Tz^* = Sz^* = z^*$.

Now we show the uniqueness. Assume that z^{**} is also a common fixed point of S and T and $z^* \neq z^{**}$. Then

$$\begin{aligned}\tau + F\big(d(z^*, z^{**})\big) = \tau + F\big(d(Sz^*, Tz^{**})\big) &\leq F\big(ad(z^*, z^{**}) + bd(z^*, Sz^*) + cd(z^{**}, Tz^{**})\big)\\ &= F\big(ad(z^*, z^{**}) + bd(z^*, z^*) + cd(z^{**}, z^{**})\big).\end{aligned}$$

By (F1), we get $(1-a)d(z^*, z^{**}) < 0$, which is a contradiction. Hence $z^* = z^{**}$. □

Example 3. *Suppose* $X = \{X_n := 2n+1,\ n \in \mathbb{N}\}$, $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ e^{|x-y|} & \text{if } x \neq y, \end{cases}$ $F(x) = \ln x$ *and* $S, T : X \to X$ *are defined by*

$$T(X_n) = \begin{cases} X_1 & \text{if } n = 1,2 \\ X_{n-1} & \text{if } n \geq 3 \end{cases} \quad \text{and} \quad S(X_n) = \begin{cases} X_1 & \text{if } n = 1 \\ X_2 & \text{if } n = 2 \\ X_{n-2} & \text{if } n \geq 3. \end{cases}$$

It can be easily verified that d is an $\mathcal{F}$-metric and F satisfies (F1)-(F2). Fix $b = c = 0$ and $(x,y) \in X \times X$. Suppose $m \neq n$, then

$$\begin{aligned}F\big(d(SX_n, TX_m)\big) &= \ln e^{|X_{n-2} - X_{m-1}|} = \ln e^{|2(n-m)-2|}\\ &< \ln(e^{-1} \cdot e^{|2(n-m)|}) = F(ad(X_n, X_m))\\ &= F\big(ad(X_n, X_m) + bd(X_n, SX_n) + cd(X_m, TX_m)\big),\end{aligned}$$

whenever $\min\{d(SX_n, TX_m), d(X_n, X_m)\} > 0$*, where $a = e^{-1}$. For $\tau \in (0, \ln e) = (0,1)$, the inequality (1) holds true. Moreover, it is clear that X_1 is the only common fixed point of S and T.*

Taking $a = 0$ in Theorem 3, we get the following result of Kannan-type F-contractions.

Corollary 1. *Suppose $(f, \alpha) \in \mathcal{B} \times [0, \infty)$ and (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $S, T : X \to X$ be self-mappings. Suppose that for $k \in [0,1)$, there exist $F \in \mathcal{B}$ and $\tau > 0$ such that*

$$\tau + F\big(d(Sx, Ty)\big) \leq F\left(\tfrac{k}{2}\big(d(x, Sx) + d(y, Ty)\big)\right)$$

with $\min\{d(Sx, Ty), d(x, Sx), d(y, Ty)\} > 0$*, for all $(x,y) \in X \times X$. Then S and T have at most one common fixed point in X.*

Replacing S with T, we get the following result of single mappings.

Corollary 2. *Suppose $(f,\alpha) \in \mathcal{B} \times [0,\infty)$ and (X,d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $T : X \to X$ be a self-mapping. Suppose that for $k \in [0,1)$, there exist $F \in \mathcal{B}$ and $\tau > 0$ such that*

$$\tau + F\big(d(Tx,Ty)\big) \leq F\left(\tfrac{k}{2}\big(d(x,Tx) + d(y,Ty)\big)\right)$$

with $\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} > 0$*, for all $(x,y) \in X \times X$. Then T has at most one fixed point in X.*

4. Fixed Points of Reich-Type F-Contractions on $\mathcal{F}$-Closed Balls

This portion of the paper deals with the fixed-point theorems of Reich-type F-contractions that hold true only on the closed balls rather than on the whole space X.

Definition 5. *Let (X,d) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $S,T : X \to X$ be self-mappings. Suppose that $a+b+c < 1$ for $a,b,c \in [0,\infty)$. Then the mapping T is called a Reich-type F-contraction on $B(x_0,r) \subseteq X$ if there exist $F \in \mathcal{B}$ and $\tau > 0$ such that*

$$\tau + F(d(Sx,Ty)) \leq F\big(ad(x,y) + bd(x,Sx) + cd(y,Ty)\big), \quad \forall x,y \in B(x_0,r). \tag{6}$$

Theorem 4. *Suppose $(f,\alpha) \in \mathcal{B} \times [0,\infty)$ and (X,d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let T be a Reich-type F-contraction on $B(x_0,r) \subseteq X$. Suppose that for $x_0 \in X$ and $r > 0$, the following conditions are satisfied:*

- *(a)* *$B(x_0,r)$ is $\mathcal{F}$-closed,*
- *(b)* *$d(x_0,x_1) \leq (1-\lambda)r$, for $x_1 \in X$ and $\lambda = \frac{a+b}{1-c}$,*
- *(c)* *There exist $0 < \epsilon < r$ such that $f\left((1-\lambda^{k+1})r\right) \leq f(\epsilon) - \alpha$, where $k \in \mathbb{N}$.*

Then S and T have at most one common fixed point in $B(x_0,r)$.

Proof. Suppose x_0 is an arbitrary point, and define a sequence (x_n) by

$$T(x_{2j}) = x_{2j+1} \text{ and } S(x_{2j+1}) = x_{2j+2}; j = 0,1,2,\ldots.$$

We need to show that (x_n) is in $B(x_0,r)$ for all $n \in \mathbb{N}$. We show it by mathematical induction. By (b), we write

$$d(x_0,x_1) < r.$$

Therefore $x_1 \in B(x_0,r)$. Suppose $x_2,\ldots,x_k \in B(x_0,r)$ for some $k \in \mathbb{N}$. Now, if $2j \leq k$, then by (6), we can write

$$\begin{aligned}
\tau + F\big(d(x_{2j},x_{2j+1})\big) &= \tau + F\big(d(Sx_{2j-1},Tx_{2j})\big) \\
&\leq F\big(ad(x_{2j-1},x_{2j}) + bd(x_{2j-1},Sx_{2j-1}) + cd(x_{2j},Tx_{2j})\big) \\
&= F\big(ad(x_{2j-1},x_{2j}) + bd(x_{2j-1},x_{2j}) + cd(x_{2j},x_{2j+1})\big).
\end{aligned}$$

By (F1), we write

$$d(x_{2j},x_{2j+1}) < \tfrac{a+b}{1-c}\, d(x_{2j-1},x_{2j}) = \lambda d(x_{2j-1},x_{2j}). \tag{7}$$

Similarly, if $2j+1 \leq k$,

$$d(x_{2j-1},x_{2j}) < \tfrac{a+b}{1-c}\, d(x_{2j-2},x_{2j-1}) = \lambda d(x_{2j-2},x_{2j-1}). \tag{8}$$

Therefore, from inequalities (7) and (8), we write

$$d(x_{2j},x_{2j+1}) < \lambda d(x_{2j-1},x_{2j}) < \ldots < \lambda^{2j} d(x_0,x_1) \tag{9}$$

and

$$d(x_{2j-1}, x_{2j}) < \lambda d(x_{2j-2}, x_{2j-1}) < \ldots < \lambda^{2j-1} d(x_0, x_1). \tag{10}$$

From (9) and (10), we write

$$d(x_k, x_{k+1}) \leq \lambda^k d(x_0, x_1). \tag{11}$$

Now, using (11), we have

$$\begin{aligned} f(d(x_0, x_{k+1})) &\leq f\left(\sum_{i=1}^{k+1} d(x_{i-1}, x_i)\right) + \alpha = f\big(d(x_0, x_1) + \ldots + d(x_k, x_{k+1})\big) + \alpha \\ &\leq f\big((1 + \lambda + \lambda^2 + \ldots + \lambda^k) d(x_0, x_1)\big) + \alpha = f\left(\tfrac{1-\lambda^{k+1}}{1-\lambda} d(x_0, x_1)\right) + \alpha. \end{aligned}$$

By (b) and (c), we can write

$$f(d(x_0, x_{k+1})) \leq f\left((1 - \lambda^{k+1}) r\right) + \alpha \leq f(\epsilon) < f(r).$$

Hence by (F1), we deduce that

$$x_{k+1} \in B(x_0, r).$$

Therefore, $(x_n) \subset B(x_0, r)$ for all $n \in \mathbb{N}$. Now by (6), we have

$$\begin{aligned} \tau + F\big(d(x_{2j+1}, x_{2j+2})\big) &= \tau + F\big(d(Tx_{2j}, Sx_{2j+1})\big) \\ &\leq F\big(ad(x_{2j}, x_{2j+1}) + bd(x_{2j}, Tx_{2j}) + c(x_{2j+1}, Sx_{2j+1})\big) \\ &= F\big(ad(x_{2j}, x_{2j+1}) + bd(x_{2j}, x_{2j+1}) + cd(x_{2j+1}, x_{2j+2})\big). \end{aligned}$$

Following the same steps of proof of Theorem 3 and using (a), we obtain that the sequence (x_n) is $\mathcal{F}$-convergent to some z^* in $B(x_0, r)$. Furthermore, z^* can be proved as common fixed point of S and T in the same way as in Theorem 3. □

Taking $S = T$ in Theorem 4, we get the following result of single mappings.

Corollary 3. *Suppose $(f, \alpha) \in \mathcal{B} \times [0, \infty)$, $(F, \tau) \in \mathcal{B} \times [0, \infty)$, (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $T : X \to X$ is a self-mapping. Suppose that $a + b + c < 1$ for $a, b, c \in [0, \infty)$. Suppose that for $x_0 \in X$ and $r > 0$, the following conditions are satisfied:*

- *(a)* *$B(x_0, r) \subseteq X$ is $\mathcal{F}$-closed,*
- *(b)* *$\tau + F(d(Tx, Ty)) \leq F\big(ad(x, y) + bd(x, Tx) + cd(y, Ty)\big)$, for all $x, y \in B(x_0, r)$,*
- *(c)* *$d(x_0, x_1) \leq (1 - \lambda) r$, for $x_1 \in X$ and $\lambda = \frac{a+b}{1-c}$,*
- *(d)* *There exist $0 < \epsilon < r$ such that $f\left((1 - \lambda^{k+1}) r\right) \leq f(\epsilon) - \alpha$, where $k \in \mathbb{N}$.*

Then T has at most one fixed point in $B(x_0, r)$.

Example 4. *Let $X = [0, \infty)$ and $F(x) = \ln x$. Define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1] \\ x^2 & \text{if } x \in (1, \infty) \end{cases}$$

and define d by

$$d(x, y) = \begin{cases} (x - y)^2 & \text{if } (x, y) \in [0, 1] \times [0, 1] \\ |x - y| & \text{if } (x, y) \notin [0, 1] \times [0, 1]. \end{cases}$$

It can be easily verified that d is an $\mathcal{F}$-metric and functions f and F satisfy (F1)-(F2). Fix $x_0 = r = \frac{1}{2}$, then $B(x_0, r) \subset [0,1]$. Clearly $B(x_0, r)$ is $\mathcal{F}$-closed so condition (a) of Corollary 3 is satisfied. Now if $a = \frac{3}{4}, b = c = 0$, then $\lambda = a$ and

$$d(x_0, x_1) = d(x_0, Tx_0) = (\tfrac{1}{2} - \tfrac{1}{4})^2 = \tfrac{1}{16} < \left(1 - \tfrac{3}{4}\right)\tfrac{1}{2} = (1-\lambda)r.$$

This shows that condition (c) is fulfilled. Furthermore, since f is an increasing function and $\lambda < 1$, for every $k \in \mathbb{N}$, we can find some $\epsilon < r$ and $\alpha \in [0, \infty)$ such that $f\big((1-\lambda^{k+1})r\big) \leq f(\epsilon) - \alpha$ is satisfied, i.e., condition (d) is fulfilled. Now if $(x, y) \in B(x_0, r) \times B(x_0, r)$, then

$$\begin{aligned} F\big(d(Tx, Ty)\big) &= \ln\left(\tfrac{x}{2} - \tfrac{y}{2}\right)^2 < \ln\tfrac{3}{4}(x-y)^2 \\ &= F\big(ad(x,y) + bd(x,Tx) + cd(y,Ty)\big) \end{aligned}$$

where $b = c = 0$ with

$$\tau \in \left(0, \ln\tfrac{3}{4}(x-y)^2 - \ln\left(\tfrac{x}{2} - \tfrac{y}{2}\right)^2\right) = (0, \ln 3).$$

Therefore, for all $(x, y) \in B(x_0, r) \times B(x_0, r)$, condition (b) is also satisfied. On the other hand, if $(x, y) \notin B(x_0, r) \times B(x_0, r)$ e.g., $x = 2$ and $y = 3$, then

$$\begin{aligned} F\big(d(Tx, Ty)\big) &= \ln|2^2 - 3^2| > \ln\tfrac{3}{4}|(2-3)| = F(ad(x,y)) \\ &= F\big(ad(x,y) + bd(x,Tx) + cd(y,Ty)\big). \end{aligned}$$

Hence, condition (b) holds only for $B(x_0, r)$ and not on $X \times X$. Moreover, $0 \in B(x_0, r)$ is the fixed point of T.

Corollary 4. *Suppose $(f, \alpha) \in \mathcal{B} \times [0, \infty), (F, \tau) \in \mathcal{B} \times [0, \infty), (X, d)$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $S, T : X \to X$ be self-mappings and $\kappa \in [0, 1)$. Suppose that for $x_0 \in X$ and $r > 0$, the following conditions are satisfied:*

- *(a)* $B(x_0, r) \subseteq X$ *is $\mathcal{F}$-closed,*
- *(b)* $\tau + F\big(d(Tx, Ty)\big) \leq F\big(\frac{\kappa}{2}(d(x, Sx) + d(y, Ty))\big)$, *for all $x, y \in B(x_0, r)$,*
- *(c)* $d(x_0, x_1) \leq (1-\lambda)r$, *for $x_1 \in X$ and $\lambda = \frac{\kappa}{2-\kappa}$,*
- *(d)* *There exist $0 < \epsilon < r$ such that $f\left((1-\lambda^{k+1})r\right) \leq f(\epsilon) - \alpha$, where $k \in \mathbb{N}$.*

Then S and T have at most one common fixed point in $B(x_0, r)$.

5. Fixed Points of Set-Valued Reich-Type F-Contractions in $\mathcal{F}$-Metric Spaces

This section is concerned with the fixed points of set-valued Reich-type F-contractions in $\mathcal{F}$-metric spaces.

Definition 6. *[1] Let (X, d) be a metric space. Let $CB(X)$ be the family of all non-empty closed and bounded subsets of X. Let $H : CB(X) \times CB(X) \to [0, \infty)$ be a function defined by*

$$H(A, B) = \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\} \text{ for all } A, B \in CB(X),$$

where $D(x, B) = \inf\{d(x, y) : y \in B\}$. Then H defines a metric on $CB(X)$ called the Hausdorff metric induced by d.

Definition 7. *Let (X, d) be an $\mathcal{F}$-metric space [23,24]. Suppose $F \in \mathcal{B}$ and $H : CB(X) \times CB(X) \to [0, \infty)$ be the Hausdorff metric function defined in Definition 6. A mapping $T : X \to CB(X)$ is known as a set-valued Reich-type F-contraction if there is some $\tau > 0$ such that*

$$2\tau + F\big(H(Tx, Ty)\big) \leq F\big(ad(x,y) + bd(x,Tx) + cd(y,Ty)\big) \tag{A}$$

for $(x, y) \in X \times X$ *and* $a, b, c \in [0, \infty)$ *such that* $a + b + c < 1$.

Theorem 5. *Let* (X, d) *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $(f, \alpha) \in \mathcal{B} \times [0, \infty)$. *If the mapping* $T : X \to CB(X)$ *is a set-valued Reich-type F-contraction such that F is right continuous, then T has a fixed point in X.*

Proof. Suppose for a fixed natural number $n_0, x_{n_0} \in X$. If $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is the fixed point of T. Let x_0 be an initial guess in X. Choose a point x_1 in X such that $x_1 \in Tx_0$ and $x_1 \notin Tx_1$, continuing in this manner, we can define a sequence (x_n) such that $x_{n+1} \in Tx_n$ and $x_{n+1} \notin Tx_{n+1}$ for all $n \geq 0$. Assume that for x_n in X, $x_n \notin Tx_n$ for all $n \geq 0$, then $Tx_{n-1} \neq Tx_n$. Since F is right continuous, there exists some real value $h > 1$ such that

$$F\big(hH(Tx_{n-1}, Tx_n)\big) \leq F\big(H(Tx_{n-1}, Tx_n)\big) + \tau.$$

By (A), we have

$$2\tau + F\big(H(Tx_{n-1}, Tx_n)\big) \leq F\big(ad(x_{n-1}, x_n) + bd(x_{n-1}, Tx_{n-1}) + cd(x_n, Tx_n)\big).$$

Since

$$D(x_n, Tx_n) \leq H(Tx_{n-1}, Tx_n) < hH(Tx_{n-1}, Tx_n),$$

we have by (F1)

$$F\big(D(x_n, Tx_n)\big) \leq F\big(hH(Tx_{n-1}, Tx_n)\big) \leq F\big(H(Tx_{n-1}, Tx_n)\big) + \tau. \tag{12}$$

Since F is right continuous, we can write by Remark 1,

$$F\big(D(x_n, Tx_n)\big) = \inf_{y \in Tx_n} F\big(d(x_n, y)\big).$$

We also have by (12),

$$\inf_{y \in Tx_n} F\big(d(x_n, y)\big) \leq F\big(H(Tx_{n-1}, Tx_n)\big) + \tau. \tag{13}$$

By (13), there exists $x_{n+1} \in Tx_n$ such that

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(hH(Tx_{n-1}, Tx_n)\big) \leq F\big(H(Tx_{n-1}, Tx_n)\big) + \tau.$$

Thus

$$\begin{aligned}
2\tau + F\big(d(x_n, x_{n+1})\big) &\leq 2\tau + F\big(H(Tx_{n-1}, Tx_n)\big) + \tau \\
&\leq F\big(ad(x_{n-1}, x_n) + bd(x_{n-1}, Tx_{n-1}) + cd(x_n, Tx_n)\big) + \tau \\
&\leq F\big(ad(x_{n-1}, x_n) + bd(x_{n-1}, x_n) + cd(x_n, x_{n+1})\big) + \tau \\
&\leq F\big((a + b)d(x_{n-1}, x_n) + cd(x_n, x_{n+1})\big) + \tau.
\end{aligned}$$

By (F1), letting $\lambda = \frac{a+b}{1-c}$, we write

$$d(x_n, x_{n+1}) < \tfrac{a+b}{1-c}\, d(x_{n-1}, x_n) = \lambda d(x_{n-1}, x_n),$$

which yields

$$d(x_n, x_{n+1}) < \lambda d(x_{n-1}, x_n) < \lambda^2 d(x_{n-2}, x_{n-1}) < \ldots < \lambda^n d(x_0, x_1), n \in \mathbb{N}.$$

Observe that

$$\sum_{k=n}^{m-1} d(x_k, x_{k+1}) < \lambda^n (1 + \lambda + \lambda^2 + \ldots + \lambda^{m-n-1}) d(x_0, x_1)$$
$$\leq \frac{\lambda^n}{1-\lambda}\, d(x_0, x_1),\ m > n.$$

Since $\lim_{n\to\infty} \frac{\lambda^n}{1-\lambda}\, d(x_0, x_1) = 0$, for any $\delta > 0$ there exists some $n' \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda}\, d(x_0, x_1) < \delta,\ n \geq n'. \tag{14}$$

Furthermore, assume $(f, \alpha) \in \mathcal{B} \times [0, \infty)$ satisfies (d3) and let $\epsilon > 0$ be fixed. By (F2), there is $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \alpha. \tag{15}$$

By (14) and (15), we write

$$f\left(\sum_{k=n}^{m-1} d(x_k, x_{k+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda}\, d(x_0, x_1)\right) < f(\epsilon) - \alpha,\ m > n \geq n'.$$

Using (d3) and the above inequality, we obtain

$$d(x_n, x_m) > 0,\ m > n > n' \Longrightarrow f(d(x_n, x_m)) < f(\epsilon)$$

which shows

$$d(x_n, x_m) < \epsilon,\ m > n \geq n'.$$

This shows that the sequence (x_n) is $\mathcal{F}$-Cauchy in X. Since (X, d) is $\mathcal{F}$-complete, there is some $z^* \in X$ such that (x_n) is $\mathcal{F}$-convergent to z^*, i.e.,$\lim_{n\to\infty} d(x_n, z^*) = 0$. We claim that $z^* \in Tz^*$. If not, i.e., $d(z^*, Tz^*) > 0$, then we can write

$$\begin{aligned} 2\tau + F\big(d(x_n, Tz^*)\big) &\leq 2\tau + F\big(H(Tx_{n-1}, Tz^*)\big) \\ &\leq F\big(ad(x_{n-1}, z^*) + bd(x_{n-1}, Tx_{n-1}) + cd(z^*, Tz^*)\big) \\ &\leq F\big(ad(x_{n-1}, z^*) + bd(x_{n-1}, x_n) + cd(z^*, Tz^*)\big). \end{aligned}$$

Using (F1) and letting $n \to \infty$, we get

$$(1 - c)d(z^*, Tz^*) < 0,$$

which is a contradiction, hence $d(z^*, Tz^*) = 0$, i.e., z^* is the fixed point of T. □

Corollary 5. *Suppose $(f, \alpha) \in \mathcal{B} \times [0, \infty)$ and (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $T : X \to CB(X)$ be a Reich-type F-contraction such that F is right continuous. Suppose that for $k \in [0, 1)$, there exist $F \in \mathcal{B}$ and $\tau > 0$ such that*

$$\tau + F\big(d(Tx, Ty)\big) \leq F\left(\frac{k}{2}\,(d(x, Tx) + d(y, Ty))\right)$$

with $\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} > 0$, *for all $(x, y) \in X \times X$. Then T has a fixed point in X.*

6. Applications

In this part, we use the obtained results, to show the existence of a unique common solution of functional equations in dynamic programming.

The problems of dynamic programming comprise two main parts. One is the state space, which is a collection of parameters representing different states, including transitional states, initial states, and action states. The other is the decision space that is the series of decisions taken to solve the problems. This setting formulates the problems of computer programming and mathematical optimization. In particular, the problems of dynamic programming are transformed into the problems of functional equations:

$$p(a) = \sup_{b\in B} \left\{ G(a,b) + g_1\big(a,b,p(\eta(a,b))\big) \right\} \text{ for } a \in A, \tag{16}$$

$$q(a) = \sup_{b\in B} \left\{ G(a,b) + g_2\big(a,b,p(\eta(a,b))\big) \right\} \text{ for } a \in A, \tag{17}$$

where U and V are Banach spaces such that $A \subseteq U$ and $B \subseteq V$ and

$$\eta : A \times B \to A,$$

$$G : A \times B \to \mathbb{R},$$

$$g_1, g_2 : A \times B \times \mathbb{R} \to \mathbb{R}.$$

Assume that the decision space and state space are A and B respectively. Our aim is to show that the Equations (16) and (17) have at most one common solution. Suppose $W(A)$ represents the collection of all real-valued bounded mappings on A. Suppose that h is an arbitrary element of $W(A)$ and define $\|h\| = \sup_{a\in A} |h(a)|$. Then $(W(A), ||\cdot||)$ is a Banach space along with the metric d given by

$$d(h,k) = \sup_{a\in A} |h(a) - k(a)|. \tag{18}$$

Assume the following conditions hold true:

(C1) G, g_1 and g_2 are bounded.
(C2) For $a \in A$ and $h \in W(A)$, define $S, T : W(A) \to W(A)$ by

$$Sh(a) = \sup_{b\in B}\{G(a,b) + g_1\big(a,b,h(\eta(a,b))\big)\} \text{ for } a \in A, \tag{19}$$

$$Th(a) = \sup_{b\in B}\{G(a,b) + g_2\big(a,b,h(\eta(a,b))\big)\} \text{ for } a \in A. \tag{20}$$

Clearly, if the functions G, g_1 and g_2 are bounded then S and T are well-defined.
(C3) For $\tau : \mathbb{R}_+ \to \mathbb{R}_+$, $(a,b) \in A \times B, h,k \in W(A)$ and $t \in A$, we have

$$|g_1\big(a,b,h(t)\big) - g_2\big(a,b,k(t)\big)| \leq e^{-\tau} M(h,k), \tag{21}$$

where

$$M(h,k) = \alpha d(h,k) + \beta d(h, Sh) + \gamma d(k, Tk)$$

for $\alpha, \beta, \gamma \in [0,\infty)$ such that $\alpha + \beta + \gamma < 1$, where $\min\{d(Sh,Tk), M(h,k)\} > 0$.

Now we prove the following theorem.

Theorem 6. *Suppose the conditions (C1)-(C3) hold, then the Equations* (16) *and* (17) *have at most one common bounded solution.*

Proof. By [6], we know that $(W(A), d)$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space, where d is given by (18). By (C1), S and T are self-mappings on $W(A)$. Suppose λ is an arbitrary positive number and $h_1, h_2 \in W(A)$. Take $a \in A$ and $b_1, b_2 \in B$ such that

$$Sh_j < G(a,b_j) + g_1\big(a,b_j,h_j(\eta(a,b_j))\big) + \lambda \tag{22}$$

$$Th_j < G(a,b_j) + g_2\big(a,b_j,h_j(\eta(a,b_j))\big) + \lambda \tag{23}$$

and

$$Sh_1 \geq G(a, b_2) + g_1\big(a, b_2, h_1(\eta(a, b_2))\big) \tag{24}$$

$$Th_2 \geq G(a, b_1) + g_2\big(a, b_1, h_2(\eta(a, b_1))\big). \tag{25}$$

Then using (22) and (25), we get

$$\begin{aligned} Sh_1(a) - Th_2(a) &< g_1\big(a, b_1, h_1(\eta(a, b_1))\big) - g_2\big(a, b_1, h_2(\eta(a, b_1))\big) + \lambda \\ &\leq |g_1\big(a, b_1, h_1(\eta(a, b_1))\big) - g_2\big(a, b_1, h_2(\eta(a, b_1))\big)| + \lambda \\ &\leq e^{-\tau} M\big(h_1(a), h_2(a)\big) + \lambda. \end{aligned}$$

Similarly, by (23) and (24), we get

$$Th_2(a) - Sh_1(a) < e^{-\tau} M\big(h_1(a), h_2(a)\big) + \lambda.$$

Combining the above two inequalities, we get

$$|Sh_1(a) - Th_2(a)| < e^{-\tau} M\big(h_1(a), h_2(a)\big) + \lambda$$

for all $\lambda > 0$. Hence

$$d(Sh_1(a), Th_2(a)) \leq e^{-\tau} M(h_1(a), h_2(a))$$

that is,

$$d(Sh_1, Th_2) \leq e^{-\tau} M(h_1, h_2)$$

for each $a \in A$. Taking logarithms, we have

$$\ln d(Sh_1, Th_2) \leq \ln e^{-\tau} M(h_1, h_2)$$

This implies that the mapping $F : \mathbb{R}_+ \to \mathbb{R}$ defined by $F(a) = \ln a$ is an element of $\mathcal{B}$, and

$$\tau + F(d(Sh_1, Th_2)) \leq F(M(h_1, h_2)).$$

Now it is clear that all the conditions of Theorem 3 are fulfilled, so by applying Theorem 3, S and T have at most one common and bounded solution of the Equations (16) and (17). $\square$

7. Conclusions

Our results extend and generalize the existence of fixed points of Reich-type F-contractions in $\mathcal{F}$-metric spaces and $\mathcal{F}$-closed balls with relaxed conditions. We get fixed-point results of set-valued Reich-type F-contractions. Finally, we applied the results to obtain a fixed-point theorem in dynamic programming.

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