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An Interactive Data-Driven (Dynamic) Multiple Attribute Decision Making Model via Interval Type-2 Fuzzy Functions

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Abstract: A new multiple attribute decision making (MADM) model was proposed in this paper in order to cope with the temporal performance of alternatives during different time periods. Although dynamic MADM problems are enjoying a more visible position in the literature, majority of the applications deal with combining past and present data by means of aggregation operators. There is a research gap in developing data-driven methodologies to capture the patterns and trends in the historical data. In parallel with the fact that style of decision making evolving from intuition-based to data-driven, the present study proposes a new interval type-2 fuzzy (IT2F) functions model in order to predict current performance of alternatives based on the historical decision matrices. As the availability of accurate historical data with desired quality cannot always be obtained and the data usually involves imprecision and uncertainty, predictions regarding the performance of alternatives are modeled as IT2F sets. These estimated outputs are transformed into interpretable forms by utilizing the vocabulary matching procedures. Then the interactive procedures are employed to allow decision makers to modify the predicted decision matrix based on their perceptions and subjective judgments. Finally, ranking of alternatives are performed based on past and current performance scores.

Keywords: dynamic multiple attribute decision making; fuzzy regression; interval type-2 fuzzy sets

1. Introduction

Managers are continuously engaged in a process of making decisions in a rapidly changing business environment. Making right decisions is crucial in order to attain organizational goals and effective use of resources. Quality of decisions relies heavily on the information processing capabilities by considering multiple and conflicting criteria. With the dramatic increase in the availability of information obtained from diverse set of resources, decision making becomes much more complicated and difficult. Multiple attribute decision making (MADM) offers a set of sophisticated techniques to help decision makers in selecting the best alternative by considering multiple, conflicting, and incommensurate criteria.

The field of MADM is rapidly expanding with the continuing proliferation of new techniques and applications. Many state of the art methods have been proposed such as multi-attribute utility theory (MAUT) [1–3], analytic hierarchy process (AHP) [4], analytic network process (ANP) [5], technique for order preference by similarity to ideal solution (TOPSIS) [6], elimination and choice translating reality (ELECTRE) [7], VlseKriterijumska Optimizacija I Kompromisno Resenje technique (VIKOR) [8], and decision-making trial and evaluation laboratory (DEMATEL) [9]. Despite many successful applications of the MADM methods currently available in the literature, the salient deficiency of these methods is their incapability to handle temporal profiles of alternatives. Unfortunately, static

MADM methods cannot deal with the temporal profiles of alternatives, that is, past performance scores are not taken into consideration. In order to overcome this deficiency, data-driven, and dynamic MADM methods have been developing along with diverse applications.

In the dynamic MADM, at least two-period decision making information is considered. In addition to the alternative and criteria dimensions, time is considered as a third dimension in the dynamic MADM. With the recent advances in the information technologies, data is becoming an indispensable part of the decision making practices, which forces the pace of a paradigm shift towards data-driven decision making. As ever-more data pour through the networks of organizations, collecting, and storing performance scores of alternatives with time stamps are not cumbersome procedures anymore. As the style of decision making evolving from intuition based to data-driven, the decision makers should be supported with relevant methodologies and tools to fully capitalize the available data. However, it is evident that the field of dynamic MADM is in its infancy and the current literature is far from meeting the requirements of a fully-fledged data-driven methodology.

In one of the earliest works on the dynamic MADM, Kornbluth [10] discussed the problem of time dependence of the criteria weights, and empirical laboratory findings were used for the analysis. Decision making teams were monitored for 12 sequential decisions and the time-dependent weights were analyzed based on different scenarios. Dong et al. [11] proposed a dynamic MADM method based on relative differences between the performance scores of the subsequent time-periods. In the study, disadvantages of using absolute differences were discussed and a numerical example was provided. Lou et al. [12] proposed a dynamic MADM model to evaluate and rank country risks based on historical data. The proposed model was aimed at predicting possible credit crises in advance. The proposed model was applied to the world economy development indicators data of 32 countries. The utilités additives discriminantes (UTADIS) method was used to rank country risk scores.

Despite many new developments, the literature of the dynamic MADM field is dominated by the aggregation-operator based models. Campanella and Ribeiro [13] proposed a framework for dynamic MADM where the aggregation operators were the main computation tools, and majority of the studies in the literature employ this framework. Xu and Yager [14] proposed dynamic intuitionistic fuzzy weighted averaging and uncertain dynamic intuitionistic fuzzy weighted averaging operators. According to their model, decision matrices of the past periods are aggregated into a decision matrix, and classical MADM techniques were implemented afterwards. Park et al. [15] proposed dynamic intuitionistic fuzzy weighted geometric and uncertain dynamic intuitionistic fuzzy weighted geometric operators for dynamic MADM problems. The past decision matrices were aggregated and then the VIKOR method was used to rank the alternatives. Zhou et al. [16] hybridized dynamic triangular fuzzy weighting average operators with fuzzy VIKOR method for quality improvement pilot program selection. Dynamic feedbacks of the customers were also incorporated into the proposed model. Bali et al. [17] employed dynamic intuitionistic fuzzy weighted averaging method with TOPSIS for multi-period third-party logistics provider selection problem. Chen and Li [18] proposed a new distance measure for triangular intuitionistic fuzzy sets with an application in dynamic MADM. The weighted arithmetic averaging operator for triangular intuitionistic fuzzy numbers was used to aggregate the decision matrices of the past periods. The ranking orders were obtained by using closeness coefficients. An investment decision making was used to illustrate the proposed method. Liang et al. [19] employed evidential reasoning approach to aggregate decision matrices with incomplete information. The enterprise evaluation in a technological zone was used to illustrate the model. Bali et al. [20] proposed an integrated model based on AHP and dynamic intuitionistic fuzzy weighted averaging operator under an intuitionistic fuzzy environment. The proposed model was implemented in a personnel promotion problem.

In some of the studies, the aggregation operators were not used to aggregate the decision matrices of the past periods at the very beginning. Xu [21] proposed dynamic weighted geometric aggregation operator along with an illustrative three-period investment decision making model. Rather than aggregating the decision matrices of the past periods, aggregation was conducted based on the closeness coefficients of the different periods. Zulueta et al. [22] proposed discrete time variable index by admitting bipolar values in the aggregation. A five-period supplier selection problem was used to illustrate the proposed aggregation operator. Lin et al. [23] used grey numbers and Minkowski distance in dealing with dynamic subcontractor selection problem. The proposed method calculated the period weighted distances to the ideal and anti-ideal solutions. Similar aggregation operator-based studies in the literature, which utilize intuitionistic fuzzy numbers [24,25], 2-tuple linguistic representation [26–30], grey numbers [31,32], etc. can be found. For more information about dynamic aggregation operators, we refer to a review paper [33]. On the other hand, non-aggregation operator based studies can be summarized as follows: Saaty [34] studied time-dependent eigenvectors and approximating functional forms of relative priorities in dynamic MADM. Hashemkhani Zolfani et al. [35] emphasized the relevance and necessity of future studies in MADM problems. In the paper, scenario-based MADM papers were reviewed and analyzed. Possible changes in the experts' evaluations were expressed by using probabilities. Orji and Wei [36] integrated fuzzy logic and system dynamics simulation to sustainable supplier selection problem. Very recently, Baykasoğlu and Gölcük [37] proposed a dynamic MADM model by learning of fuzzy cognitive maps. In the study, fuzzy cognitive maps were trained by using a metaheuristic algorithm in order to capture patterns and trends in the past data. Then, the trained model was used to generate short-, medium-, and long-term future decision matrices. Finally, past, current and future decision matrices were used to rank the alternatives. The proposed model was realized in a real-life supplier selection problem.

Although a wide range of applications have been provided in the context of dynamic MADM, the literature still lacks the following considerations:

- Decision makers are expected to fill out tedious questionnaires to articulate their preferences over alternatives at each period. This is especially very time consuming and demanding when the number of criteria and alternatives are high, and the decision points are frequent, i.e., performance evaluation, risk assessment, etc.
- The models do not provide any mechanism to help decision makers making use of past decision making matrices when articulating their preferences at the current period. An interactive mechanism is needed to facilitate preference elicitation in the light of historical performance of alternatives.

Due to the availability of accurate historical data with high quality and quantity cannot always be assured and the data is usually affected by imprecision and noise, the predictions regarding the performance of alternatives should handle uncertainty properly. Interval type-2 fuzzy (IT2F) sets are very suitable tools for manipulating and reasoning with uncertain information. For that reason, the present study makes use of IT2F regression [38] to predict the current decision matrix. In order to enhance the prediction capability of the IT2F regression, a new hybrid IT2F model is proposed on the basis of highly practical method of so called "fuzzy functions" [39]. The proposed model is able to capture nonlinearities more successfully than the traditional fuzzy regression models, due to its unique and intelligent way of integrating membership grades of data points into the prediction problem.

The proposed dynamic MADM model contributes to the literature with its following features:

- A dynamic MADM model is proposed based on a new IT2F functions approach.
- An interactive procedure is provided that the current decision making matrix is predicted in forms of IT2F sets. Moreover, vocabulary matching procedure is developed so that the predicted performance scores of alternatives are recommended to the decision makers through linguistic terms such as low, medium, high, etc.
- The proposed model interacts with decision makers whose subjective judgments are combined with the notion of "let the data speak for itself". By providing decision makers with data-driven suggestions regarding the performance of alternatives, preference elicitation effort at each period is considerably reduced.

- The proposed model does not require any technical knowledge such as fuzzy sets, t-norms, t-conorms, implication functions, etc. The proposed model can be easily integrated into the legacy systems of the firms, since the crisp values are processed when providing IT2F outputs.
- A real-life personnel promotion problem is used to demonstrate the applicability of the proposed model. Rankings of employees are calculated based on past and current performance matrices with appropriate time series weights.

This paper is organized as follows: Theoretical background on the methodologies used within the scope of this paper is given in Section 2. The proposed model is provided in Section 3. The real life application of the proposed dynamic MADM model is illustrated in Section 4. Discussions are given in Section 5. Concluding remarks are given in Section 6.

2. Theoretical Background

2.1. Traditional Dynamic Multiple Attribute Decision Making

A dynamic MADM problem under study can be described as follows. Let $A = \{A_1, A_2, \ldots, A_M\}$ be a discrete set of M feasible alternatives and $C = \{C_1, C_2, \ldots, C_N\}$ be a finite set of all attributes. The set of all periods is denoted by $t = \{t_1, t_2, \ldots, t_H\}$. Each alternative is evaluated in terms of N attributes and H periods. Each period is associated with weights that these time series weights are denoted by $\xi(t) = [\xi(t_1), \xi(t_2), \ldots, \xi(t_H)]^T$ where $\xi(t_H) \geq 0$ and $\sum_{k=1}^{H} \xi(t_k) = 1$. The weight vector of attributes are given by $[w_1(t_k), w_2(t_k), \ldots w_N(t_k)]^T$ where $w_i(t_k) \geq 0$ and $\sum_{i=1}^{N} w_i(t_k) = 1$. The decision matrix at the period t_k is denoted by $\boldsymbol{A}(t_k) = \left(a_{ij}(t_k)\right)_{N\times M}$ where $a_{ij}(t_k)$ is the value of alternative A_j with respect to attribute C_i at period t_k. Let Ω_b and Ω_c be the set of benefit and cost attributes, respectively. Due to the immensurability of the different attributes, decision matrix $\boldsymbol{A}(t_k)$ is normalized to corresponding dimensionless decision matrix $\boldsymbol{R}(t_k) = \left(r_{ij}(t_k)\right)_{N\times M}$ by using the following formulas:

$$r_{ij}(t_k) = \frac{a_{ij}(t_k)}{\max_j\left\{a_{ij}(t_k)\right\}},\ j = 1, 2, \ldots, M;\ k = 1, 2, \ldots, H;\ i \in \Omega_b, \tag{1}$$

$$r_{ij}(t_k) = \frac{\min_j\left\{a_{ij}(t_k)\right\}}{a_{ij}(t_k)},\ j = 1, 2, \ldots, M;\ k = 1, 2, \ldots, H;\ i \in \Omega_c, \tag{2}$$

Hence, the normalized decision matrix is obtained as:

$$\boldsymbol{R}(t_k) = \begin{bmatrix} r_{11}(t_k) & r_{12}(t_k) & \cdots & r_{1M}(t_k) \\ r_{21}(t_k) & r_{22}(t_k) & \cdots & r_{2M}(t_k) \\ \vdots & \vdots & \ddots & \vdots \\ r_{N1}(t_k) & r_{N2}(t_k) & \cdots & r_{NM}(t_k) \end{bmatrix}. \tag{3}$$

The overall assessment value of the jth alternative is calculated by:

$$y_j = \sum_{k=1}^{H}\sum_{i=1}^{N} \xi(t_k) w_i(t_k) r_{ij}(t_k),\ j = 1, 2, \ldots, M. \tag{4}$$

Therefore, alternatives are ranked based on y_j in which the best alternative is with the highest overall assessment value.

2.2. Possibilistic Fuzzy Regression

In this section, possibilistic fuzzy regression analysis with asymmetric fuzzy numbers is overviewed based on [38]. Although there are a variety of fuzzy regression approaches developed during the last two decades, regression models relying on possibility and necessity concepts have pivotal role in the current literature. Possibilistic models strive to minimize sum of spreads in such a way that the estimated outputs must include the given targets. It is indeed advantageous to have asymmetric fuzzy numbers in possibilistic models as the lower and upper bounds of the estimated model are not necessarily equidistant from the center, which implies superior capability to capture central tendency. A fuzzy regression model can be formalized as:

$$Y(\boldsymbol{x}) = \tilde{\beta}_0 + \tilde{\beta}_1 x_1 + \cdots + \tilde{\beta}_{nv} x_{nv} = \tilde{\boldsymbol{\beta}}\boldsymbol{x}, \tag{5}$$

where the input vector is represented by $\boldsymbol{x} = (1, x_1, \ldots, x_{nv})^t$ and $\tilde{\boldsymbol{\beta}} = \left(\tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_{nv}\right)$ is a vector of fuzzy coefficients, and $Y(\boldsymbol{x})$ is the estimated fuzzy output. The coefficients $\tilde{\beta}_i$ are denoted as $\tilde{\beta}_i = (a_i, c_i, d_i)$ can be defined by:

$$\mu_{A_i}(x) = \begin{cases} 1 - (a_i - x)/c_i, & \text{if } a_i - c_i \le x \le a_i \\ 1 - (x - a_i)/d_i, & \text{if } a_i \le x \le a_i + d_i, \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where a_i represents center, and c_i and d_i left- and right-spreads, respectively.

Given the input-output data as $\left(\boldsymbol{x}_j, y_j\right) = \left(1, x_{j,1}, \ldots, x_{j,nv}; y_j\right)$, $j = 1, 2, \ldots, nd$, where $x_{j,nv}$ being value of the variable nv of the j-th data-point among the total of nd data-points, the estimated output $Y(\boldsymbol{x}_j)$ can be calculated by using fuzzy arithmetic. Representing regression coefficients as $\tilde{\beta}_i = (a_i, c_i, d_i)$, $(i = 0, 1, \ldots, nv)$, fuzzy regression model can be expressed as:

$$Y(\boldsymbol{x}_j) = (a_0, c_0, d_0) + (a_1, c_1, d_1)x_{j,1} + \cdots + (a_n, c_n, d_n)x_{j,nv}. \tag{7}$$

Equation (7) can be written in a more compact form as given in Equation (8).

$$Y(\boldsymbol{x}_j) = \left(\theta_C\left(\boldsymbol{x}_j\right), \theta_L\left(\boldsymbol{x}_j\right), \theta_R\left(\boldsymbol{x}_j\right)\right), \tag{8}$$

where the terms $\theta_C\left(\boldsymbol{x}_j\right), \theta_L\left(\boldsymbol{x}_j\right), \theta_R\left(\boldsymbol{x}_j\right)$, are calculated as given in Equation (9).

$$\begin{aligned} \theta_C\left(\boldsymbol{x}_j\right) &= \sum_{i=0}^{n} a_i x_{ji} \\ \theta_L\left(\boldsymbol{x}_j\right) &= \sum_{x_{ji} \ge 0}^{n} c_i x_{ji} - \sum_{x_{ji} < 0}^{n} d_i x_{ji} \\ \theta_R\left(\boldsymbol{x}_j\right) &= \sum_{x_{ji} \ge 0}^{n} d_i x_{ji} - \sum_{x_{ji} < 0}^{n} c_i x_{ji} \end{aligned} . \tag{9}$$

Finally, the possibilistic fuzzy regression model can be written as:

$$\begin{aligned} J = \sum_{j=1}^{nd} \left(y_j - \boldsymbol{a}^t \boldsymbol{x}_j\right)^2 + (1-h) \sum_{j=1}^{nd} \left(\boldsymbol{c}^t \left|\boldsymbol{x}_j\right| + \boldsymbol{d}^t \left|\boldsymbol{x}_j\right|\right) + \xi\left(\boldsymbol{c}^t \boldsymbol{c} + \boldsymbol{d}^t \boldsymbol{d}\right) \\ \text{subject to} \quad \theta_C(\boldsymbol{x}_j) + (1-h)\theta_R(\boldsymbol{x}_j) \ge y_j \\ \theta_C(\boldsymbol{x}_j) - (1-h)\theta_L(\boldsymbol{x}_j) \le y_j \\ c_i \ge 0,\ d_i \ge 0,\ i = 0, 1, \ldots, nv \end{aligned} , \tag{10}$$

where ξ is a small positive number. The term $\xi\left(\boldsymbol{c}^t \boldsymbol{c} + \boldsymbol{d}^t \boldsymbol{d}\right)$ is added to objective function so that the objective function becomes a quadratic function with respect to decision variables $\boldsymbol{a}$, $\boldsymbol{c}$, and $\boldsymbol{d}$. The resulting optimization problem is a quadratic optimization problem, which involves minimizing a quadratic objective function subject to linear constrains. The possibilistic fuzzy regression analysis will be detailed in the subsequent sections.

2.3. Turksen's Fuzzy Functions Approach

Turksen [39] proposed a new fuzzy modeling technique as an alternative to classical fuzzy rule bases (FRB). Fuzzy rule bases (FRB) have been effectively used as a facilitator to decision makers' problem solving activities. FRBs are the one of the best currently available means to codify human knowledge. This knowledge is represented by "IF ... THEN" rule structures. The "IF" part represents the antecedents and "THEN" part represents consequents. In the literature, there are different FRB system modeling strategies with unique antecedent and consequent parameter formation approaches. In these systems, membership values have also different interpretations such as "degree of fire", "degree of compatibility", "degree of belongingness", or "weight or strength of local functions". The fuzzy functions approach adds a new means of membership degrees to the list by exploiting the predictor power of membership grades [40].

The representation of each unique rule of an FRB system by means of fuzzy functions is the governing idea of the fuzzy functions approach. In the fuzzy functions approach, the membership degree of each sample vector directly affects the local fuzzy functions. One of the advantages of the fuzzy functions approach is that even non-experts can build fuzzy models as there are lower steps and parameters. It is quite practical to identify and reason with the fuzzy functions approach that some technical information regarding constructing fuzzy system models is not required such as fuzzification, t-norms and t-conorms, modus ponens, etc.

A vast array of fuzzy modeling approaches has been developed in the literature where the expert knowledge is encoded to define linguistic variables characterized by fuzzy sets. However, majority of the approaches suffers from a major drawback of being subjective and not generalizable. In order to reduce expert intervention into fuzzy system modeling, more objective methods have been developed [41–45]. In these systems, membership grades are not defined by decision makers, on the contrary, they are extracted from the dataset. There are also some approaches where some sophisticated techniques are integrated into fuzzy models so that the hybrid fuzzy system models are built. The prominent examples of these methods are neuro-fuzzy systems [46] and genetic-fuzzy systems [47].

There are still enduring challenges in the mentioned fuzzy system modeling approaches. Main disadvantages of the classical fuzzy system modeling approaches can be listed as follows:

- Membership functions pertaining to antecedent and consequent parts of the fuzzy rules should be identified.
- Aggregation of antecedents requires selection of suitable conjunction and disjunction operators (t-norms, t-conorms).
- Proper implication operators should be identified for representation of the rules, which can be a challenging issue.
- A suitable defuzzification method should be identified.

The fuzzy functions approach mainly reduces the number of fuzzy operators by taking advantage of data-driven modeling. For instance, fuzzy operators in determination of the membership functions in antecedents and consequents, fuzzification, aggregation of antecedents, implication, and in aggregation of consequents. It can be said that the fuzzy functions are more practical than their counterpart FRB models. Fuzzy function can be simply described as follows:

The training dataset is partitioned into c overlapping clusters where each cluster center is represented by v_i, $i = 1, 2, \ldots, c$.

For each one of the clusters, a local fuzzy model $f_i : v_i \to \Re$ is built and one output is produced for each cluster. Here, memberships and their several transformations are added into the input space and the augmented input matrix is generated. Membership grades and their transformations are considered as new variables in the regression matrix. Practically, least square estimation is used to derive regression coefficients. Then, degree of belongingness of each given input vector is used to aggregate the local model outputs and the estimated values are produced.

General steps of the fuzzy functions approach can be given as:

Step 1: Matrix $\boldsymbol{Z}$ comprises of inputs and output of the system. Inputs and output of the system are clustered by using the fuzzy c-means clustering algorithm. Fuzzy c-means clustering method can be applied by using the formulas given as:

$$v_i = \frac{\sum\limits_{j=1}^{nd} \mu_{ij}^m \boldsymbol{z}_j}{\sum\limits_{j=1}^{nd} \mu_{ij}^m},\ i = 1,2,\ldots,c, \tag{11}$$

$$\mu_{ij} = \frac{1}{\sum\limits_{h=1}^{c} \left(\frac{\|v_i - z_j\|}{\|v_h - z_j\|} \right)^{\frac{2}{m-1}}},\ i = 1,2,\ldots,c;\ j = 1,2,\ldots,nd, \tag{12}$$

where $\|.\|$ represents the Euclidean distance between data point $\boldsymbol{z}_j$ to cluster center $\boldsymbol{v}_i$.

Step 2: In the second step, membership values of the input space are calculated. Here, the cluster centers identified in the previous step are used to calculate membership grades of the input data. Membership construction from the identified cluster centers is performed based on Equation (13).

$$\mu_{ij} = \frac{1}{\sum\limits_{h=1}^{c} \left(\frac{\|v_i - x_j\|}{\|v_h - x_j\|} \right)^{\frac{2}{m-1}}},\ i = 1,2,\ldots,c;\ j = 1,2,\ldots,nd. \tag{13}$$

Step 3: For each cluster i, membership values of each input data sample, μ_{ij} and original inputs are gathered together, and i-th local fuzzy function is obtained by predicting $\boldsymbol{Y}^{(i)} = \boldsymbol{X}^{(i)}\boldsymbol{\beta}^{(i)} + \boldsymbol{\varepsilon}^{(i)}$ based on least squares estimation. When the number of inputs is nv, $\boldsymbol{X}^{(i)}$, and $\boldsymbol{Y}^{(i)}$ matrices are as follows:

$$\boldsymbol{X}^{(i)} = \begin{bmatrix} \mu_{i,1} & x_{1,1} & \cdots & x_{nv,1} \\ \mu_{i,2} & x_{1,2} & \cdots & x_{nv,2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{i,nd} & x_{1,nd} & \cdots & x_{nv,nd} \end{bmatrix},\ \boldsymbol{Y}^{(i)} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{nd} \end{bmatrix}. \tag{14}$$

Step 4: Output values are calculated by aggregating the results of the local fuzzy functions as follows:

$$\hat{\boldsymbol{y}}_j = \frac{\sum_{i=1}^{c} \hat{y}_{ij}\mu_{ij}}{\sum_{i=1}^{c} \mu_{ij}},\ j = 1,2,\ldots,nd. \tag{15}$$

3. Developed IT2F Model

In this section, the developed IT2F model for dynamic MADM problems is given. First, the basics of the IT2F sets and necessary equations are overviewed. Afterwards, the IT2F regression model is revisited based on [38]. Then, the procedural steps of the proposed dynamic MADM model are given.

3.1. Interval Type-2 Fuzzy Sets

Definition 1 ([48,49]). *A type-2 fuzzy set $\tilde{\tilde{A}}$ in the universe of discourse X can be represented by a type-2 membership function $\mu_{\tilde{\tilde{A}}}$* as:

$$\tilde{\tilde{A}} = \left\{ ((x,u),\ \mu_{\tilde{\tilde{A}}}(x,u)) | \forall x \in X,\ \forall u \in J_x \subseteq [0,1], 0 \le \mu_{\tilde{\tilde{A}}}(x,u) \le 1 \right\}, \tag{16}$$

where J_x denotes an interval in [0,1]. Moreover, type-2 fuzzy set $\tilde{\tilde{A}}$ can also be represented as:

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{\tilde{A}}}(x,u)/(x,u), \tag{17}$$

where $J_x \subseteq [0,1]$ and $\int\int$ denotes union over all admissible x and u.

Definition 2 ([48,49])**.** *Let $\tilde{\tilde{A}}$ be a type-2 fuzzy set in the universe of discourse X represented by the type-2 membership function $\mu_{\tilde{\tilde{A}}}$. If all $\mu_{\tilde{\tilde{A}}}(x,u) = 1$, then $\tilde{\tilde{A}}$ is called an IT2F set. An IT2F set $\tilde{\tilde{A}}$, which can be regarded as a special case of a type-2 fuzzy set, is represented as follows:*

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} 1/(x,u), \tag{18}$$

where $J_x \subseteq [0,1]$.

The footprint of uncertainty (FOU) is represented by the lower and upper membership functions:

$$FOU(\tilde{\tilde{A}}) = \int_{x \in X} \left[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)\right], \tag{19}$$

where $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ represent lower and upper membership functions, respectively.

Definition 3 ([50])**.** *An IT2F set $\tilde{\tilde{A}}$ is said to be normal if $\sup\overline{\mu}_{\tilde{A}}(x) = 1$ and $\sup\underline{\mu}_{\tilde{A}}(x) = h < 1$, where h represents the height of the lower membership function. An IT2F set $\tilde{\tilde{A}}$ is said to be perfectly normal if $\sup\overline{\mu}_{\tilde{A}}(x) = \sup\underline{\mu}_{\tilde{A}}(x) = 1$.*

In this study, perfectly normal IT2F sets were employed so that the basic definitions and operational laws regarding perfectly normal triangular IT2F sets were overviewed.

Considering perfectly normal triangular IT2F numbers $\tilde{\tilde{A}} = \left(\overline{A}, \underline{A}\right) = \left((\overline{a}_1, \overline{a}_2, \overline{a}_3; 1)\left(\underline{a}_1, \underline{a}_2, \underline{a}_3; 1\right)\right)$ and $\tilde{\tilde{B}} = \left(\overline{b}, \underline{b}\right) = \left(\left(\overline{b}_1, \overline{b}_2, \overline{b}_3; 1\right)\left(\underline{b}_1, \underline{b}_2, \underline{b}_3; 1\right)\right)$, their operational laws are as follows [51]:

$$\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = \begin{pmatrix} \left(\overline{a}_1 + \overline{b}_1, \overline{a}_2 + \overline{b}_2, \overline{a}_3 + \overline{b}_3; 1\right), \\ \left(\underline{a}_1 + \underline{b}_1, \underline{a}_2 + \underline{b}_2, \underline{a}_3 + \underline{b}_3; 1\right) \end{pmatrix}, \tag{20}$$

$$\tilde{\tilde{A}} \ominus \tilde{\tilde{B}} = \begin{pmatrix} \left(\overline{a}_1 - \overline{b}_3, \overline{a}_2 - \overline{b}_2, \overline{a}_3 - \overline{b}_1; 1\right), \\ \left(\underline{a}_1 - \underline{b}_3, \underline{a}_2 - \underline{b}_2, \underline{a}_3 - \underline{b}_1; 1\right) \end{pmatrix}, \tag{21}$$

$$\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} = \begin{pmatrix} \left(\overline{a}_1 \times \overline{b}_1, \overline{a}_2 \times \overline{b}_2, \overline{a}_3 \times \overline{b}_3; 1\right), \\ \left(\underline{a}_1 \times \underline{b}_1, \underline{a}_2 \times \underline{b}_2, \underline{a}_3 \times \underline{b}_3; 1\right) \end{pmatrix}, \tag{22}$$

$$\tilde{\tilde{A}} \oslash \tilde{\tilde{B}} = \begin{pmatrix} \left(\overline{a}_1 / \overline{b}_3, \overline{a}_2 / \overline{b}_2, \overline{a}_3 / \overline{b}_1; 1\right), \\ \left(\underline{a}_1 / \underline{b}_3, \underline{a}_2 / \underline{b}_2, \underline{a}_3 / \underline{b}_1; 1\right) \end{pmatrix}, \tag{23}$$

$$k \times \tilde{\tilde{A}} = \begin{cases} \left((k \times \overline{a}_1, k \times \overline{a}_2, k \times \overline{a}_3; 1)\left(k \times \underline{a}_1, k \times \underline{a}_2, k \times \underline{a}_3; 1\right)\right), & k \geq 0 \\ \left((k \times \overline{a}_3, k \times \overline{a}_2, k \times \overline{a}_1; 1)\left(k \times \underline{a}_3, k \times \underline{a}_2, k \times \underline{a}_1; 1\right)\right), & k \leq 0 \end{cases}. \tag{24}$$

Definition 4. *The ranking value $Rank(\tilde{\tilde{A}})$ of a IT2F set $\tilde{\tilde{A}} = \left(\overline{A}, \underline{A}\right)$ can be defined via the concept of centroid as [52]:*

$$C^{L}_{\tilde{\tilde{A}}} = \min_{\xi \in [a,b]} \frac{\int_a^{\xi} x\overline{\mu}_A(x)dx + \int_{\xi}^{b} x\underline{\mu}_A(x)dx}{\int_a^{\xi} \overline{\mu}_A(x)dx + \int_{\xi}^{b} \underline{\mu}_A(x)dx}, \tag{25}$$

$$C^{R}_{\tilde{\tilde{A}}} = \max_{\xi \in [a,b]} \frac{\int_a^{\xi} x\underline{\mu}_A(x)dx + \int_{\xi}^{b} x\overline{\mu}_A(x)dx}{\int_a^{\xi} \underline{\mu}_A(x)dx + \int_{\xi}^{b} \overline{\mu}_A(x)dx}, \tag{26}$$

where $C^{L}_{\tilde{\tilde{A}}}$ and $C^{R}_{\tilde{\tilde{A}}}$ are the endpoints of the centroid. The ranking value of the IT2F set $\tilde{\tilde{A}}$ is calculated as:

$$Rank(\tilde{\tilde{A}}) = \frac{C^{L}_{\tilde{\tilde{A}}} + C^{R}_{\tilde{\tilde{A}}}}{2}, \tag{27}$$

where $Rank(\tilde{\tilde{A}})$ is the centroid-based ranking value of $\tilde{\tilde{A}}$.

Definition 5 ([53])**.** *The Jaccard similarity measure for fuzzy sets $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is defined by*

$$SM\left(\tilde{\tilde{A}}, \tilde{\tilde{B}}\right) = \frac{\int_X \min\left(\overline{\mu}_A(x), \overline{\mu}_B(x)\right)dx + \int_X \min\left(\underline{\mu}_A(x), \underline{\mu}_B(x)\right)dx}{\int_X \max\left(\overline{\mu}_A(x), \overline{\mu}_B(x)\right)dx + \int_X \max\left(\underline{\mu}_A(x), \underline{\mu}_B(x)\right)dx}, \tag{28}$$

where SM represent similarity degree of fuzzy sets with respect to Jaccard measure.

3.2. IT2F Regression Model

In this section, IT2F regression model is revisited based on [38]. IT2F regression model will be utilized within the fuzzy functions approach [39] in order to increase its performance in the next section. In this section, the necessary equations are derived for IT2F regression in a step-by-step approach. IT2F regression models can be constructed based on the concepts of possibility and necessity. Here, possibility and necessity concepts are used to build an upper approximation model (UAM) and lower approximation model (LAM), respectively [54]. Building an integrated model, UAM and LAM are used to form upper and lower membership functions of the IT2F coefficients, respectively.

In mathematical terms, LAM and UAM models can be written as:

$$\begin{array}{c} \text{LAM} : \tilde{Y}_*(\boldsymbol{x}_j) = \tilde{\beta}_{*0} + \tilde{\beta}_{*1}x_{j1} + \cdots + \tilde{\beta}_{*n}x_{j,nv} = \tilde{\boldsymbol{\beta}}_* \boldsymbol{x}_j,\ j = 1, \ldots, nd \\ \text{UAM} : \tilde{Y}^*(\boldsymbol{x}_j) = \tilde{\beta}^*_0 + \tilde{\beta}^*_1 x_{j1} + \cdots + \tilde{\beta}^*_n x_{j,nv} = \tilde{\boldsymbol{\beta}}^* \boldsymbol{x}_j,\ j = 1, \ldots, nd \end{array}, \tag{29}$$

where coefficients $\tilde{\beta}_{*j}$ and $\tilde{\beta}^*_j$ are non-symmetric triangular fuzzy numbers. The regression coefficients $\tilde{\beta}_{*j}$ and $\tilde{\beta}^*_j$ are shown in Figure 1.

As shown in Figure 1, $\tilde{\beta}_{*i}$ and $\tilde{\beta}^*_i$ can be defined as:

$$\begin{array}{l} \tilde{\beta}_{*i} = (b_i - f_i, b_i, b_i + g_i; 1) \\ \tilde{\beta}^*_i = (b_i - f_i - p_i, b_i, b_i + g_i + q_i; 1) \end{array}, \tag{30}$$

where the condition $\tilde{\beta}^*_i \supseteq \tilde{\beta}_{*i}$, is satisfied for $i = 0, \ldots, nv$.

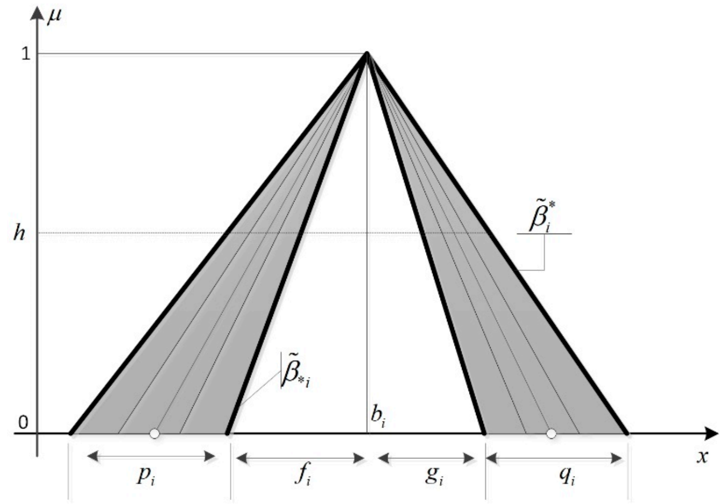

Figure 1. Representation of regression coefficients.

In order to increase readability of the formulations, conventional representation of the literature is adopted here, where (center, left_spread, right_spread) is used to show the LAM and UAM as given by $\tilde{\beta}_{*i} = (b_i, f_i, g_i)$ and $\tilde{\beta}_i^* = (b_i, f_i + p_i, g_i + q_i)$, respectively.

The inclusion relation between $\tilde{\beta}_{*i}$ and $\tilde{\beta}_i^*$ can be extended to $\tilde{Y}_*(\boldsymbol{x}_j)$ and $\tilde{Y}^*(\boldsymbol{x}_j)$, that is:

$$\tilde{Y}^*(\boldsymbol{x}) \supseteq \tilde{Y}_*(\boldsymbol{x}) \text{ for any } x = (1, x_1, \ldots, x_{nv})^t \text{ if } \tilde{\beta}_i^* \supseteq \tilde{\beta}_{*i}. \tag{31}$$

Using the coefficients of the LAM model $\tilde{A}_{*i} = (b_i, f_i, g_i)$, $\tilde{Y}_*(\boldsymbol{x}_j)$ can be expressed as:

$$\begin{aligned} \tilde{Y}_*(x_j) &= (b_0, f_0, g_0) + (b_1, f_1, g_1)x_{j1} + \cdots + (b_{nv}, f_{nv}, g_{nv})x_{j,nv} \\ &= \left(\sum_{i=0}^{n} b_i x_{ji}, \sum_{x_{ji} \geq 0} f_i x_{ji} - \sum_{x_{ji} \leq 0} g_i x_{ji}, \sum_{x_{ji} \geq 0} g_i x_{ji} - \sum_{x_{ji} \leq 0} f_i x_{ji}, \right) \quad , \\ &= \left(\boldsymbol{b}^t \boldsymbol{x}_j, \theta_{*L}(\boldsymbol{x}_j), \theta_{*R}(\boldsymbol{x}_j) \right) \end{aligned} \tag{32}$$

where $\boldsymbol{b} = (b_0, b_1, \ldots, b_{nv})^t$.

Similarly, $\tilde{Y}^*(\boldsymbol{x}_j)$ can be expressed as:

$$\begin{aligned} \tilde{Y}^*(x_j) &= (b_0, f_0, g_0) + (b_1, f_1 + p_1, g_1 + q_1)x_{j1} + \cdots + (b_{nv}, f_{nv} + p_{nv}, g_{nv} + q_{nv})x_{jnv} \\ &= \left(\sum_{i=0}^{nv} b_i x_{ji}, \sum_{x_{ji} \geq 0} f_i x_{ji} + \sum_{x_{ji} \geq 0} p_i x_{ji} - \sum_{x_{ji} \leq 0} g_i x_{ji} - \sum_{x_{ji} \leq 0} q_i x_{ji}, \sum_{x_{ji} \geq 0} g_i x_{ji} + \sum_{x_{ji} \geq 0} q_i x_{ji} - \sum_{x_{ji} \leq 0} f_i x_{ji} - \sum_{x_{ji} \leq 0} p_i x_{ji}, \right) . \\ &= \left(\boldsymbol{b}^t \boldsymbol{x}_j, \theta^*{}_L(\boldsymbol{x}_j), \theta^*{}_R(\boldsymbol{x}_j) \right) \end{aligned} \tag{33}$$

As the possibility and necessity concepts were employed, observed outputs were transformed into granular constructs by admitting a tolerance level for the left- and right-spreads. A user-defined tolerance_level was set, thereby possibilistic relationships between observed and estimated outputs could be defined. Generally, tolerance_level is a percentage type, i.e., assigning 20% of each output y_j as the corresponding spread. In this study, left- and right-spreads of the observed outputs were called tolerance levels. The tolerance level for the jth data-point is denoted by e_j.

The possibilistic model states that the resulting output from the UAM should cover all of the observed data-points within the given tolerance- and h-level. In other words, $\left[\tilde{Y}^*(\boldsymbol{x}_j)\right]_h$ should approach to $\left[Y_j\right]_h$ from the upper side; i.e., $\left[\tilde{Y}^*(\boldsymbol{x}_j)\right]_h$ should be the least interval among all feasible solutions. This brings up the following constraints:

$$\left[\tilde{Y}^*(\boldsymbol{x}_j)\right]_h \supseteq \left[Y_j\right]_h \Leftrightarrow \begin{array}{c} \left\{ \begin{array}{l} \boldsymbol{b}^t\boldsymbol{x}_j + (1-h)\theta^*_R(\boldsymbol{x}_j) \geq y_j + (1-h)e_j \\ \boldsymbol{b}^t\boldsymbol{x}_j - (1-h)\theta^*_L(\boldsymbol{x}_j) \leq y_j - (1-h)e_j \end{array} \right\}, \ j = 1, \ldots, nd \\ \boldsymbol{f} \geq 0,\ \boldsymbol{g} \geq 0,\ \boldsymbol{p} \geq 0,\ \boldsymbol{q} \geq 0, \end{array}, \tag{34}$$

where $\boldsymbol{x}_j = \left(1, x_{j,1}, x_{j,2}, \ldots, x_{j,nv}\right)$, $j = 1, \ldots, nd$ and the term $x_{j,nv}$ denotes the value of the variable nv of the jth data point.

On the other hand, according to necessity model, the h-level set of the $\tilde{Y}_*(x_j)$ should be included in the h-level set of the given output Y_j. In other words, $\left[\tilde{Y}_*(\boldsymbol{x}_j)\right]$ should approach to $\left[Y_j\right]_h$ from the lower side, i.e., $\left[\tilde{Y}_*(\boldsymbol{x}_j)\right]$ should be the greatest interval among all feasible solutions. This can be written in a constraint form as:

$$\left[\tilde{Y}_*(\boldsymbol{x}_j)\right]_h \subseteq \left[Y_j\right]_h \Leftrightarrow \begin{array}{c} \left\{ \begin{array}{l} \boldsymbol{b}^t\boldsymbol{x}_j + (1-h)\theta_{*R}(\boldsymbol{x}_j) \leq y_j + (1-h)e_j \\ \boldsymbol{b}^t\boldsymbol{x}_j - (1-h)\theta_{*L}(\boldsymbol{x}_j) \geq y_j - (1-h)e_j \end{array} \right\}, \ j = 1, \ldots, nd \\ \boldsymbol{f} \geq 0,\ \boldsymbol{g} \geq 0,\ i = 0, \ldots, n \end{array}. \tag{35}$$

Integrating the possibility and necessity models by taking into account the inclusion relation $\left[\tilde{Y}_*(\boldsymbol{x}_j)\right]_h \subseteq \left[\tilde{Y}^*(\boldsymbol{x}_j)\right]_h$ a quadratic programming formulation of the IT2F regression model is formulated as:

$$\begin{array}{l} \min\limits_{\boldsymbol{b},\boldsymbol{f},\boldsymbol{g},\boldsymbol{p},\boldsymbol{q}} J = \sum\limits_{j=1}^{nd}\left(y_j - \boldsymbol{b}^t\boldsymbol{x}_j\right)^2 + (1-h)\sum\limits_{j=1}^{nd}\left(\boldsymbol{p}^t|\boldsymbol{x}_j| + \boldsymbol{q}^t|\boldsymbol{x}_j|\right) + \xi\left(\boldsymbol{f}^t\boldsymbol{f} + \boldsymbol{g}^t\boldsymbol{g} + \boldsymbol{p}^t\boldsymbol{p} + \boldsymbol{q}^t\boldsymbol{q}\right) \\ \text{Subject to} \begin{array}{c} \left\{ \begin{array}{l} \boldsymbol{b}^t\boldsymbol{x}_j + (1-h)\theta^*_R(\boldsymbol{x}_j) \geq y_j + (1-h)e_j \\ \boldsymbol{b}^t\boldsymbol{x}_j - (1-h)\theta^*_L(\boldsymbol{x}_j) \leq y_j - (1-h)e_j \\ \boldsymbol{b}^t\boldsymbol{x}_j + (1-h)\theta_{*R}(\boldsymbol{x}_j) \leq y_j + (1-h)e_j \\ \boldsymbol{b}^t\boldsymbol{x}_j - (1-h)\theta_{*L}(\boldsymbol{x}_j) \geq y_j - (1-h)e_j \end{array} \right\}, \ j = 1, \ldots, nd \\ \boldsymbol{f} \geq 0,\ \boldsymbol{g} \geq 0,\ \boldsymbol{p} \geq 0,\ \boldsymbol{q} \geq 0 \end{array} \end{array}, \tag{36}$$

where ξ is a small positive number. The term $\xi\left(\boldsymbol{f}^t\boldsymbol{f} + \boldsymbol{g}^t\boldsymbol{g} + \boldsymbol{p}^t\boldsymbol{p} + \boldsymbol{q}^t\boldsymbol{q}\right)$ is inserted into the objective function so that the objective function becomes a quadratic function with respect to decision variables $\boldsymbol{b}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{p}$, and $\boldsymbol{q}$. The obtained UAM and LAM by solving the above integrated quadratic programming model always satisfy inclusion relation $Y_*(\boldsymbol{x}) \subseteq Y^*(\boldsymbol{x})$ at the h-level.

3.3. Dynamic MADM Model via Proposed IT2F Functions

In this section, the proposed IT2F functions approach was given in a step-by-step manner. The flowchart of the proposed model is illustrated in Figure 2.

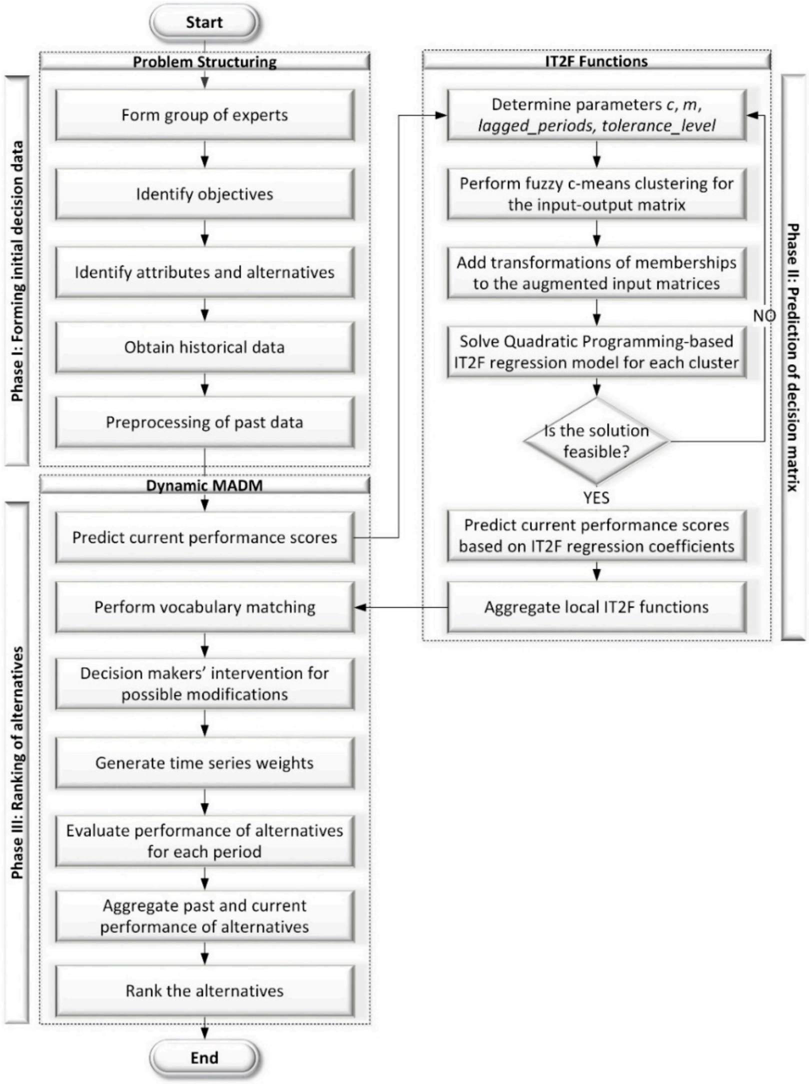

Figure 2. Flowchart of the proposed model.

3.3.1. Phase-I: Problem Structuring

Step 1: Problem-framing: In this step, a group of experts decided on the objective of the study, and the attributes and alternatives were identified. Here, expert opinions and the literature surveys helped to arrive at problem-framing.

Step 2: Obtaining historical data: Historical records were identified and the past data were fetched from the databases. Past data contained performance values of alternatives with respect to attributes at different periods as given in Equation (37).

$$\boldsymbol{A}(t_1) = \begin{bmatrix} a_{11}(t_1) & a_{12}(t_1) & \cdots & a_{1M}(t_1) \\ a_{21}(t_1) & a_{22}(t_1) & \cdots & a_{2M}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}(t_1) & a_{N2}(t_1) & \cdots & a_{NM}(t_1) \end{bmatrix}, \ldots, \boldsymbol{A}(t_H) = \begin{bmatrix} a_{11}(t_H) & a_{12}(t_H) & \cdots & a_{1M}(t_H) \\ a_{21}(t_H) & a_{22}(t_H) & \cdots & a_{2M}(t_H) \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}(t_H) & a_{N2}(t_H) & \cdots & a_{NM}(t_H) \end{bmatrix}, \quad (37)$$

where $\boldsymbol{A}(t_1)$ and $\boldsymbol{A}(t_H)$ are the decision matrices at the first and last period of the historical data, respectively.

This data can be unstructured so that the preprocessing is required.

Step 3: Preprocessing of historical data: In order to ensure accurate and meaningful analysis, data cleaning and preprocessing techniques are implemented in this step. Bad or missing data are eliminated by removing or replacing. Abrupt changes and local optima values are also identified. Smoothing or de-trending methods can be applied to remove noise.

Moreover, the historical decision making matrices are arranged as time series data. Here, for each alternative and attribute pair, time series data are formed. Mathematically speaking, the performance scores for a particular alternative and attribute $\left(a_{ij}(t_1), a_{ij}(t_2), \ldots, a_{ij}(t_H)\right)$ are collected from each period and the time series $\boldsymbol{y} = (y_1, y_2, \ldots, y_H)^t$ is formed, where the number of points is equal to number of periods H.

Then, the lagged matrices are constructed where number of lagged periods is denoted by p. Note that jth data point in the input matrix $\boldsymbol{x}_j = \left(x_{t-1,j}, x_{t-2,j}, \ldots, x_{t-p,j}\right)^t$ will be used to estimate y_j, $j = 1, 2, \ldots, nd$, where nd is equal to $H - p$. The inputs and the outputs of the system are given as:

$$\boldsymbol{X} = \begin{bmatrix} x_{t-1,1} & x_{t-2,1} & \cdots & x_{t-p,1} \\ x_{t-1,2} & x_{t-2,2} & \cdots & x_{t-p,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{t-1,nd} & x_{t-2,nd} & \cdots & x_{t-p,nd} \end{bmatrix}, \boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{nd} \end{bmatrix}, \tag{38}$$

where $x_{t-p,nd}$ represents the value of variable $t - p$ of the data point nd.

Let the number of past decision matrices are four, and the lagged periods are determined as two. Suppose that decision makers are concerned with the past performance of the second alternative with respect to the first attribute. The corresponding time series data, input and output matrices are illustrated in Table 1.

Table 1. Illustration of the four-period example.

Period	**Past Decision Matrices**	**Corresponding Time Series**	**Input Matrix**	**Output Matrix**
t_1	$\begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ C_1\begin{bmatrix} \cdots & a_{12}(t_1) = 5 & \cdots \end{bmatrix} \end{array}$			
t_2	$\begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ C_1\begin{bmatrix} \cdots & a_{12}(t_2) = 3 & \cdots \end{bmatrix} \end{array}$	$\begin{bmatrix} y_1 = 5 \\ y_2 = 3 \\ y_3 = 6 \\ y_4 = 8 \end{bmatrix}$	$\boldsymbol{X} = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 6 \end{bmatrix}$	$\boldsymbol{Y} = \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$
t_3	$\begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ C_1\begin{bmatrix} \cdots & a_{12}(t_3) = 6 & \cdots \end{bmatrix} \end{array}$			
t_4	$\begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ C_1\begin{bmatrix} \cdots & a_{12}(t_4) = 8 & \cdots \end{bmatrix} \end{array}$			

Note that because the lagged periods are two, number of data points is $H - p = 4 - 2 = 2$.

3.3.2. Phase-II: Training of Fuzzy Functions Approach

Step 4: Determining parameters of the IT2F functions model: In this step, parameters of the fuzzy c-means clustering algorithm were determined. The number of clusters c, fuzzification coefficient m, lagged_periods, and tolerance_level for calculating possibility- and necessity-based constraints in IT2F regression were defined.

Step 5: Performing fuzzy c-means clustering to input-output model: In this step, inputs and outputs of the system were used to carry out fuzzy c-means clustering. Having the inputs of the system in the form of lagged variables, the next step was to form the input–output matrix $\boldsymbol{Z}$. The matrix

$\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ is composed of the input matrix $\boldsymbol{X}$ and output matrix $\boldsymbol{Y}$. Then, elements of the $\boldsymbol{Z}$ matrix $\boldsymbol{z}_j$ were clustered by using the FCM algorithm. FCM was applied by using the following formulas:

$$\boldsymbol{v}_i = \frac{\sum_{k=1}^{nd} \mu_{ij}^m \boldsymbol{z}_j}{\sum_{j=1}^{nd} \mu_{ij}^m}, \; i = 1, 2, \ldots, c, \tag{39}$$

$$\mu_{ij} = \frac{1}{\sum_{h=1}^{c} \left(\frac{\|\boldsymbol{v}_i - \boldsymbol{z}_j\|}{\|\boldsymbol{v}_h - \boldsymbol{z}_j\|} \right)^{\frac{2}{m-1}}}, \; i = 1, 2, \ldots, c; \; j = 1, 2, \ldots, nd, \tag{40}$$

where ||.|| represents the Euclidian distance.

Step 6: Generating augmented input matrices: The augmented input matrix is obtained by adding memberships and their transformations into the original input matrix. Based on the cluster centers found in the Step 5, membership values of the input space are calculated as:

$$\mu_{ij} = \frac{1}{\sum_{h=1}^{c} \left(\frac{\|\boldsymbol{v}_i - \boldsymbol{x}_j\|}{\|\boldsymbol{v}_h - \boldsymbol{x}_j\|} \right)^{\frac{2}{m-1}}}, \; i = 1, 2, \ldots, c; \; j = 1, 2, \ldots, nd, \tag{41}$$

where $\boldsymbol{x}$ denotes the input matrix.

Membership values of each input data sample,μ_{ij} and their transformations are augmented to the original input matrix for the ith cluster as:

$$\boldsymbol{\phi}_i = \begin{bmatrix} 1 & \mu_{i,1} & \exp(\mu_{i,1}) & (\mu_{i,1})^p & x_{t-1,1} & \cdots & x_{t-p,1} \\ 1 & \mu_{i,2} & \exp(\mu_{i,2}) & (\mu_{i,2})^p & x_{t-1,2} & \cdots & x_{t-p,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mu_{i,j} & \exp(\mu_{i,j}) & (\mu_{i,j})^p & x_{t-1,j} & \cdots & x_{t-p,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mu_{i,nd} & \exp(\mu_{i,nd}) & (\mu_{i,nd})^p & x_{t-1,nd} & \cdots & x_{t-p,nd} \end{bmatrix}, \tag{42}$$

where j-th data point is represented by $\boldsymbol{\phi}_{i,j} = \left(1, \mu_{i,j}, \exp(\mu_{i,j}), \mu_{i,j}^2, x_{t-1,j}, \ldots, x_{t-p,j}\right)^t$.

The schematic representation of the proposed IT2F functions is given in Figure 3.

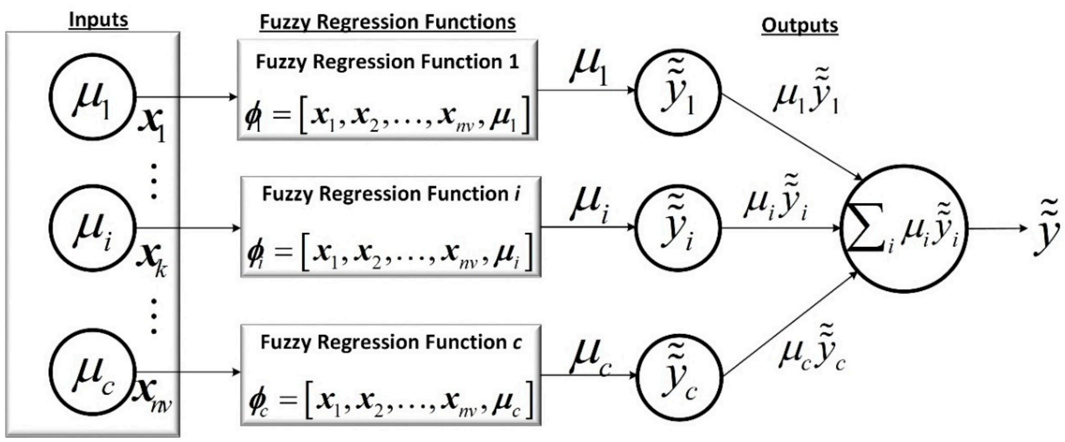

Figure 3. Proposed IT2F functions approach.

Step 7: Solving quadratic programming model for each cluster: Fuzzy regression coefficients are calculated for each cluster by solving a quadratic programming model:

$$\begin{gathered}\min_{\boldsymbol{b},\boldsymbol{f},\boldsymbol{g},\boldsymbol{p},\boldsymbol{q}} J=\sum_{j=1}^{nd}\left(y_j-\boldsymbol{b}^t\boldsymbol{\phi}_{i,j}\right)^2+(1-h)\sum_{j=1}^{nd}\left(\boldsymbol{p}^t|\boldsymbol{\phi}_{i,j}|+\boldsymbol{q}^t|\boldsymbol{\phi}_{i,j}|\right)+\xi\left(\boldsymbol{f}^t\boldsymbol{f}+\boldsymbol{g}^t\boldsymbol{g}+\boldsymbol{p}^t\boldsymbol{p}+\boldsymbol{q}^t\boldsymbol{q}\right)\\ \text{Subject to}\ \left\{\begin{array}{l}\boldsymbol{b}^t\boldsymbol{\phi}_{i,j}+(1-h)\theta^*_R(\boldsymbol{\phi}_{i,j})\geq y_j+(1-h)e_j\\ \boldsymbol{b}^t\boldsymbol{\phi}_{i,j}-(1-h)\theta^*_L(\boldsymbol{\phi}_{i,j})\leq y_j-(1-h)e_j\\ \boldsymbol{b}^t\boldsymbol{\phi}_{i,j}+(1-h)\theta_{*R}(\boldsymbol{\phi}_{i,j})\leq y_j+(1-h)e_j\\ \boldsymbol{b}^t\boldsymbol{\phi}_{i,j}-(1-h)\theta_{*L}(\boldsymbol{\phi}_{i,j})\geq y_j-(1-h)e_j\end{array}\right\},\ j=1,\ldots,nd\\ \boldsymbol{f}\geq 0,\ \boldsymbol{g}\geq 0,\ \boldsymbol{p}\geq 0,\ \boldsymbol{q}\geq 0\end{gathered}\quad, \tag{43}$$

where regression coefficients are IT2F numbers represented by $\tilde{\tilde{\boldsymbol{\beta}}}=((\boldsymbol{b},\boldsymbol{f}+\boldsymbol{p},\boldsymbol{g}+\boldsymbol{q}),(\boldsymbol{b},\boldsymbol{f},\boldsymbol{g}))$,

Step 8: Collecting predictions of local fuzzy functions: Predicted output values are calculated as:

$$\tilde{\tilde{\boldsymbol{y}}}_i=\boldsymbol{\phi}_i\otimes\tilde{\tilde{\boldsymbol{\beta}}}_i, \tag{44}$$

where $\tilde{\tilde{\boldsymbol{y}}}_i=\left(\tilde{\tilde{y}}_{i,1},\tilde{\tilde{y}}_{i,2},\ldots,\tilde{\tilde{y}}_{i,j},\ldots,\tilde{\tilde{y}}_{i,nd}\right)^t$, $\tilde{\tilde{\beta}}_i$ is the regression coefficients of the ith local fuzzy function and $\otimes$ denotes the fuzzy matrix multiplication.

Step 9: Aggregating local IT2F functions: Finally, outputs of the local fuzzy functions $\tilde{\tilde{\boldsymbol{y}}}_i$ are weighted by the corresponding membership values and predicted IT2F output is calculated:

$$\tilde{\tilde{\boldsymbol{Y}}}_j=\frac{\sum_i^c\tilde{\tilde{y}}_{i,j}\mu_{i,j}}{\sum_{i=1}^c\mu_{i,j}},\ j=1,2,\ldots,nd, \tag{45}$$

where $\tilde{\tilde{\boldsymbol{Y}}}_j$ is the predicted value of the jth data point.

3.3.3. Phase-III: Ranking of Alternatives

Step 10: Performing vocabulary matching: The resulting values of the IT2F functions were inherently IT2F sets. Since experts often linguistically evaluate the objects in the decision making applications and it is difficult to analytically interpret the obtained numerical values, there was a need for transforming IT2F functions results into the linguistic terms. For that aim, similarity-based vocabulary matching was implemented in this step.

Let $\boldsymbol{V}\in\{V_1,V_2,\ldots,V_U\}$ represents the vocabulary of the linguistic terms, i.e., V_U denotes the linguistic term very good, V_{U-1} denotes the term good, etc. The linguistic outputs of the IT2F functions approach can be given as:

$$\tilde{\tilde{\boldsymbol{A}}}'(t_C)=\begin{bmatrix}\tilde{\tilde{a}}'_{11}(t_C) & \tilde{\tilde{a}}'_{12}(t_C) & \cdots & \tilde{\tilde{a}}'_{1M}(t_C)\\ \tilde{\tilde{a}}'_{21}(t_C) & \tilde{\tilde{a}}'_{22}(t_C) & \cdots & \tilde{\tilde{a}}'_{2M}(t_C)\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{\tilde{a}}'_{N1}(t_C) & \tilde{\tilde{a}}'_{N2}(t_C) & \cdots & \tilde{\tilde{a}}'_{NM}(t_C)\end{bmatrix}. \tag{46}$$

The $\tilde{\tilde{a}}'_{ij}$ values are calculated as:

$$\tilde{\tilde{a}}'_{ij}=\underset{l\in\{1,2,\ldots,U\}}{\operatorname{argmax}} SM\left(\tilde{\tilde{a}}'_{ij},V_l\right), \tag{47}$$

where SM represents the Jaccard similarity measure given earlier, and V_l is the lth linguistic term in the vocabulary.

Step 11: Modifying solutions if necessary: In this step, the results of IT2F functions in the form of linguistic variables were presented to the decision makers. In other words, the current decision matrix was automatically generated based on the past data. The decision makers evaluated the results and made necessary modifications if needed. The illustration of this process is given in Figure 4. Here, the decision makers' perceptions had a pivotal role. For example, decision makers might decide on the fact that the performance of the first alternative with respect to the first and second attributes needs to be modified. Then the decision matrix takes the form as given in Equation (48).

$$\tilde{\tilde{A}}''(t_C) = \begin{bmatrix} \tilde{\tilde{a}}''_{11}(t_C) & \tilde{\tilde{a}}'_{12}(t_C) & \cdots & \tilde{\tilde{a}}'_{1M}(t_C) \\ \tilde{\tilde{a}}''_{21}(t_C) & \tilde{\tilde{a}}'_{22}(t_C) & \cdots & \tilde{\tilde{a}}'_{2M}(t_C) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\tilde{a}}'_{N1}(t_C) & \tilde{\tilde{a}}'_{N2}(t_C) & \cdots & \tilde{\tilde{a}}'_{NM}(t_C) \end{bmatrix}, \tag{48}$$

where $\tilde{\tilde{a}}''_{ij}(t_C)$ represents the subjective judgments of the decision makers and $\tilde{\tilde{a}}'_{ij}(t_C)$ denotes the IT2F function result.

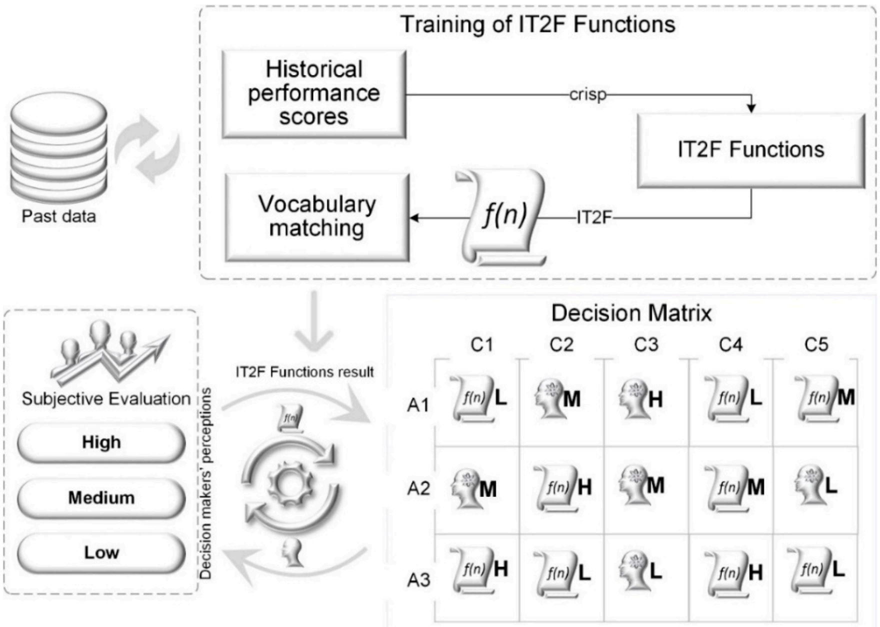

Figure 4. Interactive process of the proposed model.

Step 12: Generating time series weights: In this step, time series weights were generated. A basic unit-interval monotonic (BUM) function was used to generate weights. Yager [55] defined the BUM function as $Q : [0, 1] \to [0, 1]$, where the weights are considered as quantifiers underlying the information fusion process.

- $Q(0) = 0$,
- $Q(1) = 1$,
- $Q(x) \geq Q(y)$, if $x > y$, where $Q(x)$ is a monotonically non-decreasing function defined in the unit interval $[0, 1]$.

Based on the BUM function, the time series weights are generated as:

$$\xi(t_k) = Q\left(\frac{k}{p}\right) - Q\left(\frac{k-1}{p}\right),\ k = 1, 2, \ldots, p, \tag{49}$$

where $Q(x) = \frac{e^{\alpha x}-1}{e^{\alpha}-1}$, $\alpha > 0$.

According to the BUM function, the more the period is closer to the current period, the higher the weight of that period, which is a desired behavior for real-world applications.

Step 13: Evaluating performance of alternatives at each period: Performance of each alternative is calculated for different periods separately. In this step, variety of MADM methods can be employed to obtain a performance indicator. For the sake of simplicity, well-known closeness coefficient measures are used to evaluate performance of alternatives at each period. As the current decision matrix comprises of IT2F evaluations, computational steps for the IT2FSs are given in this section in order to avoid repetition. Note that the required computations for the past data are the same, except the fact that numerical values are crisp. First, the decision matrices related to past periods are normalized as given earlier in Equations (1) and (2). As the current decision matrix consists of IT2F evaluations, normalization is conducted based on the Equations (50) and (51).

$$\tilde{\tilde{r}}_{ij}(t_k) = \left(\begin{array}{c} \left(\frac{\overline{a}_{ij1}(t_k)}{\max\limits_j\{\overline{a}_{ij3}(t_k)\}}, \frac{\overline{a}_{ij2}(t_k)}{\max\limits_j\{\overline{a}_{ij3}(t_k)\}}, \frac{\overline{a}_{ij3}(t_k)}{\max\limits_j\{\overline{a}_{ij3}(t_k)\}}; 1 \right), \\ \left(\frac{\underline{a}_{ij1}(t_k)}{\max\limits_j\{\overline{a}_{ij3}(t_k)\}}, \frac{\underline{a}_{ij2}(t_k)}{\max\limits_j\{\overline{a}_{ij3}(t_k)\}}, \frac{\underline{a}_{ij3}(t_k)}{\max\limits_j\{\overline{a}_{ij3}(t_k)\}}; 1 \right) \end{array} \right), \text{if } i \in \Omega_b, \tag{50}$$

$$\tilde{\tilde{r}}_{ij}(t_k) = \left(\begin{array}{c} \left(\frac{\min\limits_j\{\overline{a}_{ij1}(t_k)\}}{\overline{a}_{ij3}(t_k)}, \frac{\min\limits_j\{\overline{a}_{ij1}(t_k)\}}{\overline{a}_{ij2}(t_k)}, \frac{\min\limits_j\{\overline{a}_{ij1}(t_k)\}}{\overline{a}_{ij1}(t_k)}; 1 \right), \\ \left(\frac{\min\limits_j\{\overline{a}_{ij1}(t_k)\}}{\underline{a}_{ij3}(t_k)}, \frac{\min\limits_j\{\overline{a}_{ij1}(t_k)\}}{\underline{a}_{ij2}(t_k)}, \frac{\min\limits_j\{\overline{a}_{ij1}(t_k)\}}{\underline{a}_{ij1}(t_k)}; 1 \right) \end{array} \right), \text{if } i \in \Omega_c. \tag{51}$$

Then, the weighted normalized decision matrices are calculated as:

$$\tilde{\tilde{v}}_{ij}(t_k) = \tilde{\tilde{r}}_{ij}(t_k) \times w_i,\ i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M,\ k = 1, 2, \ldots, H. \tag{52}$$

When the weighted normalized decision matrices are constructed, the next step is to calculate the positive ideal solutions (PIS) and negative ideal solutions (NIS) as:

$$\begin{aligned} \text{PIS} &= \left(v_1^+, v_2^+, \ldots, v_N^+\right) \\ &= \left\{ \left(\max_j\left\{Rank\left(\tilde{\tilde{v}}_{ij}(t_k)\right)\right\}\right) \middle| i \in \Omega_b, \left(\min_j\left\{Rank\left(\tilde{\tilde{v}}_{ij}(t_k)\right)\right\}\right) \middle| i \in \Omega_c \right\},\ k = 1, 2, \ldots, H \end{aligned} \tag{53}$$

$$\begin{aligned} \text{NIS} &= \left(v_1^-, v_2^-, \ldots, v_N^-\right) \\ &= \left\{ \left(\min_j\left\{Rank\left(\tilde{\tilde{v}}_{ij}(t_k)\right)\right\}\right) \middle| i \in \Omega_b, \left(\max_j\left\{Rank\left(\tilde{\tilde{v}}_{ij}(t_k)\right)\right\}\right) \middle| i \in \Omega_c \right\},\ k = 1, 2, \ldots, H \end{aligned} \tag{54}$$

Then, separation measures are calculated by using the Euclidean distance as:

$$D_j^+(t_k) = \sqrt{\sum_{i=1}^{N} \left(Rank\left(\tilde{\tilde{v}}_{ij}(t_k)\right) - v_i^+(t_k)\right)^2}, \tag{55}$$

$$D_j^-(t_k) = \sqrt{\sum_{i=1}^{N} \left(Rank\left(\tilde{\tilde{v}}_{ij}(t_k)\right) - v_i^-(t_k)\right)^2}. \tag{56}$$

Finally, closeness coefficients are calculated as:

$$CC_j(t_k) = \frac{D_j^-(t_k)}{\left(D_j^+(t_k) + D_j^-(t_k)\right)}, \tag{57}$$

where $CC_j(t_k)$ represents the closeness coefficient of the jth alternative at period t_k.

Step 14: Aggregating past and current performance of alternatives: Finally, a dynamic weighted averaging (DWA) operator was utilized to obtain final ranking values of alternatives.

$$DWA_{\xi(t)}\Big(CC_j(t_1), CC_j(t_2), \ldots, CC_j(t_H)\Big) = \sum_{k=1}^{H} \xi(t_k) CC_j(t_k). \tag{58}$$

Step 15: Rank the alternatives: When the past and current performance scores of alternatives were aggregated, alternatives were ranked based on their ranking values. Higher ranking value implies superiority of an alternative.

4. Case Study

One of the most important assets of a company is undoubtedly the human resources (HRs). Regardless of how the other resources are managed in an organization, inadequacies in the management of HRs result in poor performance of many operations. Therefore, firms have steadily recognized the importance of HRs and have been taking necessary actions to increase overall performance.

Personnel promotion is a significant task in HR management that aims to select the right person for the right job. Despite its similarity with the personnel selection problem, personnel promotion problem deals with selecting appropriate personnel for higher positions within the firms' current personnel rather than evaluating the applicants from outside the firm. Personnel promotion problem can be defined as selecting the most qualified employee among the available candidates for a vacant position by considering their performance during their employment at the firm. Hence, the personnel promotion problem is an inherently dynamic MADM problem as the temporal performance of employees are taken into consideration with respect to predetermined attributes. Unfortunately, many enterprises are not aware of the methodologies and tools to utilize historical records in their HR practices. As evaluating personnel with respect to their performance on a diverse set of criteria during their employment is cognitively demanding, firms should be supported with relevant data-driven tools.

In this study, a real-life personnel promotion problem is considered and the proposed model is implemented. The company, which was contacted within the scope of this study, was a medium sized firm involved in automobile subsidiary industry. Due to the firm policy, it was named as company A. Company A had hired three students as a part-time employee for their continuous improvement project. Upon graduation, at most two students with qualified skills will be offered to full-time job so that the supervisors have been evaluating students' performance in a monthly basis. Therefore, performance data was used to demonstrate the procedural steps of the proposed model.

4.1. Structering Personnel Promotion Problem

Step 1: In this step, the experts were identified based on their professional backgrounds. The heterogeneity of the experts was assured and three experts were identified. The experts were the directors of HR and production and quality control departments. After identification of the experts, participants were elucidated about the scope and details of the study. The evaluation criteria determined by the experts were employed within the scope of this study. The evaluation criteria consisted of content-specific knowledge (C_1), communication skills (C_2), job involvement (C_3), organizational commitment (C_4), and problem solving skills (C_5). The decision hierarchy is given in Figure 5.

Step 2: Historical performance data of the employees with respect to predefined attributes were obtained. The company had a performance evaluation system that assigned performance scores between 0–10, 0 and 10 represented the worst and the best values, respectively. In the scope of this study, performance values of the past 1.5 years (18 months) were directly used. The historical data of the performance scores of the employee 1 with respect to decision attributes is given in Table 2. Similarly, the historical records for the employee 2 and employee 3 were obtained.

Step 3: Historical data was organized in tables, and missing values were sought for. No missing values were identified within the 18-month data period. On the other hand, it was decided not to implement normalization techniques as the proposed model could handle numeric values in the range 0–10. The historical decision matrices were also transformed into time series vectors so that the IT2F functions method could be implemented.

Afterwards, the data set was divided into training and test sets. The training set consisted of the first 80% part of the historical data and the rest of the data points were allocated to the test set.

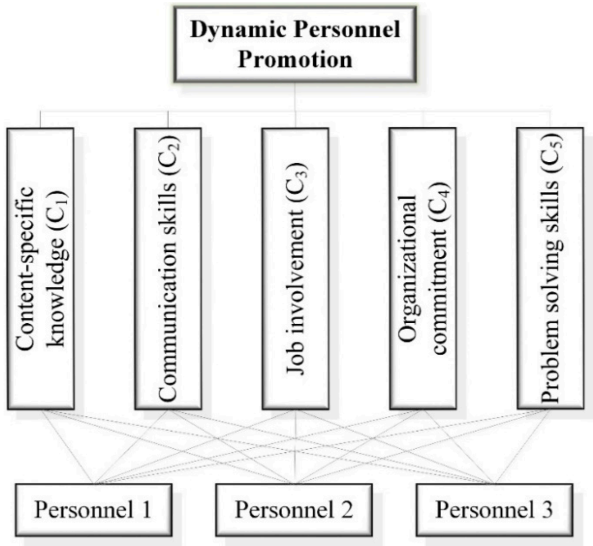

Figure 5. Decision hierarchy.

Table 2. Historical data of employee 1.

Period	C_1	C_2	C_3	C_4	C_5
1	6	7	3	3	7
2	6	7	2	2	6
3	7	8	3	3	5
4	6	9	4	5	5
5	6	8	3	6	5
6	6	9	4	8	4
7	7	8	4	8	4
8	6	7	5	7	4
9	6	6	6	8	5
10	6	5	6	8	5
11	7	4	7	9	6
12	6	3	6	10	6
13	5	4	6	10	6
14	5	4	8	8	6
15	4	4	6	7	8
16	4	3	6	7	7
17	4	3	6	7	8
18	4	2	6	9	8

4.2. Estimating the Current Decision Matrix

Having obtained the historical data, the next step was to employ the proposed IT2F functions approach to estimate the current decision matrix based on past data.

Step 4: In this step, parameters of the model were identified. The model parameters were the number of lagged_periods, tolerance_levels, number of cluster, and the degree of fuzzification. Instead of using cluster validity indices in order to select the optimum c and m parameters, the model parameters were selected by using a grid search for each parameter based on Root Mean Square

Error (RMSE) performance metric. The tolerance_levels were set to 30% for being able to find feasible solutions for every combination of parameters when solving quadratic programming models.

The identified parameters are given in Table 3.

Table 3. Parameters of the model.

Alternative	Criteria	Parameters		
		c	m	Lagged_Periods
1	1	2	1.6	5
	2	5	1.6	5
	3	4	2.1	4
	4	5	2.1	5
	5	4	1.6	4
2	1	4	1.6	5
	2	5	2.1	5
	3	2	1.1	5
	4	5	2.1	5
	5	5	1.6	5
3	1	2	2.1	5
	2	2	1.6	5
	3	3	1.6	5
	4	5	2.1	5
	5	5	1.6	5

Step 5: In this step, the input and output matrices were combined and the FCM clustering algorithm was performed based on the parameters identified in the previous step. By utilizing FCM clustering algorithm, cluster centers were identified. These cluster centers had a key role in the Turksen's fuzzy system model in which the clusters were used as a fuzzification engine of the classical FRB systems.

Step 6: Having identified cluster centers, the next step was to find membership grades of the input data and to form the augmented input matrices by integrating membership transformations as explanatory variables. In this study, integration of membership degrees, exponentials of the memberships, and the square of the memberships were found to exhibit good performance so that these transformations were used to augment the input space.

Step 7: When the input and output matrices were formed by means of membership grades, quadratic programming model was solved in order to obtain fuzzy regression coefficients. The models were written in MATLAB 9.5.0 and quadratic programming models were solved via quadratic programming solver function cplexqp of the Cplex Optimization Studio 12.8. As the regression coefficients were defined as the distance to the center (b) earlier, their corresponding IT2F number representations are given in Tables 4 and 5 for the case of first alternative and first criteria.

Similarly, the same computations were performed for all of the alternative and criteria pairs. Note that number of clusters was based on the Table 3.

Step 8: When all of the IT2F regression coefficients were calculated, the output was predicted by using the obtained regression coefficients for each cluster. For the time series data of the alternative 1 and criteria 1, 2 local fuzzy functions were calculated. Since there were five clusters in the time series data of the alternative 1 and criteria 2, the total of five local fuzzy function results were obtained.

Step 9: In this step, the local fuzzy functions were aggregated and the estimated outputs were calculated. Performance of the proposed IT2F functions approach was compared with the IT2F regression model by means of RMSE and Mean Absolute Percentage Error (MAPE) metrics. Table 6 shows the performance comparison of the proposed IT2F functions approach.

Table 4. Cluster 1 results for the 1st alternative and 1st criteria.

Variable	IT2F Regression Coefficients					IT2F Coefficients
	b	f	g	p	q	
1	10.572	8.65×10^{-9}	0.031	9.30×10^{-11}	9.13×10^{-11}	((10.572, 10.572, 10.603;1), (10.572, 10.572, 10.603;1))
μ	−8.524	3.39×10^{-9}	0.070	6.45×10^{-11}	6.44×10^{-11}	((−8.524, −8.524, −8.454;1), (−8.524, −8.524, −8.454;1))
$\exp(\mu)$	−4.550	2.04×10^{-9}	0.016	2.79×10^{-11}	2.78×10^{-11}	((−4.55, −4.55, −4.534;1), (−4.55, −4.55, −4.534;1))
μ^2	13.466	7.27×10^{-9}	0.085	7.34×10^{-11}	7.34×10^{-11}	((13.466, 13.466, 13.551;1), (13.466, 13.466, 13.551;1))
x_{t-1}	0.017	0.146043	0.010	2.67×10^{-11}	2.59×10^{-11}	((−0.129, 0.017, 0.027;1), (−0.129, 0.017, 0.027;1))
x_{t-2}	−0.613	1.24×10^{-9}	0.005	0.1	0.185714	((−0.713, −0.613, −0.423;1), (−0.613, −0.613, −0.609;1))
x_{t-3}	−0.211	1.44×10^{-9}	0.004	1.67×10^{-11}	1.63×10^{-11}	((−0.211, −0.211, −0.207;1), (−0.211, −0.211, −0.207;1))
x_{t-4}	0.822	0.078111	0.126	1.48×10^{-11}	1.42×10^{-11}	((0.744, 0.822, 0.948;1), (0.744, 0.822, 0.948;1))
x_{t-5}	0.038	1.44×10^{-9}	0.006	1.36×10^{-11}	1.34×10^{-11}	((0.038, 0.038, 0.044;1), (0.038, 0.038, 0.044;1))

Table 5. Cluster-2 results for the 1st alternative and 1st criteria.

Variable	IT2F Regression Coefficients					IT2F Coefficients
	b	f	g	p	q	
1	−10.907	1.28	1.018	4.25×10^{-9}	4.59×10^{-9}	((−12.184, −10.907, −9.889;1), (−12.184, −10.907, −9.889;1))
μ	−18.964	3.03×10^{-9}	0.000	1.44×10^{-8}	1.44×10^{-8}	((−18.964, −18.964, −18.964;1), (−18.964, −18.964, −18.964;1))
$\exp(\mu)$	15.077	1.79×10^{-9}	0.000	3.39×10^{-9}	3.61×10^{-9}	((15.077, 15.077, 15.077;1), (15.077, 15.077, 15.077;1))
μ^2	−3.447	3.12×10^{-9}	0.000	7.20×10^{-1}	1.16	((−4.167, −3.447, −2.289;1), (−3.447, −3.447, −3.447;1))
x_{t-1}	−0.107	1.50×10^{-9}	0.000	8.08×10^{-10}	8.56×10^{-10}	((−0.107, −0.107, −0.107;1), (−0.107, −0.107, −0.107;1))
x_{t-2}	−0.725	9.06×10^{-10}	0.000	3.64×10^{-7}	0.024974	((−0.725, −0.725, −0.7;1), (−0.725, −0.725, −0.725;1))
x_{t-3}	−0.262	3.18×10^{-9}	0.000	6.04×10^{-10}	6.37×10^{-10}	((−0.262, −0.262, −0.262;1), (−0.262, −0.262, −0.262;1))
x_{t-4}	0.735	3.18×10^{-9}	0.000	5.26×10^{-10}	5.27×10^{-10}	((0.735, 0.735, 0.735;1), (0.735, 0.735, 0.735;1))
x_{t-5}	0.150	1.16×10^{-5}	0.001	5.74×10^{-10}	6.10×10^{-10}	((0.15, 0.15, 0.151;1), (0.15, 0.15, 0.151;1))

Despite its practicality, IT2F regression, which does not make use of FCM clustering and augmented input matrix as in the case of the proposed IT2F functions, cannot capture the trends and patterns in the historical dataset to the desired extent. Hence, both of the performance metrics were quite high. On the other hand, the proposed IT2F functions approach had successfully captured the patterns in the data thanks to its membership processing mechanisms. Figure 6 shows the estimated performance scores of the alternative 1 with respect to five criteria by means of the IT2F functions approach.

Table 6. Performance comparison of the proposed method and IT2F regression.

Alternative	Criteria	Proposed IT2F Functions		IT2F Regression	
		RMSE	MAPE	RMSE	MAPE
1	1	0.2307	2.3883	0.6974	11.4445
	2	0.1948	3.3149	0.7844	15.1122
	3	0.2519	4.0679	0.9828	14.0217
	4	0.375	3.4459	0.6952	7.2826
	5	0.3606	5.6463	0.8976	16.7376
2	1	0.2894	6.0687	0.4449	13.6456
	2	0.2392	2.4306	0.6793	7.5079
	3	0.3733	4.2811	0.9343	11.083
	4	0.4283	5.1677	0.5934	7.8132
	5	0.3035	4.0027	0.5657	7.9788
3	1	0.1457	1.722	0.7285	8.0463
	2	0.1686	3.9912	0.5833	17.4631
	3	0.2417	2.6302	0.6196	6.6876
	4	0.2142	2.135	0.3221	3.4379
	5	0.2205	2.7494	0.5567	7.9517

Note that the produced results were IT2F numbers. Another critical point is about whether the produced results satisfy the imposed constraints by possibility and necessity relationships. Figure 7 digs into the first data points of the predictions, which are illustrated in Figure 6. Note that the lower and upper tolerances covered the lower membership function, which was the property of the necessity constraints in the mathematical model. Furthermore, the lower and upper tolerances were covered by the upper membership function, which was due to the possibility constraints. It can be seen that both of the lower and upper tolerances lied within the FOU of the IT2F outputs. The estimated outputs were also inside the FOU and the centers of the IT2F outputs were quite close to the target values, which indicates the predictive power of the IT2F functions.

Based on the trained IT2F functions model, the current decision matrix was predicted. The resulting decision matrix is shown in Table 7.

As it is difficult for experts to interpret the produced results, they were transformed into linguistic terms in the next section.

Table 7. Resulting decision matrix of the IT2F functions.

Criteria	Alternatives		
	A_1	A_2	A_3
C_1	((1.694, 2.99, 4.515;1), (2.064, 2.99, 3.822;1))	((2.163, 3.73, 5.516;1), (2.864, 3.73, 4.204;1))	((5.223, 7.902, 11.27;1), (5.601, 7.902, 10.396;1))
C_2	((1.354, 2.255, 3.33;1), (1.891, 2.255, 2.623;1))	((4.992, 8.419, 12.522;1), (5.964, 8.419, 10.802;1))	((2.662, 3.919, 5.449;1), (3.204, 3.919, 4.596;1))
C_3	((2.986, 5.358, 8.46;1), (4.13, 5.358, 6.362;1))	((2.194, 4.143, 5.991;1), (3.144, 4.143, 5.238;1))	((6.016, 9.229, 12.788;1), (6.68, 9.229, 11.589;1))
C_4	((7.187, 10.353, 14.08;1), (8.084, 10.353, 12.123;1))	((0.469, 1.617, 3.242;1), (0.933, 1.617, 2.178;1))	((5.678, 8.415, 11.417;1), (6.071, 8.415, 10.508;1))
C_5	((2.409, 5.252, 9.04;1), (3.372, 5.252, 7.208;1))	((4.427, 6.446, 9.052;1), (5.168, 6.446, 7.487;1))	((3.715, 6.246, 8.94;1), (4.566, 6.246, 7.472;1))

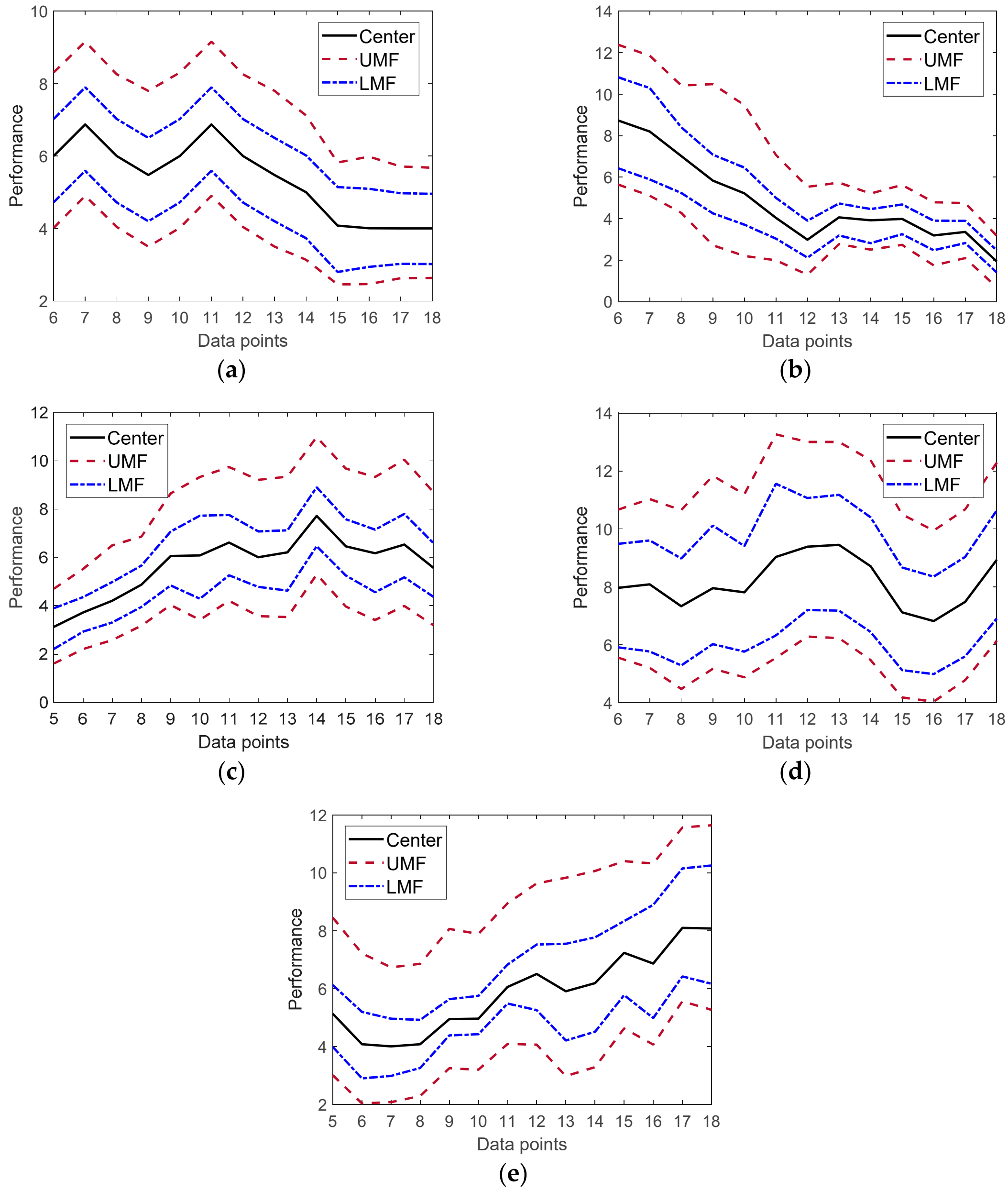

Figure 6. Estimations for the 1st alternative. (**a**) 1st criterion; (**b**) 2nd criterion; (**c**) 3rd criterion; (**d**) 4th criterion and (**e**) 5th criterion.

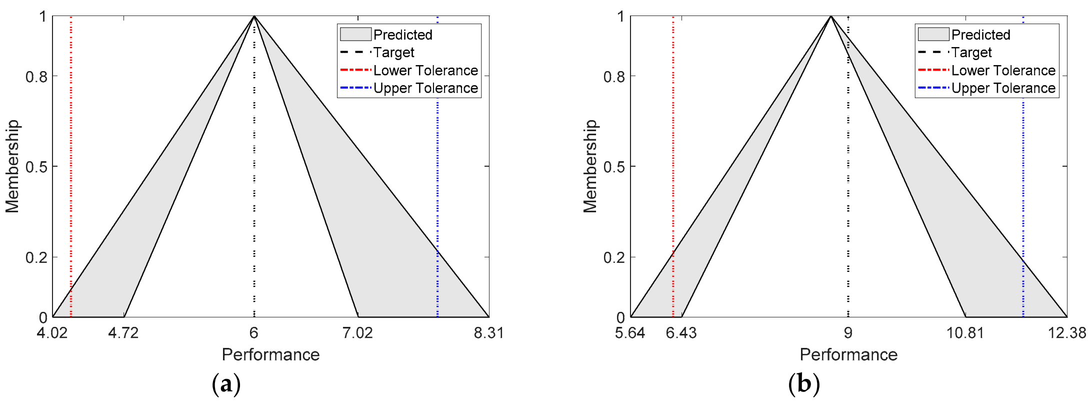

Figure 7. *Cont.*

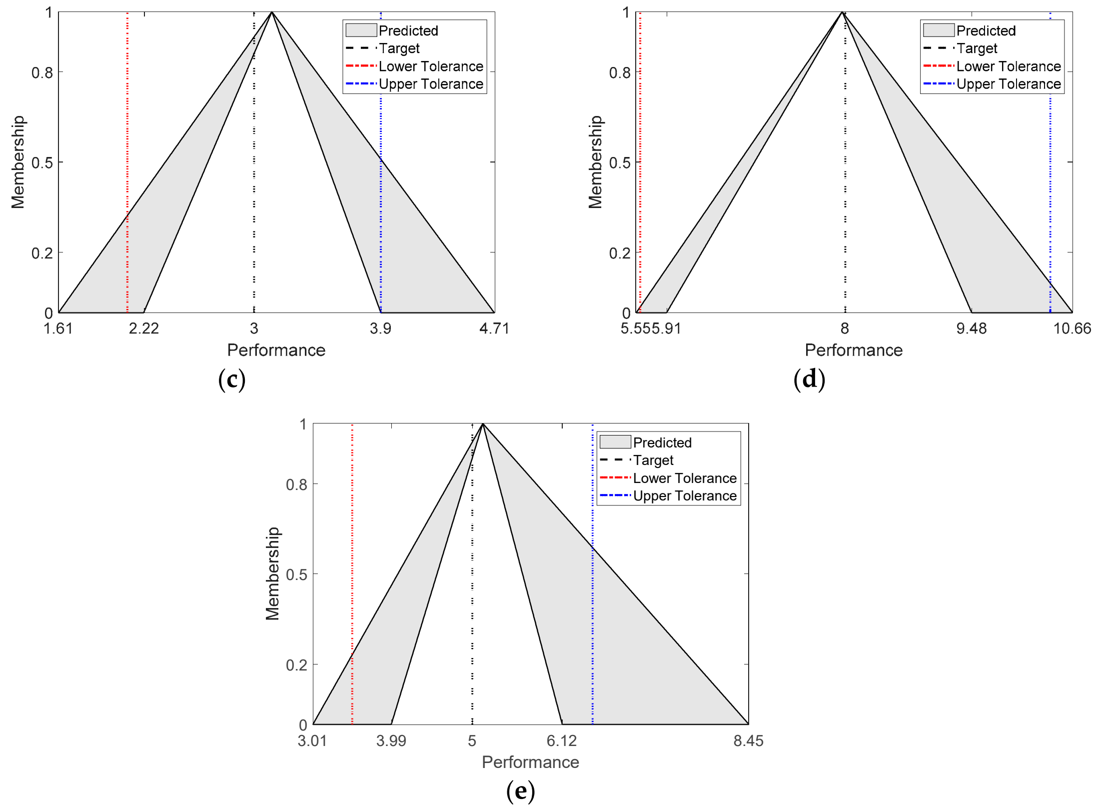

Figure 7. Details of the predictions for the 1st alternative. (**a**) 6th data point w.r.t. 1st criterion; (**b**) 6th data point w.r.t. 2nd criterion; (**c**) 5th data point w.r.t. 3rd criterion; (**d**) 6th data point w.r.t. 4th criterion and (**e**) 5th data point w.r.t. 5th criterion.

4.3. Ranking of Employees

Step 10: In this step, a similarity-based vocabulary matching was performed. First, the linguistic terms and their corresponding IT2F number were determined. Table 8 shows the linguistic terms of the vocabulary.

Based on the linguistic terms given in Table 8, similarity-based vocabulary matching was carried out. As a result, the linguistic decision matrix is given in Table 9.

Step 11: When the linguistic decision matrix was formed, the results were presented to the decision makers. Then, the decision makers were asked to change the performance values of the alternatives based on their perceptions. The decision makers might accept the decision matrix as it is or alter the whole performance values depending on their judgments. In this study, decision makers decided to change the performance values of the second and third alternatives with respect to first, fourth, and fifth criteria. The modified decision matrix is represented in Table 10. The modified linguistic terms are highlighted by gray shading.

Table 8. Vocabulary of linguistic terms.

Linguistic Terms		IT2F Number
Symbol	**Explanation**	
VL	Very Low	((0, 0, 1;1), (0, 0, 0.5;1))
L	Low	((0, 1, 3;1), (0.5, 1, 2;1))
ML	Medium Low	((1, 3, 5;1), (2, 3, 4;1))
M	Medium	((3, 5, 7;1), (4, 5, 6;1))
MH	Medium High	((5, 7, 9;1), (6, 7, 8;1))
H	High	((7, 9, 10;1), (8, 9, 9.5;1))
VH	Very High	((9, 10, 10;1), (9.5, 10, 10;1))

Table 9. Linguistic decision matrix.

Criteria	Alternatives		
	A_1	**A_2**	**A_3**
C_1	ML	ML	MH
C_2	ML	H	ML
C_3	M	M	H
C_4	H	L	H
C_5	M	MH	MH

Table 10. Modified decision matrix.

Criteria	Alternatives		
	A_1	**A_2**	**A_3**
C_1	ML	MH	MH
C_2	ML	H	ML
C_3	M	M	H
C_4	H	L	MH
C_5	M	H	MH

The decision makers accepted the linguistic terms produced by the IT2F functions model for the rest of the evaluations. By this way, required expert judgments were decreased by 80%, which dramatically increased the speed of decision making.

Step 12: In this step, time series weights were generated based on the BUM function. The only parameter required by BUM function was α. Figure 8 illustrates the time series weights for different α values. In this study, α was selected as 0.5. Furthermore, results for the different α values were examined as well.

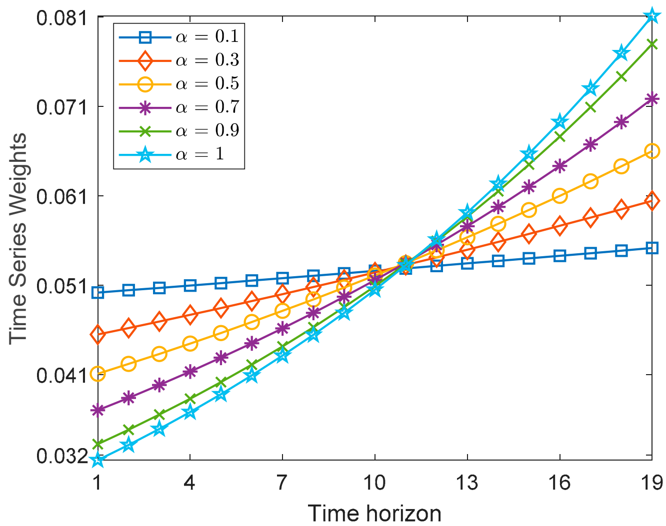

Figure 8. Time series weights.

Step 13: In this step, performance of each employee was evaluated at each period. Here, past and current decision matrices were evaluated one-by-one separately. As mentioned earlier, closeness coefficient is very practical to use so that for each period distance to positive and negative ideal solutions were calculated. The weights of criteria were assumed to be fixed throughout the periods and given as 0.10, 0.10, 0.15, 0.25, and 0.35, respectively. The separation measures are given in Tables 11 and 12.

Table 11. Distances between the performance of employees and the positive ideal solution.

Periods	Distance to Positive Ideal Solution		
	Employee 1	**Employee 2**	**Employee 3**
t_1	0.1458	0.1101	0.1111
t_2	0.2259	0.1569	0.1084
t_3	0.2155	0.1546	0.1161
t_4	0.1604	0.1203	0.0850
t_5	0.1730	0.1313	0.0938
t_6	0.1700	0.0919	0.0833
t_7	0.1700	0.0977	0.0750
t_8	0.1714	0.0709	0.0643
t_9	0.1256	0.0872	0.0750
t_{10}	0.1303	0.0874	0.0750
t_{11}	0.0960	0.0797	0.1086
t_{12}	0.1146	0.0901	0.1397
t_{13}	0.1204	0.0961	0.1393
t_{14}	0.1004	0.0914	0.1050
t_{15}	0.1226	0.1401	0.0797
t_{16}	0.1264	0.1330	0.0972
t_{17}	0.1422	0.1873	0.1001
t_{18}	0.1404	0.2335	0.0981
t_C	0.1725	0.1955	0.1143

Table 12. Distances between the performance of employees and the negative ideal solution.

Periods	Distance to Negative Ideal Solution		
	Employee 1	**Employee 2**	**Employee 3**
t_1	0.1409	0.1556	0.1541
t_2	0.1169	0.1524	0.2462
t_3	0.1161	0.1334	0.2155
t_4	0.0898	0.0698	0.1613
t_5	0.0995	0.0805	0.1762
t_6	0.1011	0.1636	0.1836
t_7	0.0983	0.1278	0.1857
t_8	0.0744	0.1697	0.1807
t_9	0.0684	0.1259	0.1385
t_{10}	0.0563	0.0950	0.1336
t_{11}	0.0811	0.1102	0.0717
t_{12}	0.0997	0.1397	0.0750
t_{13}	0.0817	0.1311	0.0750
t_{14}	0.0680	0.1050	0.0914
t_{15}	0.0927	0.0850	0.1247
t_{16}	0.0810	0.1000	0.1244
t_{17}	0.1218	0.1050	0.1719
t_{18}	0.2143	0.1173	0.1966
t_C	0.1873	0.1631	0.1751

Based on the distances from positive and negative ideal solutions, closeness coefficients were calculated as given in Table 13.

Step 14: In this step, closeness coefficients pertaining to the past and current performance scores of employees were aggregated by taking into account the time series weights. As there were numerous aggregation operators, the most practical one, namely the DWA operator, was used in this study. As a result of the DWA operator, rankings of employees were obtained. Moreover, for the purpose of comparison, different model components were activated and the results were examined. Table 14 summarizes the computational setting with different model components.

The proposed model integrates all of the model components. Different from the proposed model, the predicted decision matrix was not modified in model 1. In model 2, neither the linguistic decision matrix was generated nor the decision makers were allowed to change the decision matrix. Finally, model 3 made use of the IT2F functions, vocabulary matching, and modified preferences, but only the decision matrix of the current period was considered for ranking of alternatives, which corresponded to the conventional MADM approach.

Step 15: The employees were ranked and the results were interpreted in this step. As can be seen from the Table 15, employee 3 dominated the other candidates regardless of the model configurations. However, the performance of employee 1 needed more attention here as the ranking order differed by the static or dynamic MADM aspects. The performance of employee 1 has been visualized in Figure 9.

It was observed that the performance of employee 2 was better than employee 1 for all of the dynamic MADM models. However, considering only the current decision matrix in model 3, static MADM ranked employee 1 as second and employee 2 as third.

Although the performance of the employee 1 was higher than the employee 2 in the current period, considering historical performance values changed the overall rankings. Therefore, the present study provided policy makers with a broader view regarding performance of employees by considering different modeling aspects.

Table 13. Closeness coefficients of employees.

Periods	**Closeness Coefficients**		
	Employee 1	**Employee 2**	**Employee 3**
t_1	0.4915	0.5856	0.5811
t_2	0.3411	0.4928	0.6942
t_3	0.3501	0.4633	0.6499
t_4	0.3588	0.3671	0.6549
t_5	0.3652	0.3801	0.6527
t_6	0.3729	0.6405	0.6878
t_7	0.3664	0.5666	0.7123
t_8	0.3028	0.7053	0.7376
t_9	0.3525	0.5907	0.6487
t_{10}	0.3016	0.5208	0.6404
t_{11}	0.4580	0.5802	0.3978
t_{12}	0.4652	0.6078	0.3493
t_{13}	0.4043	0.5769	0.3500
t_{14}	0.4039	0.5347	0.4653
t_{15}	0.4304	0.3776	0.6099
t_{16}	0.3906	0.4291	0.5614
t_{17}	0.4613	0.3593	0.6321
t_{18}	0.6041	0.3345	0.6671
t_C	0.5206	0.4547	0.6051

Table 14. Different model configurations.

Model Components	**Models**			
	Proposed Model	**Model 1**	**Model 2**	**Model 3**
IT2F Functions	Yes	Yes	Yes	Yes
Vocabulary Matching	Yes	Yes	No	Yes
Modified preferences (interactive)	Yes	No	No	Yes
DWA Operator	Yes	Yes	Yes	No

Table 15. Overall results.

Model	Overall Assessment					
	Employee 1		Employee 2		Employee 3	
	CC	Rank	CC	Rank	CC	Rank
Proposed Model	0.4138	3	0.4984	2	0.5895	1
Model 1	0.4176	3	0.4926	2	0.5967	1
Model 2	0.4185	3	0.4905	2	0.5951	1
Model 3	0.5206	2	0.4547	3	0.6051	1

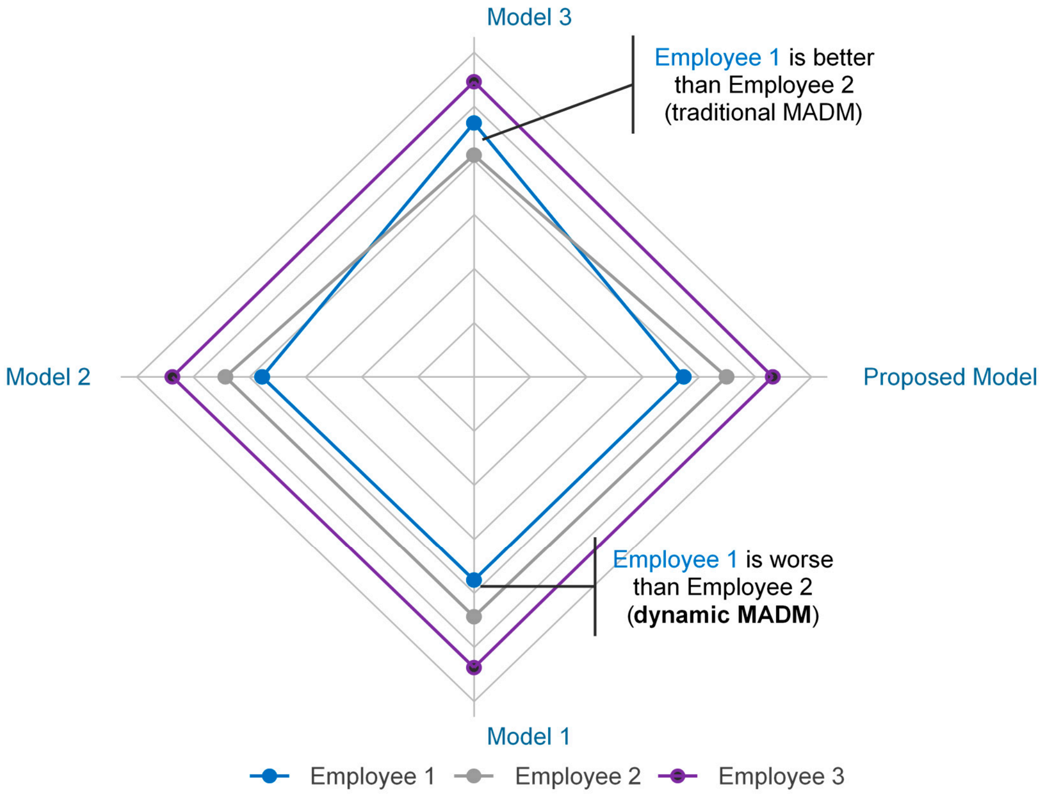

Figure 9. Changes in the ranking of employee 1.

5. Discussion

According to proposed dynamic MADM model, past performance values of employees were used to form current decision matrix. Hence, the current decision matrix was automatically generated based on the proposed IT2F functions model and the decision makers modified three performance scores out of 15. Therefore, preference elicitation procedures were dramatically improved.

The results of the case study had also shown that employee 3 had ranked first according to all of the dynamic MADM models. Although employee 1 had better performance than employee 2 in terms of current period performance, consideration of the past data made employee 1 the worst candidate for the full-time position among other employees. Although employee 2 had the worst performance without considering the past performance, with the dynamic character of the proposed model, employee 2 had become the second ranked alternative. As a result, taking into account the past and current decision matrices together provides more information to the decision makers and leads to more accurate rankings.

6. Conclusions

In this study, a new interactive dynamic MADM model, which combines IT2F regression and fuzzy functions approaches, was proposed. The proposed model exhibited desirable properties that helped overcome the drawbacks of the traditional dynamic MADM approaches. The proposed model was realized in a real-life personnel promotion problem. The proposed IT2F functions approach had improved the performance of IT2F regression by successfully capturing the trends and patterns in the historical records.

The advantage of the proposed model lies in its data-driven character, that is to say, the past data is used to guide preference elicitation processes. In classical MADM approaches, the decision makers are asked to fill out the entire decision matrix, which can be time consuming and cognitively demanding. On the other hand, the proposed model generates an IT2F decision matrix based on past data and allows decision makers to alter automatically generated decision matrix by using linguistic terms. Fuzziness in both the linguistic expressions and the past data were modeled by the proposed approach.

The main limitation of the presented study was that the results produced by the proposed model were highly dependent on the data obtained from the case company. Accordingly, results were difficult to be generalized. Secondly, prediction performance highly depends on available data. Nevertheless, the presented study can be improved in terms of many aspects. Hyper-parameter optimization can be used to tune the parameters of the model automatically. Different strategies can also be imposed to generate time series weights, and results can be compared. Last but not least, different MADM approaches can be used to rank the alternatives. In future studies, these considerations will be at the top of our agenda.

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