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Sums of A Pair of Orthogonal Frames

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Abstract: Frames are more stable as compared to bases under the action of a bounded linear operator. Sums of different frames under the action of a bounded linear operator are studied with the help of analysis, synthesis and frame operators. A simple construction of frames from the existing ones under the action of such an operator is presented here. It is shown that a frame can be added to its alternate dual frames, yielding a new frame. It is also shown that the sum of a pair of orthogonal frames is a frame. This provides an easy construction of a frame where the frame bounds can be computed easily. Moreover, for a pair of orthogonal frames, the necessary and sufficient condition is presented for their alternate dual frames to be orthogonal. This allows for an easy construction of a large number of new frames.

Keywords: frames; orthogonal frames; canonical dual; alternate dual

MSC: 42C15

1. Introduction

Frames are alternatives to a Riesz or orthonormal basis in Hilbert spaces. Frame theory plays an important role in signal processing, image processing, data compression and many other applied areas. Ole Christensen's book [1] provides a good source of theory of frames and its applications. Constructing frames and their dual frames has always been a critical point in applications. Frames are associated with operators. The properties of those operators can be found in [2] and orthogonality of frames can be found in [3–6]. Sums of frames are studied in [7,8]. Sums considered in this paper consist of frames and dual frames. The sum of a pair of orthogonal frames is a frame and also the frame bounds are shown to be easy to compute. Therefore, the orthogonality of a pair of frames plays an important role in this setting and hence we characterize the orthogonality of alternate dual frames in order to obtain new frames as a sum. This allows the construction of a large number of frames from the given ones.

Note: throughout this paper the sequence of scalars will be denoted by $(c_j)_j$ and the sequence of vectors will be denoted by $\{x_j\}_j$. Operators involved are linear and the space $\mathbb{H}$ is separable.

Let $\mathbb{H}$ be a separable Hilbert space and let $\mathbb{J}$ be a countable index set. A sequence $\mathbb{X} = \{x_j\}_j$, $j \in \mathbb{J}$, in $\mathbb{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that for all $f \in \mathbb{H}$,

$$\sum_j |\langle f, x_j \rangle|^2 \leq B||f||^2.$$

$\mathbb{X}$ is said to be a frame if there exist constants $0 < A \leq B$ such that for all $f \in \mathbb{H}$,

$$A||f||^2 \leq \sum_j |\langle f, x_j \rangle|^2 \leq B||f||^2.$$

A and B are called the frame bounds. $\mathbb{X}$ is called a tight frame or an A tight frame if $A = B$. It's called a normalized tight frame if $A = B = 1$. If $\mathbb{X}$ is an orthonormal basis, it is a normalized tight frame. The left inequality in the definition of a frame implies that the sequence $\mathbb{X}$ is complete i.e., $\langle f, x_j \rangle = 0$ for all x_j implies $f = 0$. This implies that $\overline{span(\mathbb{X})} = \mathbb{H}$.

A complete sequence $\mathbb{X}$ in a Hilbert space $\mathbb{H}$ is a Riesz basis [2] if there exist constants $0 < A \leq B$, such that for all finite sequences (c_j),

$$A \sum_j |c_j|^2 \leq ||\sum_j c_j x_j||^2 \leq B \sum_j |c_j|^2.$$

It turns out that it is precisely the image of an orthonormal basis under the action of a bounded bijective operator in a Hilbert space [1] (Chapter 3).

Let $\mathbb{X}$ be a Bessel sequence. The analysis and synthesis operators, denoted respectively by $T^*_{\mathbb{X}} : \mathbb{H} \to l_2(\mathbb{J})$ and $T_{\mathbb{X}} : l_2(\mathbb{J}) \to \mathbb{H}$, are defined respectively by

$$T^*_{\mathbb{X}} : f \to (\langle f, x_j \rangle)_j;$$

and

$$T_{\mathbb{X}} : (c_j)_j \to \sum_{j \in \mathbb{J}} c_j x_j.$$

The analysis operator is actually the Hilbert space adjoint operator to the synthesis operator. These operators are well defined and bounded because $\mathbb{X}$ is a Bessel sequence [1] (Lemma 5.2.1). It turns out that $\mathbb{X}$ is a frame if and only if the analysis operator is injective. Also, it is a frame if and only if the synthesis operator is surjective [2] (Proposition 4.1, 4.2).

The frame operator, denoted by $S_{\mathbb{X}}$, is defined by $S_{\mathbb{X}} := T_{\mathbb{X}} T^*_{\mathbb{X}} : \mathbb{H} \to \mathbb{H}$, and is given by

$$S_{\mathbb{X}} f = \sum_{j \in \mathbb{J}} \langle f, x_j \rangle x_j. \tag{1}$$

It is known that if $\mathbb{X}$ is a frame, the series (1) converges unconditionally, the operator $S_{\mathbb{X}}$ is bounded, self adjoint, positive and has a bounded inverse [1] (Lemma 5.1.5). Thus we have the following reconstruction formula,

$$f = \sum_{j \in \mathbb{J}} \langle f, S^{-1}_{\mathbb{X}} x_j \rangle x_j = \sum_{j \in \mathbb{J}} \langle f, x_j \rangle S^{-1}_{\mathbb{X}} x_j. \tag{2}$$

Let $\mathbb{Y} = \{y_j\}_j$ be another Bessel sequence in $\mathbb{H}$. If the operator $S_{\mathbb{X},\mathbb{Y}} := T_{\mathbb{X}} T^*_{\mathbb{Y}}$ given by

$$T_{\mathbb{X}} T^*_{\mathbb{Y}} f = \sum_{j \in \mathbb{J}} \langle f, y_j \rangle x_j$$

is an identity, then the Bessel sequences $\mathbb{X}$ and $\mathbb{Y}$ are actually frames and are called dual frames [6]. In this case, the reconstruction formula takes the form

$$f = \sum_{j \in \mathbb{J}} \langle f, y_j \rangle x_j.$$

So it follows from (2) that the sequence $S^{-1}_{\mathbb{X}}(\mathbb{X}) := \{S^{-1}_{\mathbb{X}}(x_j)\}_j$, $j \in \mathbb{J}$ is a dual frame to $\mathbb{X}$, called the canonical dual frame. Besides the canonical dual, a frame has many dual frames known as alternate dual frames.

Two Bessel sequences $\mathbb{X}$ and $\mathbb{Y}$ in a Hilbert space $\mathbb{H}$ are said to be orthogonal [3–6] if $ran(T^*_{\mathbb{X}}) \perp ran(T^*_{\mathbb{Y}})$. This is equivalent to

$$S_{\mathbb{Y},\mathbb{X}} = T_{\mathbb{Y}}T^*_{\mathbb{X}} = 0 \text{ or to } S_{\mathbb{X},\mathbb{Y}} = T_{\mathbb{X}}T^*_{\mathbb{Y}} = 0.$$

These are equivalent to

$$\sum_{j\in\mathbb{J}} \langle f, x_j\rangle y_j = 0 \text{ for all } f \in \mathbb{H} \text{ or to } \sum_{j\in\mathbb{J}} \langle f, y_j\rangle x_j = 0 \text{ for all } f \in \mathbb{H}.$$

The sum of a pair of frames is not always a frame [9] (Proposition 6.6). For example, simply consider the sum $\mathbb{X} + (-\mathbb{X})$ to see that the sum of the frames is not a frame. Moreover, here are two examples that motivate the work of this paper.

Example 1. *Let $\mathbb{X} = \{e_j\}_j$ be an orthonormal basis for a Hilbert space $\mathbb{H}$. Let L be a shift operator defined by $L(e_j) = e_{j-1},\ j > 1$, and $L(e_1) = 0$. Then $L(\mathbb{X})$is a frame for $\mathbb{H}$ but L is not invertible.*

Example 2. *Let $f = \kappa_{[0,1]}$ and $g = \kappa_{[1,2]}$. Then $\{E_m T_n f\}_{m,n\in\mathbb{Z}}$ and $\{E_m T_n g\}_{m,n\in\mathbb{Z}}$ form frames for $L_2(\mathbb{R})$ (in fact, they are orthonormal basis), where $E_m f(t) = e^{2\pi imt} f(t)$, and $T_n f(t) = f(t-n)$. But the sum $\{E_m T_n (f+g)\}_{m,n\in\mathbb{Z}}$ fails to be a frame [8].*

Some known results about sums being frames are provided in Section 2. Section 2.1 provides conditions under which the sum of a pair of frames is a frame. In Section 2.2, in particular, it is shown that the sum of an orthogonal pair of frames is a frame and also the frame bounds are given. We provide an easy proof of this through the use of analysis and synthesis operators (Theorem 1). This improves the result presented in [8] (Proposition 3.1). In addition to that more sums under the action of a surjective operator are also provided. Moreover, it is shown that a frame can be added to its alternate dual frames to get a new frame (Theorem 3). An easy way to construct frames from sums is to add a pair of orthogonal frames. It is known that the canonical dual frames of a pair of orthogonal frames are orthogonal [10]. Alternate dual frames of a pair of orthogonal frames need an extra condition to be an orthogonal pair. This condition is provided here (Theorem 2). This generalizes the results provided in [10] (Lemma 2 and 3). This provides a large number of a pair orthogonal frames which can be added to get new frames. Some examples are provided in support of the results.

2. Sums of Frames

Frames are considerably more stable than the basis upon the action of operators [1]. For example let $L : \mathbb{H} \to \mathbb{H}$ be a bounded operator, $\mathbb{X} = \{x_i\}_i$, $i \in \mathbb{J}$, be an orthonormal basis, and let $L(\mathbb{X}) := \{L(x_i)\}_i$. Then $L(\mathbb{X})$ is an orthonormal basis for $\mathbb{H}$ if L is a unitary operator, $L(\mathbb{X})$ is a Riesz basis for $\mathbb{H}$ if L is a bounded bijective operator, $L(\mathbb{X})$ is a Bessel sequence in $\mathbb{H}$ if L is a bounded operator, $L(\mathbb{X})$ is a frame sequence (a frame sequence is a frame for its span) for $\mathbb{H}$ if L is a bounded operator with closed range, and $L(\mathbb{X})$ is a frame for $\mathbb{H}$ if L is a bounded surjective operator.

The operators associated with a frame are useful in the study of frames [2,3]. Let L_1 and L_2 be two bounded operators on the Hilbert space $\mathbb{H}$. Sums of Gabor frames are studied in [8]. The authors of [7] provide a condition under which the sequences $\mathbb{X} + L(\mathbb{X})$ and $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ form a Riesz basis for the space $\mathbb{H}$, where $\mathbb{X}$ and $\mathbb{Y}$ are Bessel sequences. In [9], the authors study sums of frame sequences in a Hilbert space that are strongly disjoint, disjoint, a complementary pair, and weekly disjoint. The authors in [11] study sums of frames under the same conditions.

Let $\mathbb{X}$ be a frame for $\mathbb{H}$, with frame bounds A and B and L be a bounded surjective operator. Then $L(\mathbb{X})$ is a frame for $\mathbb{H}$ with the frame bounds $A||L^{\dagger}||^{-2}$ and $B||L||^2$, where $L^{\dagger}$ is the pseudo-inverse of L [7]. The associated analysis, synthesis and frame operators of $L(\mathbb{X})$ are given by the following lemma.

Lemma 1. *The analysis, synthesis and frame operators for the frame $L(\mathbb{X})$ are given by $T^*_{\mathbb{X}}L^*$, $LT_{\mathbb{X}}$, and $LS_{\mathbb{X}}L^*$ respectively.*

Proof. Simple calculations

$$\begin{aligned} T^*_{L(\mathbb{X})}f &= (\langle f, L(x_i)\rangle)_i = (\langle L^*f, x_i\rangle)_i = T^*_{\mathbb{X}}L^*f, \\ T_{L(\mathbb{X})}(c_k)_k &= \sum_k c_k L(x_k) = \sum_k L(c_k x_k) = LT_{\mathbb{X}}(c_k)_k, \text{ and} \\ S_{L(\mathbb{X})}f &= \sum_k \langle f, L(x_k)\rangle L(x_k) = L\sum_k \langle L^*f x_k\rangle x_k = LS_{\mathbb{X}}L^*f \end{aligned}$$

establish the lemma. □

Since the analysis operator is injective, $T^*_{\mathbb{X}}L^*$ is injective and $LT_{\mathbb{X}}$ is surjective. So it follows that $L(\mathbb{X})$ is a frame iff L is surjective as in [7] (Proposition 2.3). Moreover, the sequences $L(\mathbb{X})$ and $L^*(\mathbb{X})$ are both frames iff the operator L is invertible. It is shown that the sequence $\mathbb{X} + L(\mathbb{X})$ is a frame iff the operator $(I + L)$ is invertible [8] (Proposition 2.1), however it turns out to be the case when the operator is simply surjective. Calculations similar to the ones in Lemma 1 prove the following lemma [7,8].

Lemma 2. *The analysis, synthesis, and the frame operators for the frame $\mathbb{X} + L(\mathbb{X})$ are given by $T^*_{\mathbb{X}}(I + L^*)$, $(I + L)T_{\mathbb{X}}$, and $(I + L)S_{\mathbb{X}}(I + L^*)$ respectively.*

This lemma and the remarks before the previous lemma reveal that the frame bounds for the frame $\mathbb{X} + L(\mathbb{X})$ are $A||(I + L)^{\dagger}||^{-2}$ and $B||I + L||^2$. A special case of the above lemma is that $\{\mathbb{X} + S_{\mathbb{X}}(\mathbb{X})\}$ is a also a frame. To each frame $\mathbb{X}$, there is a naturally associated tight frame $S^{-1/2}_{\mathbb{X}}(\mathbb{X})$, known as canonical Parseval frame. The system $\{\mathbb{X} + S^{-1}_{\mathbb{X}}(\mathbb{X})\}$, where the given frame is being added to its canonical dual and the system $\{\mathbb{X} + S^{-1/2}_{\mathbb{X}}(\mathbb{X})\}$, where the frame is being added to its canonical Parseval frame, are all frames.

For the sum to be a Riezs basis, the following proposition is taken from [7] (Proposition 2.8).

Proposition 1. *Let $L : \mathbb{H} \to \mathbb{H}$ be a bounded operator and $\mathbb{X}$ be a Riesz basis for $\mathbb{H}$, where $T^*_{\mathbb{X}}, T_{\mathbb{X}}, S_{\mathbb{X}}$ are respectively the analysis, synthesis and frame operators with Riesz basis bounds $A \leq B$. Then $\mathbb{X} + L(\mathbb{X})$ is a Riesz basis for $\mathbb{H}$ with bounds $A||(I + L)^{-1}||^{-2}$ and $B||I + L||^2$ iff $I + L$ is invertible on $\mathbb{H}$.*

Proof. Since $\mathbb{X}$ is a Riesz basis, $T^*_{\mathbb{X}}$ is an invertible operator. If $I + L$ is invertible, then the operator $T^*_{\mathbb{X}}(L^* + I)$ is also invertible. But this is the analysis operator for the sequence $I + L(\mathbb{X})$. Hence, the sequence $\mathbb{X} + L(\mathbb{X})$ is a Riesz basis. If $\mathbb{X} + L(\mathbb{X})$ is a Riesz basis, then the analysis operator $T^*_{\mathbb{X}}(L^* + I)$ is invertible. Since $T^*_{\mathbb{X}}$ is invertible, so is the operator $I + L$. □

The following proposition, mentioned incorrectly in [8] (Proposition 3.1), is corrected in [7] (Proposition 2.12).

Proposition 2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Bessel sequences in $\mathbb{H}$ with analysis operators $T^*_{\mathbb{X}}$, $T^*_{\mathbb{Y}}$ and frame operators $S_{\mathbb{X}}, S_{\mathbb{Y}}$ respectively. Let $L_1, L_2 : \mathbb{H} \to \mathbb{H}$ be bounded operators. Then the following statements are equivalent.*

(A) $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ *is a Riesz basis for* $\mathbb{H}$.

(B) $T^*_{\mathbb{X}}L^*_1 + T^*_{\mathbb{Y}}L^*_2$ *is an invertible operator on* $\mathbb{H}$.

A sum of two frames is not a frame in general. The fact that a sequence is a frame is equivalent to its analysis operator being injective or the synthesis operator being surjective [3]. The following proposition provides condition under which the sum in the above proposition is a frame.

Proposition 3. *Let L_1 and L_2 be bounded operators, and $\mathbb{X}$ and $\mathbb{Y}$ be Bessel sequences in a Hilbert space $\mathbb{H}$. Then the following statements are equivalent.*

(A) $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ *is a frame.*
(B) $L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}}$ *is surjective.*

The frame operator is given by $S = L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^* + L_1 T_{\mathbb{X}} T_{\mathbb{Y}}^* L_2^* + L_2 T_{\mathbb{Y}} T_{\mathbb{X}}^* L_1^*$.

Proof. The synthesis operator for the sequence $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is $L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}}$. It therefore follows that (A) and (B) are equivalent. □

Corollary 1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Bessel sequences. Then the following are equivalent.*

(A) $\mathbb{X} + \mathbb{Y}$ *is a frame.*
(B) $T_{\mathbb{X}} + T_{\mathbb{Y}}$ *is surjective.*

The frame operator is given by $S = S_{\mathbb{X}} + S_{\mathbb{Y}} + T_{\mathbb{X}} T_{\mathbb{Y}}^* + T_{\mathbb{Y}} T_{\mathbb{X}}^*$.

It is still difficult to verify the conditions of Proposition 3 or its corollary. The sum happens to be a frame if we impose an extra condition of orthogonality of the sequences. Assuming the Bessel sequences to be orthogonal, the following proposition is easily established.

Proposition 4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Bessel sequences such that the frame operator $S_{\mathbb{X},\mathbb{Y}}$ is a zero operator. Let $L_1, L_2 : \mathbb{H} \to \mathbb{H}$ be bounded operators, and let $\mathbb{Z} = L_1(\mathbb{X}) + L_2(\mathbb{Y})$. Then the following statements are equivalent.*

(A) $\mathbb{Z}$ *is a frame for* $\mathbb{H}$.

(B) $L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}}$ *is surjective.*

(C) $L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^*$ *is an invertible positive operator on* $\mathbb{H}$.

Proof. Since $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal frames, we have

$$T_{\mathbb{Y}} T_{\mathbb{X}}^* = 0 = T_{\mathbb{X}} T_{\mathbb{Y}}^*,$$

i.e., the frame operator $S_{\mathbb{X},\mathbb{Y}} = 0$. The analysis operator for the sequence $\mathbb{Z}$ is $T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*$, and the frame operator $S_{\mathbb{Z}}$ is given by

$$\begin{aligned} S_{\mathbb{Z}} &= (L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}})(T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*) \\ &= L_1 T_{\mathbb{X}} T_{\mathbb{X}}^* L_1^* + L_1 T_{\mathbb{X}} T_{\mathbb{Y}}^* L_2^* + L_2 T_{\mathbb{Y}} T_{\mathbb{X}}^* L_1^* + L_2 T_{\mathbb{Y}} T_{\mathbb{Y}}^* L_2^* \\ &= L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^*. \end{aligned}$$

$(A) \Leftrightarrow (C)$. Let $\mathbb{Z}$ be a frame. Then its frame operator $S_{\mathbb{Z}}$ in an invertible and positive operator. So (C) follows. It is straightforward to show that (C) implies (A). $(A) \Leftrightarrow (B)$ because the synthesis operator of a frame is surjective. □

In fact, one of operators L_1 or L_2 being surjective is enough, as the following Lemma states.

Lemma 3. *If $\mathbb{X}$ and $\mathbb{Y}$ are a pair of orthogonal frames and if either L_1 or L_2 is surjective, then $L_1T_{\mathbb{X}} + L_2T_{\mathbb{Y}}$ is surjective.*

Proof. Since $\mathbb{X}$ is a frame, the operator $T_{\mathbb{X}}T_{\mathbb{X}}^*$ is invertible and since $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal, we have $T_{\mathbb{X}}T_{\mathbb{Y}}^* = T_{\mathbb{Y}}T_{\mathbb{X}}^* = 0$. Let L_1 be surjective. Then for each $f \in \mathbb{H}$ there exists a $g \in \mathbb{H}$ such that $f = L_1g$. Let $(c_j)_j \in l_2(\mathbb{J})$ be such that $(c_j)_j = T_{\mathbb{X}}^*(T_{\mathbb{X}}T_{\mathbb{X}}^*)^{-1}(g)$. But then,

$$\begin{aligned}
(L_1T_{\mathbb{X}} + L_2T_{\mathbb{Y}})(c_j)_j &= (L_1T_{\mathbb{X}} + L_2T_{\mathbb{Y}})T_{\mathbb{X}}^*(T_{\mathbb{X}}T_{\mathbb{X}}^*)^{-1}(g) \\
&= L_1T_{\mathbb{X}}T_{\mathbb{X}}^*(T_{\mathbb{X}}T_{\mathbb{X}}^*)^{-1}(g) \\
&= L_1(g) \\
&= f.
\end{aligned}$$

So the operator $L_1T_{\mathbb{X}} + L_2T_{\mathbb{Y}}$ is surjective. □

2.1. Sums of Orthogonal Frames

Let $\mathbb{X}$ and $\mathbb{Y}$ be a pair of orthogonal frames. Let A_1 and A_2 be the lower and B_1 and B_2 be the upper frame bounds for the frames $\mathbb{X}$ and $\mathbb{Y}$ respectively. Then the following theorem provides a frame as a sum of two given frames and also provides the frame bounds.

Theorem 1. *If the pair $\mathbb{X}$ and $\mathbb{Y}$ is orthogonal and if one of L_1 or L_2 is surjective, then $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is a frame whose frame operator is $L_1S_{\mathbb{X}}L_1^* + L_2S_{\mathbb{Y}}L_2^*$, the upper bound is $B_1||L_1||^2 + B_2||L_2||^2$, and the lower bound is $A_i||L_i^{\dagger *}||^{-2}$ where $L_i (i = 1, 2)$ is surjective.*

Proof. Proposition 4 and Lemma 3 are enough for the above sum to be a frame. The bounds can be computed too. Let L_1 be surjective. We note that, $T_{\mathbb{X}}(e_i) = \{x_i\}_i$ and $T_{\mathbb{Y}}(e_i) = \{y_i\}_i$. Also, since $\mathbb{X}$ and $\mathbb{Y}$ are frames, we have

$$||T_{\mathbb{X}}^*f||^2 \geq A_1||f||^2, \text{ and } ||T_{\mathbb{Y}}^*f||^2 \geq A_2||f||^2 \text{ for all } f \in \mathbb{H}, \text{ and we also have, } T_{\mathbb{X}}T_{\mathbb{Y}}^* = 0 = T_{\mathbb{Y}}T_{\mathbb{X}}^*.$$

$$\begin{aligned}
\sum_{i\in\mathbb{J}} |\langle f, L_1(x_i) + L_2(y_i)\rangle|^2 &= \sum_{i\in\mathbb{J}} |\langle L_1^*f, x_i\rangle + \langle L_2^*f, y_i\rangle|^2 \\
&= \sum_{i\in\mathbb{J}} |\langle L_1^*f, T_{\mathbb{X}}(e_i)\rangle + \langle L_2^*f, T_{\mathbb{Y}}(e_i)\rangle|^2 \\
&= \sum_{i\in\mathbb{J}} |\langle (T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f, e_i\rangle|^2 \\
&= ||(T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f||^2 \\
&= \langle (T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f, (T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f\rangle \\
&= \langle T_{\mathbb{X}}^*L_1^*f, T_{\mathbb{X}}^*L_1^*f\rangle + \langle T_{\mathbb{Y}}^*L_2^*f, T_{\mathbb{Y}}^*L_2^*f\rangle \\
&= ||T_{\mathbb{X}}^*L_1^*f||^2 + ||T_{\mathbb{Y}}^*L_2^*f||^2 \\
&\geq A_1||L_1^*f||^2 + A_2||L_2^*f||^2 \\
&\geq A_1||L_1^*f||^2 \\
&\geq A_1||f||^2||L_1^{\dagger *}||^{-2},
\end{aligned}$$

since $L_1L_1^\dagger = I$, $L_1^{\dagger *}L_1^* = I$, we have

$$||f|| = ||L_1^{\dagger *}L_1^*f|| \leq ||L_1^{\dagger *}||\,||L_1^*f||, \text{ and } ||L_1^*f|| \geq ||f|| \, ||L_1^{\dagger *}||^{-1}.$$

Note: if both L_1 and L_2 are surjective, then larger of $||f|| \, ||L_1^{\dagger *}||^{-1}$ and $||f|| \, ||L_2^{\dagger *}||^{-1}$ serves as the lower bound. For the upper bound,

$$\begin{aligned} \sum_{i \in \mathbb{J}} |\langle f, L_1(x_i) + L_2(y_i) \rangle|^2 &= ||T_{\mathbb{X}} L_1^* f||^2 + ||T_{\mathbb{Y}} L_2^* f||^2 \\ &\leq B_1 ||L_1^* f||^2 + B_2 ||L_2^* f||^2 \\ &\leq (B_1 ||L_1^*||^2 + B_2 ||L_2^*||^2) ||f||^2. \end{aligned}$$

So the upper bound is $B_1 ||L_1||^2 + B_2 ||L_2||^2$. □

In particular if $L_1 = I$, the sum $\mathbb{X} + L_2(\mathbb{X})$ is a frame iff $(I + L_2)T_{\mathbb{X}}$ is surjective. In addition, if $L_1 = L_2 = I$ in Theorem 1, then the frame operator is simply $S_{\mathbb{X}} + S_{\mathbb{Y}}$. We can also obtain a Parseval frame as a sum as the following corollary suggests.

Corollary 2. *If $\mathbb{X}$ and $\mathbb{Y}$ are a pair of orthogonal Parseval frames, the sum $\mathbb{Z} = \mathbb{X} + \mathbb{Y}$ is a Parseval frame if and only if the operators L_1 and L_2 are scaled unitary operators i.e $L_1 = \frac{U_1}{\sqrt{2}}$, and $L_2 = \frac{U_2}{\sqrt{2}}$ where U_1 and U_2 are unitary operators.*

The frame operator in this case is

$$S_{\mathbb{Z}} = L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^* = \frac{U_1 U_1^*}{2} + \frac{U_2 U_2^*}{2} = \frac{I}{2} + \frac{I}{2} = I.$$

This generalizes to any finite sum.

Corollary 3. *Let $\mathbb{X}_1, \cdots, \mathbb{X}_k$ be pairwise orthogonal Parseval frames. Let $U_1, \cdots U_k$ be unitary operators. Then the sum $\mathbb{Z} = L_1(\mathbb{X}_1) + \cdots + L_k(\mathbb{X}_k)$ is a Parseval frame, where $L_i = \frac{U_i}{\sqrt{k}}$.*

The following example takes a pair of orthogonal frames for $l_2(\mathbb{J})$ from [12].

Example 3. *A sum of discrete Gabor frames in $l_2(\mathbb{J})$.*

Let $\{e_i\}$ be the standard orthonormal basis for $\mathbb{H} = l_2(\mathbb{J})$. Let $g = \frac{1}{\sqrt{3}}(e_1 + e_2)$, $h = \frac{1}{\sqrt{3}}(e_3 + e_4)$, let $g(n)$ denote the nth coordinate of g, and let

$$g_{k,m}(n) := e^{\frac{2\pi i k n}{3}} g(n - 2m), \text{ and } h_{k,m}(n) := e^{\frac{2\pi i k n}{3}} h(n - 2m).$$

Here $g_{k,m}$ is the sequence in $l_2(\mathbb{J})$ whose n^{th} coordinate is $e^{\frac{2\pi i k n}{3}} g(n - 2m)$. Likewise for the system $h_{k,m}$. Then the systems $\{g_{k,m} : 0 \leq k \leq 2, m \in \mathbb{Z}\}$ and $\{h_{k,m} : 0 \leq k \leq 2, m \in \mathbb{Z}\}$ form Parseval frames for the space $\mathbb{H}$, since $\mathbb{H} = \oplus_{m \in \mathbb{Z}} M_m$ is the orthogonal direct sum of $M_m = span\{e_{1+2m}, e_{2+2m}\}$ and for each fixed m the system $\{g_{k,m}\}_{k=0}^2$ is a Parseval frame for M_m [12]. Similar is the case for the system $\{h_{k,m}\}_{k=0}^2$. It turns out that the two systems form an orthogonal pair of frames for $\mathbb{H}$ [12] (Theorem 1.4). The above corollary implies that the sum $s = \frac{1}{\sqrt{2}} g + \frac{1}{\sqrt{2}} h = \frac{1}{\sqrt{6}}(e_1 + e_2 + e_3 + e_4)$, provides a Parseval frame for $\mathbb{H}$ as well, i.e, the system

$$s_{k,m}(n) = e^{\frac{\pi i k n}{3}} s(n - 2m), \; 0 \leq k \leq 5, \; m \in \mathbb{Z}$$

forms a Parseval frame for $\mathbb{H}$. This can be verified by the argument from [12] (Example 1.3).

Example 4. *A sum of Gabor frames in $L_2(\mathbb{R})$.*

For $x, y \in \mathbb{R}$, let E_x, and T_y be operators defined on $L_2(\mathbb{R})$ by

$$E_x(f(t)) = e^{2\pi ixt} f(t) \quad \text{and} \quad T_y(f(t)) = f(t-y).$$

Since the polynomial $1 + pz$ doesn't have root on the unit circle for $p \leq 1$, the set $[0,1) \cup [1,2) = [0,2)$ forms a Gabor frame wavelet set [13,14]. Likewise, the set $[-2,-1) \cup [-1,0) = [-2,0)$ forms a Gabor frame wavelet set. Let

$$g_1(t) = \chi_{[0,1)} + p\chi_{[1,2)}, \text{ and } g_2(t) = \chi_{[-1,0)} + p\chi_{[-2,-1)}.$$

The families

$$\mathbb{X} = \{E_m T_n g_1(t)\}_{m,n\in\mathbb{Z}}, \text{ and } \mathbb{Y} = \{E_m T_n g_2(t)\}_{m,n\in\mathbb{Z}}$$

form frames for the space $L_2(\mathbb{R})$.

Since the $support(\mathbb{X}) \cap support(\mathbb{Y}) = \emptyset$ for all $m, n \in \mathbb{Z}$, it follows that for all $f \in L_2(\mathbb{R})$, we have

$$\sum_{m,n\in\mathbb{Z}} \langle f(t), E_m T_n g_1(t)\rangle \, E_m T_n g_2(t) = 0,$$

So $\mathbb{X}$ and $\mathbb{Y}$ form a pair of orthogonal frames for the space $L_2(\mathbb{R})$. Therefore the sum

$$h(t) = g_1(t) + g_2(t) = \chi_{[-1,0)} + p\chi_{[-2,1)} + \chi_{[0,1)} + p\chi_{[1,2)}$$

forms a frame for $L_2(\mathbb{R})$.

Lemma 4. *Let $\tilde{\mathbb{X}}$ be the dual frame of $\mathbb{X}$. Then $L_1^{\dagger *}(\tilde{\mathbb{X}})$ is a dual to $L_1(\mathbb{X})$, where L_1 is a surjective operator.*

Proof. Since L_1 is surjective, the operator $L_1 L_1^*$ is invertible. Let $L_1^\dagger = L_1^*(L_1 L_1^*)^{-1}$. So

$$S_{L_1(\mathbb{X}), L_1^{\dagger *}(\tilde{\mathbb{X}})} = L_1 T_{\mathbb{X}} T_{\tilde{\mathbb{X}}}^* L^\dagger = L_1 L_1^\dagger = I,$$

and

$$S_{L_1^{\dagger *}(\tilde{\mathbb{X}}), L_1(\mathbb{X})} = L_1^{\dagger *} T_{\tilde{\mathbb{X}}} T_{\mathbb{X}}^* L_1^* = L_1^{\dagger *} L_1^* = I.$$

This completes the proof. □

If the operator L is invertible, we have the following result.

Corollary 4. *Let $\tilde{\mathbb{X}}$ be a dual frame of $\mathbb{X}$. Then $L_1^{-1^*}(\tilde{\mathbb{X}})$ is dual to $L_1(\mathbb{X})$.*

As a consequence of Lemma 1, we have the following theorem for a pair of orthogonal frames, where the operator L is assumed to be surjective.

Corollary 5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be a pair of orthogonal frames for $\mathbb{H}$. Then the frames $L(\mathbb{X})$ and $L(\mathbb{Y})$ are orthogonal too.*

Proof. From Lemma 1, it follows that the frame operator $S_{L(\mathbb{X}),L(\mathbb{Y})}$ is given by

$$S_{L(\mathbb{X}),L(\mathbb{Y})} = L T_{\mathbb{X}} T_{\mathbb{Y}}^* L^* = 0.$$

So $L(\mathbb{X})$ and $L(\mathbb{Y})$ are orthogonal. □

The following is proved in [11], but Lemma 1 provides a very simple proof.

Corollary 6. *Let $\mathbb{X}_1, \mathbb{X}_2 \cdots \mathbb{X}_k$ be pairwise orthogonal frames. If L_i is surjective, then $L_i^{\dagger *}(\mathbb{X}_i)$ is dual to $L_1(\mathbb{X}_1) + L_2(\mathbb{X}_2) + \cdots + L_k(\mathbb{X}_k)$.*

Proof. Use of Lemma 1 establishes this. The synthesis operator of the sum is $L_1\mathbb{X}_1 + \cdots L_k\mathbb{X}_k$, and the analysis operator of $L_i^{\dagger *}(\mathbb{X}_i)$ is $T^*_{\mathbb{X}_i} L_i^\dagger$, it turns out that the composition is

$$(L_1\mathbb{X}_1 + \cdots L_k\mathbb{X}_k)T^*_{\mathbb{X}_i} L_i^\dagger = L_i T_{\mathbb{X}_i} T^*_{\mathbb{X}_i} L_i^\dagger = L_i L_i^\dagger = I.$$

In general, $L_i^{\dagger *}(\mathbb{X}_i)$ is dual to $L_i(\mathbb{X}_i) + \sum_{j \neq i} d_j L_j(\mathbb{X}_j)$, where $d_j = 0$ or 1. □

2.2. Orthogonality of Alternate Dual Frames

Alternate dual frames of a frame $\mathbb{X}$ are given by $\tilde{\mathbb{X}} = \{S_{\mathbb{X}}^{-1}(x_j) + \psi^*(e_j)\}_j$, where $\psi \in B(\mathbb{H}, l_2(\mathbb{J}))$ (the space of bounded linear operators) such that $T_{\mathbb{X}}\psi = 0$, and $\{e_j\}_{j\in\mathbb{J}}$ is the standard orthonormal basis of $l_2(\mathbb{J})$ [15]. It is also known that $\{\psi^*(e_j)\}_{j\in\mathbb{J}}$ is a Bessel sequence in $\mathbb{H}$ [15]. The authors of [10] have studied the orthogonality of canonical dual frames of a pair of orthogonal frames. However, alternate dual frames of a pair of orthogonal frames need not be orthogonal. The following theorem establishes the conditions needed for the orthogonality of alternate dual frames of a pair of orthogonal frames.

Theorem 2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be a pair of orthogonal frames and Let $\tilde{\mathbb{X}} = \{S_{\mathbb{X}}^{-1}(x_j) + \psi^*(e_j)\}_j$, and $\tilde{\mathbb{Y}} = \{S_{\mathbb{Y}}^{-1}(y_j) + \phi^*(e_j)\}_i$ respectively be their corresponding alternate dual frames, where $\psi, \phi \in B(\mathbb{H}, l_2(\mathbb{J}))$ such that $T_{\mathbb{X}}\psi = 0$, and $T_{\mathbb{Y}}\phi = 0$. Then*

(A) The pair $\mathbb{X}$ and $\tilde{\mathbb{Y}}$ is orthogonal if and only if $T_{\mathbb{X}}\phi = 0$.
(B) The pair $\tilde{\mathbb{X}}$ and $\mathbb{Y}$ is orthogonal if and only if $T_{\mathbb{Y}}\psi = 0$.
(C) The pair $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ is orthogonal if $T_{\mathbb{X}}\phi = 0$, $T_{\mathbb{Y}}\psi = 0$ and $\phi^*\psi = 0$.
(C′) If $\mathbb{X}$ is orthogonal to $\tilde{\mathbb{Y}}$ and $\tilde{\mathbb{X}}$ is orthogonal to $\mathbb{Y}$, then $\tilde{\mathbb{X}}$ is orthogonal to $\tilde{\mathbb{Y}}$ if and only if $\phi^*\psi = 0$.

Proof. (A) Let $\Psi := \{\psi^*(e_j)\}_{j\in\mathbb{J}}$, such that $T_{\mathbb{X}}\psi = 0$ for some $\psi \in B(\mathbb{H}, l_2(\mathbb{J}))$; $\Phi := \{\phi^*(e_j)\}_{j\in\mathbb{J}}$, for some $\phi \in B(\mathbb{H}, l_2(\mathbb{J}))$, such that $T_{\mathbb{Y}}\phi = 0$. Then

$$\tilde{\mathbb{X}} = \{S_{\mathbb{X}}^{-1}(\mathbb{X}) + \Psi\}, \text{ and } \tilde{\mathbb{Y}} = \{S_{\mathbb{Y}}^{-1}(\mathbb{Y}) + \Phi\}.$$

For each $f \in \mathbb{H}$, the sequence Ψ provides

$$T^*_{\Psi}(f) = (\langle f, \psi^*(e_j)\rangle)_j = (\langle \psi(f), e_j\rangle)_i = (\psi(f))_j,$$

and for each $(c_j) \in l_2(\mathbb{J})$,

$$T_{\Psi}(c_j) = \sum_i c_i \psi^*(e_i) = \sum_i \psi^*(c_i e_i) = \psi^*\left(\sum_i c_i e_i\right) = \psi^*(c_j).$$

(A) The frame operator $S_{\mathbb{X},\tilde{\mathbb{Y}}}$ is given by

$$S_{\mathbb{X},\tilde{\mathbb{Y}}} = T_{\mathbb{X}} T^*_{\tilde{\mathbb{Y}}} = T_{\mathbb{X}}(T^*_{\mathbb{Y}} S_{\mathbb{Y}}^{-1} + T_{\Phi}) = T_{\mathbb{X}} T^*_{\mathbb{Y}} S_{\mathbb{Y}}^{-1} + T_{\mathbb{X}} T^*_{\Phi} = T_{\mathbb{X}} T^*_{\Phi},$$

and since, $T_{\mathbb{X}} T^*_{\Phi} f = T_{\mathbb{X}}\phi(f)$, (A)is established.

(B) The frame operator $S_{\tilde{\mathbb{X}},\mathbb{Y}}$ is

$$S_{\tilde{\mathbb{X}},\mathbb{Y}} = T_{\tilde{\mathbb{X}}} T^*_{\mathbb{Y}} = (S_{\mathbb{X}}^{-1} T_{\mathbb{X}} + T_{\Psi}) T^*_{\mathbb{Y}} = S_{\mathbb{X}}^{-1} T_{\mathbb{X}} T^*_{\mathbb{Y}} + T_{\Psi} T^*_{\mathbb{Y}} = T_{\Psi} T^*_{\mathbb{X}}$$

and $T_\Psi T_\mathbb{X}^* f = \psi^* T_\mathbb{X}^* f$. Since $\langle \psi^* T_\mathbb{X}^* f, g \rangle = \langle f, T_\mathbb{X} \psi g \rangle$, it follows that $\psi^* T_\mathbb{X}^* f = 0$ for all f iff $\mathbb{T}_\mathbb{X} \psi = 0$. This establishes (B).

(C) We notice that

$$\begin{aligned} S_{\tilde{\mathbb{Y}},\tilde{\mathbb{X}}} &= (S_\mathbb{Y}^{-1} T_\mathbb{Y} + T_\Phi)(T_\mathbb{X}^* S_\mathbb{X}^{-1} + T_\Psi^*) \\ &= S_\mathbb{Y}^{-1} T_\mathbb{Y} T_\mathbb{X}^* S_\mathbb{X}^{-1} + S_\mathbb{Y}^{-1} T_\mathbb{Y} T_\Psi^* + T_\Phi T_\mathbb{X}^* S_\mathbb{X}^{-1} + T_\Phi T_\Psi^* \\ &= S_\mathbb{Y}^{-1} T_\mathbb{Y} T_\Psi^* + T_\Phi T_\mathbb{X}^* S_\mathbb{X}^{-1} + T_\Phi T_\Psi^* \\ &= S_\mathbb{Y}^{-1} T_\mathbb{Y} \psi + \phi^* T_\mathbb{X}^* S_\mathbb{X}^{-1} + \phi^* \psi, \end{aligned}$$

since $T_\Phi T_\Psi^* f = \phi^* \psi(f)$. So (C) follows using (A) and (B). (C′) follows from (A), (B) and (C). □

Corollary 7. *Let $\mathbb{X}$ and $\mathbb{Y}$ be orthogonal frames with canonical dual $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ respectively. Then the sum $\frac{\mathbb{X}+L(\mathbb{Y})}{\sqrt{2}}$ is a dual to $\frac{\tilde{\mathbb{X}}+L^{\dagger *}(\tilde{\mathbb{Y}})}{\sqrt{2}}$.*

Proof. Lemma 1 and Theorem 2 establish this. □

Let $\tilde{\mathbb{X}} = S_\mathbb{X}^{-1}(\mathbb{X}) + \Psi$ be an alternate dual to $\mathbb{X}$, such that $T_\mathbb{X} \psi = 0$ as in Theorem 2. We now show that a frame can be added to any of its alternate dual frames to yield a new frame.

Theorem 3. *The sum $\mathbb{Z} = \mathbb{X} + \tilde{\mathbb{X}}$ is a frame.*

Proof. Let $T_\mathbb{Z} = T_\mathbb{X} + S_\mathbb{X}^{-1} T_\mathbb{X} + T_\Psi$. From Proposition 4, it suffices to show that the operator $T_\mathbb{Z}$ is surjective. Since $S_\mathbb{X}$ is a positive operator, the operator $S_\mathbb{X} + I$ is invertible. Therefore for each $f \in \mathbb{H}$, there exists $g \in \mathbb{H}$ such that $(S_\mathbb{X} + I)(g) = f$. Let $T_\mathbb{X}^*(g) = (d_i)$. Now,

$$\begin{aligned} (T_\mathbb{X} + S_\mathbb{X}^{-1} T_\mathbb{X} + T_\Psi)(d_i) &= (T_\mathbb{X} + S_\mathbb{X}^{-1} T_\mathbb{X} + T_\Psi) T_\mathbb{X}^*(g) \\ &= T_\mathbb{X} T_\mathbb{X}^*(g) + S_\mathbb{X}^{-1} T_\mathbb{X} T_\mathbb{X}^*(g) + T_\Psi T_\mathbb{X}^*(g) \\ &= S_\mathbb{X}(g) + g + \psi^* T_\mathbb{X}^*(g) \\ &= (S_\mathbb{X} + I)(g) \\ &= f. \end{aligned}$$

This proves the theorem. □

3. Conclusions

Motivated by the earlier work on the sums of frames [7,8], an easy construction of a frame via a sum of frames is established with the aid of analysis and synthesis operators. It is also shown that a frame can be added to its alternate dual frame to yield a frame. A pair of orthogonal frames can be added to provide the sum as a frame as well. Therefore, a condition for the orthogonality of alternate dual frames for a pair of orthogonal frames is presented. Under this condition, many pairwise orthogonal frames can be constructed and their sum is always a frame. This enables us to construct a large number of frames and also allows us to compute the frame bounds.

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