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Pre-Dual of Fofana's Spaces

Hans G. Feichtinger [1,*] and Justin Feuto [2]

1 Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Patz 1, 1090 Wien, Austria
2 Laboratoire de Mathématiques Fondamentales, UFR Mathématiques et Informatique, Université Félix Houphouët-Boigny-Cocody, 22 B.P 1194 Abidjan, Ivory Coast; justfeuto@yahoo.fr
* Correspondence: hans.feichtinger@univie.ac.at

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Abstract: The purpose of this paper is to characterize the pre-dual of the spaces introduced by I. Fofana on the basis of Wiener amalgam spaces. These spaces have a specific dilation behaviour similar to the spaces $L^{\alpha}(\mathbb{R}^d)$. The characterization of the pre-dual will be based on the idea of minimal invariant spaces (with respect to such a group of dilation operators).

Keywords: amalgam spaces; Fofana spaces; pre-dual

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1. Introduction

Let d be a fixed positive integer and $\mathbb{R}^d$ the d-dimensional Euclidean space, equipped with its Lebesgue measure dx. For $1 \leq q, p \leq \infty$, the amalgam of L^q and L^p is the space (L^q, ℓ^p) of function $f : \mathbb{R}^d \to \mathbb{C}$, which are located in L^q, so that the sequence $\left\{\left\|f\chi_{I_k}\right\|_q\right\}_{k\in\mathbb{Z}^d}$ belongs to $\ell^p(\mathbb{Z}^d)$, where $I_k = k + [0,1)^d$. The map $f \mapsto \|f\|_p$ denotes the usual norm on the Lebesgue space $L^p(\mathbb{R}^d)$ on $\mathbb{R}^d$ while χ_E stands for the characteristic function of the subset E of $\mathbb{R}^d$.

Amalgam spaces were introduced by N. Wiener in 1926 (see [1]); however, their systematic study began with the work of F. Holland [2] in 1975. Since then, they have been widely studied (see [3–8] and the references therein). It is easy to see that the usual Lebesgue space L^q coincides with the amalgam space (L^q, ℓ^q). Furthermore, the Lebesgue space L^q is known to be invariant under dilations. In fact, for $\rho > 0$, the *dilation operator* $St_\rho^{(q)} : f \mapsto \rho^{-\frac{d}{q}} f(\rho^{-1}\cdot)$ is isometric. Proper amalgam spaces do not possess this property. Worse still, for $q \neq p$ we cannot find $\alpha > 0$, such that $\sup_{\rho>0}\left\|St_\rho^{(\alpha)} f\right\|_{q,p} < \infty$, although $St_\rho^{(\alpha)} f \in (L^q, \ell^p)$ for all $f \in (L^q, \ell^p)$, $\rho > 0$ and $\alpha > 0$ (see e.g., [9] or [10]). In order to compensate for this shortfall, in the 1980s, Fofana introduced (see [11]) the functions spaces $(L^q, \ell^p)^\alpha$, which consist of $f \in (L^q, \ell^p)$ satisfying $\sup_{\rho>0}\left\|St_\rho^{(\alpha)} f\right\|_{q,p} < \infty$ (see Section 2 for more precision).

These spaces can be viewed as certain generalized Morrey spaces, and we will always refer to them as *Fofana spaces*.

Many classical results for Lebesgue and the classical Morrey spaces have been extended to the setting of the spaces $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ (see [11–20]). Although the dual space of Lebesgue spaces $L^\alpha(\mathbb{R}^d)$ $(1 \leq \alpha < \infty)$ and amalgam spaces $(L^q, \ell^p)(\mathbb{R}^d)$ $(1 \leq q, p < \infty)$ are well known ($L^{\alpha'}$ and $(L^{q'}, \ell^{p'})$ respectively with $\frac{1}{p'} + \frac{1}{p} = 1$), that of $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ is still unknown. However, in the case $p = \infty$ and $q < \alpha$, four characterizations of the pre-dual of $(L^q, \ell^\infty)^\alpha(\mathbb{R}^d)$ (see [21–24]) which are equal have already been made.

The purpose of this paper is to determine the pre-dual of $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ for $1 < q < \alpha < p < \infty$.

In doing so, we will make use of the idea of minimal invariant Banach spaces of functions, which has already been used previously and has been shown to be useful in a variety of situations, see [25–29] or [30].

The paper is organized as follows: In Section 2, we give some basic facts about *amalgams* and minimal Banach spaces. The third section is devoted to the pre-dual of *Fofana spaces*, as well as certain properties of these spaces.

2. Some Basic Facts about Amalgam and Minimal Banach Spaces

For any normed space E, we denote its topological dual space as E^*. Given $1 \leq p, q \leq \infty$, the amalgam space (L^q, ℓ^p) is equipped with the norm $\|f\|_{q,p} = \left\| \left\{ \|f\chi_{I_k}\|_q \right\} \right\|_{\ell^p}$. For any $\rho > 0$, we put:

$$ {}_\rho\|f\|_{q,p} = \begin{cases} \left[\sum_{k\in\mathbb{Z}^d} \left\| f\chi_{I_k^\rho} \right\|_q^p \right]^{\frac{1}{p}} & \text{if} \quad p < \infty \\ \sup_{k\in\mathbb{Z}^d} \left\| f\chi_{I_k^\rho} \right\|_q & \text{if} \quad p = \infty \end{cases} \tag{1} $$

with $I_k^\rho = \Pi_{j=1}^d \left[k_j\rho, (k_j+1)\rho \right)$ if $k = (k_j)_{1\leq j\leq d} \in \mathbb{Z}^d$. It is clear that $\|f\|_{q,p} = {}_1\|f\|_{q,p}$.

We present the following well known properties (see, for example, [7]).

(1) For $0 < \rho < \infty$, $f \mapsto {}_\rho\|f\|_{q,p}$ is a norm on $(L^q, \ell^p)(\mathbb{R}^d)$ equivalent to $f \mapsto \|f\|_{q,p}$. With respect to these norms, the amalgam spaces $(L^q, \ell^p)(\mathbb{R}^d)$ are Banach spaces.

(2) The spaces (strictly) increase with the global exponent q and (strictly) decrease with a growing local exponent q; more precisely,

$$ \|f\|_{q,p} \leq \|f\|_{q_1,p} \quad \text{if} \quad q < q_1 \leq \infty, \tag{2} $$
$$ \|f\|_{q,p} \leq \|f\|_{q,p_1} \quad \text{if} \quad 1 \leq p_1 < p. \tag{3} $$

(3) For $0 < \rho < \infty$, Hölder's inequality is fulfilled:

$$ \|fg\|_1 \leq {}_\rho\|f\|_{q,p} \; {}_\rho\|g\|_{q',p'}, \quad f, g \in L^1_{loc}(\mathbb{R}^d), \tag{4} $$

where q' and p' are conjugate exponents of q and p, respectively: $\frac{1}{q} + \frac{1}{q'} = 1 = \frac{1}{p} + \frac{1}{p'}$. When $1 \leq q, p < \infty$, $(L^{q'}, \ell^{p'})(\mathbb{R}^d)$ is isometrically isomorphic to the dual $(L^q, \ell^p)(\mathbb{R}^d)^*$ of $(L^q, \ell^p)(\mathbb{R}^d)$ in the sense that for any element T of $(L^q, \ell^p)(\mathbb{R}^d)^*$, there is an unique element $\phi(T)$ of $(L^{q'}, \ell^{p'})(\mathbb{R}^d)$ such that,

$$ \langle T, f \rangle = \int_{\mathbb{R}^d} f(x)\phi(T)(x)dx, \quad f \in (L^q, \ell^p)(\mathbb{R}^d) $$

and furthermore,

$$ \|\phi(T)\|_{q',p'} = \|T\|. \tag{5} $$

We recall that $\|T\| := \sup \left\{ |\langle T, f \rangle| \;\middle|\; f \in L^q_{loc}(\mathbb{R}^d) \text{ and } \|f\|_{q,p} \leq 1 \right\}$.

Next we summarize a couple of properties of dilation operators. We assume that $1 \leq \alpha \leq \infty$.

(1) For any real number $\rho > 0$, $St_\rho^{(\alpha)}$ maps $L^1_{loc}(\mathbb{R}^d)$ into itself.

(2) $St_1^{(\alpha)} f = f$, $f \in L^1_{loc}(\mathbb{R}^d)$.

(3) $St_{\rho_1}^{(\alpha)} \circ St_{\rho_2}^{(\alpha)} = St_{\rho_1\rho_2}^{(\alpha)} = St_{\rho_2}^{(\alpha)} \circ St_{\rho_1}^{(\alpha)}$.

In other words, $(St_\rho^{(\alpha)})_{\rho>0}$ is a commutative group of operators on $L^1_{loc}(\mathbb{R}^d)$, isomorphic to the multiplicative group $(0, \infty)$. As mentioned in the introduction, we have for $1 \leq \alpha \leq \infty$:

$$ \left\| St_\rho^{(\alpha)} f \right\|_\alpha = \|f\|_\alpha, \quad 0 < \rho < \infty \text{ and } f \in L^1_{loc}(\mathbb{R}^d). \tag{6} $$

In other words, each of those normalizations is isometric to exactly one of the family of L^r-spaces.

For amalgam spaces, direct computations (see for example (2.1) and Proposition 2.2 in [16]) give the following results:

$$\left\| St_\rho^{(\alpha)} f \right\|_{q,p} = \rho^{-d(\frac{1}{\alpha}-\frac{1}{q})} {}_{\rho^{-1}}\|f\|_{q,p}, \; 0<\rho<\infty \text{ and } f\in L^1_{loc}(\mathbb{R}^d)$$

and therefore,

$$\|f\|_{q,p,\alpha} := \sup_{\rho>0} \left\| St_\rho^{(\alpha)} f \right\|_{q,p} = \sup_{\rho>0} \rho^{d(\frac{1}{\alpha}-\frac{1}{q})} {}_{\rho}\|f\|_{q,p}. \tag{7}$$

It follows that the space $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ can be defined by:

$$(L^q, \ell^p)^\alpha(\mathbb{R}^d) = \left\{ f\in L^1_{loc}(\mathbb{R}^d) \mid \|f\|_{q,p,\alpha} < \infty \right\}. \tag{8}$$

We recall that for $q<\alpha$, the space $(L^q, \ell^\infty)^\alpha(\mathbb{R}^d)$ is exactly the classical Morrey space $\mathcal{M}_{q,\frac{dq}{\alpha}}(\mathbb{R}^d)$ introduced by Morrey in 1938, see [31], and defined for $\lambda = \frac{dq}{\alpha}$ by:

$$\mathcal{M}_{q,\lambda}(\mathbb{R}^d) = \left\{ f\in L^q_{loc}(\mathbb{R}^d) \mid \sup_{x\in\mathbb{R}^d, r>0} r^{\frac{\lambda-d}{q}} \left\| f\chi_{B(x,r)} \right\|_q < \infty \right\}. \tag{9}$$

Fofana spaces have the following properties (cf. [11,17]):

(1) $\left((L^q, \ell^p)^\alpha(\mathbb{R}^d), \|\cdot\|_{q,p,\alpha}\right)$ is a Banach space which is non trivial if and only if $q\le\alpha\le p$,

(2) if $\alpha\in\{p,q\}$, then $(L^q, \ell^p)^\alpha(\mathbb{R}^d) = L^\alpha(\mathbb{R}^d)$ with equivalent norms,

(3) if $q<\alpha<p$, then $L^\alpha(\mathbb{R}^d) \subsetneq L^{\alpha,\infty}(\mathbb{R}^d) \subsetneq (L^q, \ell^p)^\alpha(\mathbb{R}^d) \subsetneq (L^q, \ell^p)(\mathbb{R}^d)$, where $L^{\alpha,\infty}(\mathbb{R}^d)$ is the weak Lebesgue space on $\mathbb{R}^d$ defined by:

$$L^{\alpha,\infty}(\mathbb{R}^d) = \left\{ f\in L^1_{loc}(\mathbb{R}^d) \mid \|f\|^*_{\alpha,\infty} < \infty \right\},$$

with $\|f\|^*_{\alpha,\infty} := \sup_{\lambda>0} \left| \left\{ x\in\mathbb{R}^d \mid |f(x)|>\lambda \right\} \right|^{\frac{1}{\alpha}}$. We denote by $|E|$, the Lebesgue measure of a measurable subset E of $\mathbb{R}^d$.

For any $\rho>0$, the dilation $St_\rho^{(\alpha)}$ isometrically maps the space $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ to itself. More precisely,

$$\left\| St_\rho^{(\alpha)} f \right\|_{q,p,\alpha} = \|f\|_{q,p,\alpha}, \; 0<\rho<\infty \text{ and } f\in L^1_{loc}(\mathbb{R}^d). \tag{10}$$

Remark 1. *It is easy to see that, if $0<\rho<\infty$ and (f,g) is an element of $L^1_{loc}(\mathbb{R}^d)\times L^1_{loc}(\mathbb{R}^d)$ such that $(St_\rho^{(\alpha)} f)g$ belongs to $L^1(\mathbb{R}^d)$, then $f(St_{\rho^{-1}}^{(\alpha)} g)$ belongs to $L^1(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} \left(St_\rho^{(\alpha)} f \right)(x) g(x) dx = \int_{\mathbb{R}^d} f(x) \left(St_{\rho^{-1}}^{(\alpha')} g \right)(x) dx. \tag{11}$$

Searching for a dilation invariant version of the Segal algebra $S_0(\mathbb{R}^d)$ (introduced in [27]) Feichtinger and Zimmermann introduced a certain *exotic minimal space* in [32]. The results of that paper will be used later.

Theorem 1 ([32], Theorem 2.1)**.** *Let* **B** *be a Banach space, and $\Phi = (\varphi_j)_{j\in J}$ a (not necessarily countable) bounded family in* **B**. *Define*

$$\mathcal{B} = \mathcal{B}_\Phi := \left\{ f = \sum_{j\in J} a_j \varphi_j : \sum_{j\in J} |a_j| < \infty \right\},$$

and let

$$\|f\|_{\mathcal{B}} = \inf\left\{\sum_{j\in J}|a_j| \mid f = \sum_{j\in J} a_j\varphi_j\right\}.$$

Then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ *is a Banach space continuously embedded into* $\mathbf{B}$.

3. A Pre-Dual of Fofana Spaces

Definition 1. *Let* $1 \le q \le \alpha \le p \le \infty$. *The space* $\mathcal{H}(q, p, \alpha)$ *is defined as the set of all elements* f *of* $L^1_{loc}(\mathbb{R}^d)$ *for which there exist a sequence* $\{(c_n, \rho_n, f_n)\}_{n\ge1}$ *of elements of* $\mathbb{C} \times (0, \infty) \times (L^{q'}, \ell^{p'})(\mathbb{R}^d)$ *such that*

$$\begin{cases} \sum_{n\ge1}|c_n| < \infty \\ \|f_n\|_{q',p'} \le 1,\ n \ge 1 \\ f := \sum_{n\ge1} c_n St^{(\alpha')}_{\rho_n} f_n \text{ in the sense of } L^1_{loc}(\mathbb{R}^d) \end{cases}. \tag{12}$$

We will always refer to any sequence $\{(c_n, \rho_n, f_n)\}_{n\ge1}$ of elements of $\mathbb{C} \times (0, \infty) \times (L^{q'}, \ell^{p'})(\mathbb{R}^d)$ satisfying (12) as $\mathfrak{h}$-decomposition of f.

For any element f of $\mathcal{H}(q, p, \alpha)$, we set

$$\|f\|_{\mathcal{H}(q,p,\alpha)} := \inf\left\{\sum_{n\ge1}|c_n|\right\}, \tag{13}$$

where the infimum is taken over all $\mathfrak{h}$-decomposition of f.

Proposition 1. *Let* $1 \le q \le \alpha \le p \le \infty$. $\mathcal{H}(q, p, \alpha)$ *endowed with* $\|\cdot\|_{\mathcal{H}(q,p,\alpha)}$ *is a Banach space continuously embedded into* $L^{\alpha'}(\mathbb{R}^d)$.

Proof. Since $1 \le p' \le \alpha' \le q' \le \infty$, it comes from (2) and (3) that for any element $f \in (L^{q'}, \ell^{p'})(\mathbb{R}^d)$,

$$f \in L^{\alpha'} \text{ and } \|f\|_{\alpha'} \le \|f\|_{q',p'}.$$

Therefore, by (6)

$$St^{(\alpha')}_{\rho} f \in L^{\alpha'} \text{ and } \left\|St^{(\alpha')}_{\rho} f\right\|_{\alpha'} \le \|f\|_{q',p'}.$$

The result follows from Theorem 1, using the Banach space $\mathbf{B} = L^{\alpha'}(\mathbb{R}^d)$ and its bounded subset

$$\Phi = \left\{St^{(\alpha')}_{\rho} f \mid \rho > 0 \text{ and } \|f\|_{q',p'} \le 1\right\}.$$

□

For $1 < p < \infty$ and $0 < \lambda < d$. Zorko proved in [24] that the Morrey space $\mathcal{M}_{p,\lambda}$ is the dual space of the space, $\mathcal{Z}_{p',\lambda}$, which consists of all the functions f on $\mathbb{R}^d$ which can be written in the form $f := \sum_k c_k \mathfrak{a}_k$, where $\{c_k\}$ is a sequence in ℓ^1, and $\{\mathfrak{a}_k\}$ is a sequence of functions on $\mathbb{R}^d$ satisfying for each k,

- supp $\mathfrak{a}_k \subset$ a ball B_k,
- $\|\mathfrak{a}_k\|_{p'} \le 1/|B_k|^{\frac{d-\lambda}{dp}}$.

Notice that for $p = \infty$ and $q < \alpha$, the space $\mathcal{Z}_{q',\frac{dq}{\alpha}}$ is continuously embedded into $\mathcal{H}(q, \infty, \alpha)$. In fact, let $f = \sum_k c_k \mathfrak{a}_k \in \mathcal{Z}_{q',\frac{dq}{\alpha}}$ with the sequences $\{c_k\}$ and $\{\mathfrak{a}_k\}$ as in the Introduction. For any $k \in \mathbb{N}$, we define,

$$u_k = St^{(\alpha')}_{r_k^{-1}}(\mathfrak{a}_k),$$

where r_k is the radius of the ball B_k associated to $\mathfrak{a}_k$. There exists a constant $C > 0$ (one can take $C = |B(0,1)|^{\frac{1}{\alpha}-\frac{1}{q}}$), such that

$$\left\|2^{-d}C^{-1}u_k\right\|_{q',1} \leq 1 \text{ and } f = \sum_{k\geq 1}(2^d Cc_k)St_{r_k}^{(\alpha')}(2^{-d}C^{-1}u_k).$$

We prove next that for $1 \leq q \leq \alpha \leq p \leq \infty$, the space $\mathcal{H}(q,p,\alpha)$ is a certain minimal Banach space.

Proposition 2. *For $1 \leq q \leq \alpha \leq p \leq \infty$, the space $\mathcal{H}(q,p,\alpha)$ is a minimal Banach space, isometrically invariant under the family $(St_\rho^{(\alpha')})_{\rho>0}$ and such that*

$$(L^{q'},\ell^{p'})(\mathbb{R}^d) \subset \mathcal{H}(q,p,\alpha) \subset L^1_{loc}(\mathbb{R}^d),$$

with continuous inclusions.

Proof. (1) Let us first prove that $\mathcal{H}(q,p,\alpha)$ is isometrically invariant by $St_\rho^{(\alpha')}$ for all $\rho > 0$.

Let $f \in \mathcal{H}(q,p,\alpha)$. For $0 < \rho < \infty$ and any $\mathfrak{h}$-decomposition $\{(c_n,\rho_n,f_n)\}_{n\geq 1}$ of f, we have

$$St_\rho^{(\alpha')}f = \sum_{n\geq 1} c_n St_{\rho\rho_n}^{(\alpha')}f_n$$

so that

$$St_\rho^{(\alpha')}f \in \mathcal{H}(q,p,\alpha) \text{ and } \left\|St_\rho^{(\alpha')}f\right\|_{\mathcal{H}(q,p,\alpha)} = \|f\|_{\mathcal{H}(q,p,\alpha)}. \tag{14}$$

Thus, $St_\rho^{(\alpha')}$ is an isometric automorphism of $\mathcal{H}(q,p,\alpha)$.

(2) First, we verify that $(L^{q'},\ell^{p'})$ is continuously embedded into $\mathcal{H}(q,p,\alpha)$. For any $0 \neq f \in (L^{q'},\ell^{p'})$ we have

$$f = \|f\|_{q',p'} St_1^{(\alpha_1)}(\|f\|_{q',p'}^{-1}f) \text{ and } \left\|\|f\|_{q',p'}^{-1}f\right\|_{q',p'} = 1$$

and, therefore, f belongs to $\mathcal{H}(q,p,\alpha)$ and satisfies

$$\|f\|_{\mathcal{H}(q,p,\alpha)} \leq \|f\|_{q',p'}. \tag{15}$$

Thus, our claim is verified.

(3) It remains to be proved that this space is minimal. Let X be another Banach space continuously embedded into $L^1_{loc}(\mathbb{R}^d)$ such that

- (a) X is isometrically invariant by $(St_\rho^{(\alpha')})_{\rho>0}$, i.e., if $f \in X$ then $St_\rho^{(\alpha')}f \in X$ and $\left\|St_\rho^{(\alpha')}f\right\|_X = \|f\|_X$, $\rho > 0$,
- (b) $(L^{q'},\ell^{p'})(\mathbb{R}^d)$ is continuously embedded into X, i.e., there exists $K > 0$ such that for $f \in (L^{q'},\ell^{p'})(\mathbb{R}^d)$, $f \in X$ and $\|f\|_X \leq K_1 \|f\|_{q',p'}$.

For $f \in \mathcal{H}(q,p,\alpha)$ and any $\mathfrak{h}$-decomposition $\{(c_n,\rho_n,f_n)\}_{n\geq 1}$

$$c_n St_{\rho_n}^{(\alpha')}f_n \in X \text{ and } \left\|c_n St_{\rho_n}^{(\alpha')}f_n\right\|_X = |c_n|\,\|f_n\|_X \leq K|c_n|,\ n \geq 1.$$

Thus

$$\sum_{n\geq 1}\left\|c_n St_{\rho_n}^{(\alpha')}f_n\right\|_X \leq K\sum_{n\geq 1}|c_n|$$

and therefore, as X is a Banach space,

$$f = \sum_{n\geq 1} c_n St^{(\alpha')}_{\rho_n} f_n \in X \text{ and } \|f\|_X \leq K \sum_{n\geq 1} |c_n|.$$

From this and (13) it follows that

$$f \in X \text{ and } \|f\|_X \leq K \|f\|_{\mathcal{H}(q,p,\alpha)}.$$

Thus, $\mathcal{H}(q,p,\alpha)$ is continuously included in X.
□

Our main result can be stated as follows.

Theorem 2. *Let $1 < q \leq \alpha \leq p \leq \infty$. The operator $g \mapsto T_g$ defined by (17) is an isometric isomorphism of $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ into $\mathcal{H}(q,p,\alpha)^*$.*

For the proof, we need some intermediate results.

Remark 2. *It is clear that if $\{(c_n, \rho_n, f_n)\}_{n\geq 1}$ is a $\mathfrak{h}$-decomposition of $f \in \mathcal{H}(q,p,\alpha)$, then $(\sum_{n=1}^m c_n St^{(\alpha')}_{\rho_n} f_n)_{m\geq 1}$ is a sequence of elements of $(L^{q'}, \ell^{p'})(\mathbb{R}^d)$ which converges to f in $\mathcal{H}(q,p,\alpha)$. Hence $(L^{q'}, \ell^{p'})(\mathbb{R}^d)$ is a dense subspace of $\mathcal{H}(q,p,\alpha)$.*

Proposition 3. *Let $1 \leq q \leq \alpha \leq p \leq \infty$, $f \in \mathcal{H}(q,p,\alpha)$ and $g \in (L^q, \ell^p)^\alpha$. Then fg belongs to $L^1(\mathbb{R}^d)$ and*

$$\left| \int_{\mathbb{R}^d} f(x) g(x) dx \right| \leq \|f\|_{\mathcal{H}(q,p,\alpha)} \|g\|_{q,p,\alpha}. \tag{16}$$

Proof. Let $\{(c_n, \rho_n, f_n)\}_{n\geq 1}$ be a $\mathfrak{h}$-decomposition of f.
By using successively (11), (4) and (7), we obtain for any $n \geq 1$,

$$\begin{aligned} \left| \int_{\mathbb{R}^d} St^{(\alpha')}_{\rho_n} f_n(x) g(x) dx \right| &= \left| \int_{\mathbb{R}^d} f_n(x) St^{(\alpha)}_{\rho_n^{-1}} g(x) dx \right| \leq \int_{\mathbb{R}^d} \left| f_n(x) St^{(\alpha)}_{\rho_n^{-1}} g(x) \right| dx \\ &\leq \|f_n\|_{q',p'} \left\| St^{(\alpha)}_{\rho_n^{-1}} g \right\|_{q,p} \leq \left\| St^{(\alpha)}_{\rho_n^{-1}} g \right\|_{q,p} \leq \|g\|_{q,p,\alpha}. \end{aligned}$$

Therefore, we have

$$\sum_{n\geq 1} \int_{\mathbb{R}^d} \left| c_n St^{(\alpha')}_{\rho_n} f_n(x) g(x) \right| dx \leq \|g\|_{q,p,\alpha} \sum_{n\geq 1} |c_n|.$$

This implies that $fg = \sum_{n\geq 1} c_n St^{(\alpha')}_{\rho_n} f_n g$ belongs to $L^1(\mathbb{R}^d)$ and

$$\left| \int_{\mathbb{R}^d} f(x) g(x) dx \right| \leq \int_{\mathbb{R}^d} |f(x) g(x)| dx \leq \|g\|_{q,p,\alpha} \sum_{n\geq 1} |c_n|.$$

Taking the infimum with respect to all $\mathfrak{h}$-decompositions of f, we get

$$\left| \int_{\mathbb{R}^d} f(x) g(x) dx \right| \leq \int_{\mathbb{R}^d} |f(x) g(x)| dx \leq \|g\|_{q,p,\alpha} \|f\|_{H(q,p,\alpha)}$$

□

Remark 3. *Let $1 \leq q \leq \alpha \leq p \leq \infty$ and set:*

$$\langle T_g, f \rangle = \int_{\mathbb{R}^d} f(x)g(x)dx,\ g \in (L^q, \ell^p)^\alpha(\mathbb{R}^d) \text{ and } f \in \mathcal{H}(q,p,\alpha). \tag{17}$$

By Proposition 3 and the fact that $\varphi \mapsto \int_{\mathbb{R}^d} \varphi(x)dx$ belongs to $L^1(\mathbb{R}^d)^*$, that:

(1) *for any element g of $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$, $f \mapsto \langle T_g, f \rangle$ belongs to $\mathcal{H}(q,p,\alpha)^*$,*

(2) *$T : g \mapsto T_g$ is linear and bounded mapping from $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ into $\mathcal{H}(q,p,\alpha)^*$, satisfying $\|T\| \leq 1$, that is:*

$$\|T_g\| \leq \|g\|_{q,p,\alpha},\ g \in (L^q, \ell^p)^\alpha(\mathbb{R}^d). \tag{18}$$

Now we can prove our main result.

Proof of Theorem 2. We know (see Remark 3) that $g \mapsto T_g$ is a bounded linear application of $(L^q, \ell^p)^\alpha(\mathbb{R}^d)$ into $\mathcal{H}(q,p,\alpha)^*$ such that

$$\|T_g\| \leq \|g\|_{q,p,\alpha},\ g \in (L^q, \ell^p)^\alpha(\mathbb{R}^d).$$

Let T be an element of $\mathcal{H}(q,p,\alpha)^*$. From (15), it follows that the restriction T_0 of T to $(L^{q'}, \ell^{p'})(\mathbb{R}^d)$ belongs to $(L^{q'}, \ell^{p'})(\mathbb{R}^d)^*$. Furthermore, we have $1 \leq p' \leq \alpha' \leq q' < \infty$. So, by (5), there is an element g of $(L^q, \ell^p)(\mathbb{R}^d)$ such that

$$\langle T, f \rangle = \int_{\mathbb{R}^d} f(x)g(x)dx,\ f \in (L^{q'}, \ell^{p'})(\mathbb{R}^d). \tag{19}$$

Hence, for $f \in (L^{q'}, \ell^{p'})(\mathbb{R}^d)$ and $\rho > 0$ we have

$$\int_{\mathbb{R}^d} St_\rho^{(\alpha)} g(x) f(x)dx = \int_{\mathbb{R}^d} g(x) St_{\rho^{-1}}^{(\alpha')} f(x)dx = \left\langle T, St_{\rho^{-1}}^{(\alpha')} f \right\rangle$$

by (4) and (11), and

$$\left| \int_{\mathbb{R}^d} St_\rho^{(\alpha)} g(x) f(x)dx \right| \leq \|T\| \left\| St_{\rho^{-1}}^{(\alpha')} f \right\|_{\mathcal{H}(q,p,\alpha)} = \|T\| \, \|f\|_{\mathcal{H}(q,p,\alpha)} \leq \|T\| \, \|f\|_{q',p'}$$

by (14) and (15). From this and (5) it follows that

$$St_\rho^{(\alpha)} g \in (L^q, \ell^p)(\mathbb{R}^d) \text{ and } \left\| St_\rho^{(\alpha)} g \right\|_{q,p} \leq \|T\|,\ \rho \in (0,\infty)$$

and therefore, by (7),

$$\|g\|_{q,p,\alpha} \leq \|T\| \text{ and } g \in (L^q, \ell^p)^\alpha(\mathbb{R}^d).$$

From (19), the density of $(L^{q'}, \ell^{p'})(\mathbb{R}^d)$ in $\mathcal{H}(q,p,\alpha)$ (see Remark 2) and Proposition 3, we get

$$\langle T, f \rangle = \int_{\mathbb{R}^d} f(x)g(x)dx,\ f \in \mathcal{H}(q,p,\alpha).$$

This completes the proof. □

We end this paper by stating some interesting properties of the spaces $\mathcal{H}(q,p,\alpha)$.

Proposition 4. *Let $1 \leq q \leq \alpha \leq p \leq \infty$.*

(1) *The space $\mathcal{H}(q,p,\alpha)$ is a Banach $L^1(\mathbb{R}^d)$-module:*
for $(f,\varphi) \in \mathcal{H}(q,p,\alpha) \times L^1(\mathbb{R}^d)$

$$f * \varphi \in \mathcal{H}(q,p,\alpha) \text{ and } \|f * \varphi\|_{\mathcal{H}(q,p,\alpha)} \leq \|f\|_{\mathcal{H}(q,p,\alpha)} \|\varphi\|_1 . \tag{20}$$

(2) *If $1 < q$ and $\varphi \in L^1(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\|\varphi\|_1 = 1$ then*

$$\lim_{\epsilon \to 0} \left\| f * St_{\epsilon}^{(1)} \varphi - f \right\|_{\mathcal{H}(q,p,\alpha)} = 0, \ \ f \in \mathcal{H}(q,p,\alpha). \tag{21}$$

Consequently, $\mathcal{H}(q,p,\alpha)$ is an essential Banach $L^1(\mathbb{R}^d)$-module.

(3) *For $1 < p,q \leq \infty$ the Schwartz space of rapidly decreasing functions or the space of compactly supported C^∞-functions are dense subspaces of $\mathcal{H}(q,p,\alpha)$.*

(4) *For $1 < p,q \leq \infty$, the Banach space $\mathcal{H}(q,p,\alpha)$ is separable.*

Proof. Let $f \in \mathcal{H}(q,p,\alpha)$ and $\varphi \in L^1(\mathbb{R}^d) \setminus \{0\}$.

(1) Let $\{(c_n, \rho_n, f_n)\}_{n \geq 1}$ be a $\mathfrak{h}$-decomposition of f. A direct computation shows that

$$(St_{\rho_n}^{(\alpha')} f_n) * \varphi = St_{\rho_n}^{(\alpha')} \left[f_n * St_{\rho_n^{-1}}^{(1)} \varphi \right], \ n \geq 1.$$

Hence, combining the fact that $(L^{q'}, \ell^{p'})(\mathbb{R}^d)$ is a $L^1(\mathbb{R}^d)$ Banach module and Relation (6), we obtain

$$\left\| f_n * St_{\rho_n^{-1}}^{(1)} \varphi \right\|_{q',p'} \leq \|f_n\|_{q',p'} \|\varphi\|_1 , \ n \geq 1.$$

This implies that

$$\left\| \|\varphi\|_1^{-1} f_n * St_{\rho_n^{-1}}^{(1)} \varphi \right\|_{q',p'} \leq \|f_n\|_{q',p'} \leq 1, \ n \geq 1.$$

Moreover,

$$\begin{aligned} \sum_{n \geq 1} \left\| c_n (St_{\rho_n}^{(\alpha')} f_n) * \varphi \right\|_{\alpha'} &\leq \sum_{n \geq 1} |c_n| \left\| St_{\rho_n}^{(\alpha')} f_n \right\|_{\alpha'} \|\varphi\|_1 = \sum_{n \geq 1} |c_n| \, \|f_n\|_{\alpha'} \|\varphi\|_1 \\ &\leq \sum_{n \geq 1} |c_n| \, \|f_n\|_{q',p'} \|\varphi\|_1 \leq (\sum_{n \geq 1} |c_n|) \|\varphi\|_1 < \infty. \end{aligned}$$

Therefore,

$$f * \varphi = \sum_{n \geq 1} c_n (St_{\rho_n}^{(\alpha')} f_n) * \varphi = \sum_{n \geq 1} c_n \|\varphi\|_1 St_{\rho_n}^{(\alpha')} (\|\varphi\|_1^{-1} f_n * St_{\rho_n^{-1}}^{(1)} \varphi)$$

in the sense of $L^1_{loc}(\mathbb{R}^d)$, with

$$\sum_{n \geq 1} |c_n \|\varphi\|_1| = (\sum_{n \geq 1} |c_n|) \|\varphi\|_1 < \infty \text{ and } \left\| \|\varphi\|_1^{-1} f_n * St_{\rho_n^{-1}}^{(1)} \varphi \right\|_{q',p'} \leq 1.$$

That is $f * \varphi$ belongs to $\mathcal{H}(q,p,\alpha)$ and satisfies

$$\|f * \varphi\|_{\mathcal{H}(q,p,\alpha} \leq \|f\|_{\mathcal{H}(q,p,\alpha)} \|\varphi\|_1 .$$

(2) Let us assume that $1 < q$ and $\|\varphi\|_1 = 1$. For any real number $\epsilon > 0$, we set $\varphi_\epsilon = St_\epsilon^{(1)}\varphi$. Let us consider $\{(c_n, \rho_n, f_n)\}_{n\geq 1}$, a $\mathfrak{h}$-decomposition of f. b We know that the sequence $(f^m)_{m\geq 1}$ is defined by:

$$f^m = \sum_{n=1}^{m} c_n St_{\rho_n}^{(\alpha')} f_n, \;\; m \geq 1$$

converges to f in $\mathcal{H}(q,p,\alpha)$. Let us fix any real number $\delta > 0$. There is a positive integer m_δ satisfying:

$$\|f - f^{m_\delta}\| < \frac{\delta}{3}.$$

Moreover, for any real number $\epsilon > 0$, we have

$$\begin{aligned} \|f * \varphi_\epsilon - f\|_{\mathcal{H}(q,p,\alpha)} &\leq \|f * \varphi_\epsilon - f^{m_\delta} * \varphi_\epsilon\|_{\mathcal{H}(q,p,\alpha)} + \|f^{m_\delta} * \varphi_\epsilon - f^{m_\delta}\|_{\mathcal{H}(q,p,\alpha)} \\ &+ \|f^{m_\delta} - f\|_{\mathcal{H}(q,p,\alpha)} \\ &\leq 2\,\|f^{m_\delta} - f\|_{\mathcal{H}(q,p,\alpha)} + \left\| \sum_{n=1}^{m_\delta} c_n (St_{\rho_n}^{(\alpha')} f_n) * \varphi_\epsilon - \sum_{n=1}^{m_\delta} c_n St_{\rho_n}^{(\alpha')} f_n \right\|_{\mathcal{H}(q,p,\alpha)}. \end{aligned}$$

The last term is not greater than

$$\frac{2\delta}{3} + \left\| \sum_{n\in N(m_\delta)} c_n \left\| f_n * \varphi_{\epsilon\rho_n^{-1}} - f_n \right\|_{q',p'} St_{\rho_n}^{(\alpha')} \left(\frac{f_n * \varphi_{\epsilon\rho_n^{-1}} - f_n}{\left\| f_n * \varphi_{\epsilon\rho_n^{-1}} - f_n \right\|_{q',p'}} \right) \right\|_{\mathcal{H}(q,p,\alpha)},$$

where,

$$N(m_\delta) = \left\{ n/1 \leq n \leq m_\delta \text{ and } \left\| f_n * \varphi_{\epsilon\rho_n^{-1}} - f_n \right\|_{q',p'} \neq 0 \right\}.$$

Therefore,

$$\|f * \varphi_\epsilon - f\|_{\mathcal{H}(q,p,\alpha)} \leq \frac{2\delta}{3} + \sum_{n\in N(m_\delta)} |c_n| \left\| f_n * \varphi_{\epsilon\rho_n^{-1}} - f_n \right\|_{q',p'}.$$

From the hypothesis, we have $1 \leq p' \leq q' < \infty$ and, therefore,

$$\lim_{t\to 0} \|g * \varphi_t - g\|_{q',p'} = 0, \; g \in (L^{q'}, \ell^{p'})(\mathbb{R}^d).$$

Therefore, we have

$$\lim_{\epsilon\to 0} \left\| f_n * \varphi_{\epsilon\rho_n^{-1}} - f_n \right\|_{q',p'} = 0, \; n \in N(m_\delta)$$

thus,

$$\overline{\lim_{\epsilon\to 0}} \|f * \varphi_\epsilon - f\|_{\mathcal{H}(q,p,\alpha)} \leq \frac{2\delta}{3}.$$

Should this inequality be true for any real number $\delta > 0$, we actually have $\lim_{\epsilon\to 0} \|f * \varphi_\epsilon - f\|_{\mathcal{H}(q,p,\alpha)} = 0$.

(3) Approximating a given function in $\mathcal{H}(q,p,\alpha)$, first by some function with compact support and then convolving it by some compactly supported, infinitely differentiable test function, provides an approximation by a test function, which also belongs to the Schwartz space.

(4) For $1 < q \leq p \leq \infty$, we have $1 \leq p' \leq q' < \infty$. However, it is well known that $(L^{q'}, \ell^{p'})$ is separable. Thus, the result follows from the density of $(L^{q'}, \ell^{p'})$ in $\mathcal{H}(q,p,\alpha)$.

□

For $1 < q \leq \alpha \leq p \leq \infty$, we have defined a pre-dual of the Fofana space $(L^q, \ell^p)^\alpha$ using a specific atomic decomposition method developed by Feichtinger, and proved that for $p = \infty$ and $q < \alpha$, the pre-dual of classical Morrey space is embedded in our space. However, our further goal is to find the dual space of $(L^q, \ell^p)^\alpha$ and their interpolation spaces. This appears to be a more complex task and has been left for future work.

4. Conclusions

In summary, we have used techniques that concern minimal invariant Banach spaces of functions in order to characterize the pre-dual of certain Fofana spaces which had, to date, remained unknown. Starting from a characterization of a Fofana space as a (dense) subspace of a Wiener amalgam space under a certain group of (suitably normalized) dilation operators, one can generate the pre-dual space by starting from the pre-dual of the mentioned Wiener amalgam spaces and then describing the pre-dual via atomic decompositions, using the (adjoint) group of dilation operators.

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