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Article

Existence Result and Uniqueness for Some Fractional Problem

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Abstract: In this article, by the use of the lower and upper solutions method, we prove the existence of a positive solution for a Riemann–Liouville fractional boundary value problem. Furthermore, the uniqueness of the positive solution is given. To demonstrate the serviceability of the main results, some examples are presented.

Keywords: positive solution; green function; fractional differential equation; the method of lower and upper solutions

1. Introduction

The aim of this work is to study the existence and uniqueness of the positive solution for the following problem:

$$\begin{cases} D^{\alpha}_{0+}k(t) = j(t,k(t)), \quad 1 < \alpha \leq 2, \quad 0 < t < 1 & , \\ k(0) = 0, \quad \beta k(1) - \gamma k(\eta) = 0, \quad \eta \in [0,1] & , \end{cases} \tag{1}$$

where β, γ, and η are positive real numbers such that $\beta - 2\gamma\eta^{\alpha-1} > 0$, j is a nonnegative continuous function on $[0,1] \times [0,\infty)$, and D^{α}_{0+} is the fractional derivative in the sense of Riemann–Liouville. This type of equation is important in many disciplines such as chemistry, aerodynamics, polymer rheology, etc.

Different techniques are used in such problems to obtain the existence of solutions, for example the variational method, the Adomian decomposition method, etc.; we refer the reader to [1–8] and references therein.

Existence results of nonlinear fractional problems are given by the use of fixed point theorems; see [9–15]. More precisely, the authors in [12] gave the existence of positive solutions for the following equation:

$$D^{\alpha}_{0+}k(t) + j(t,k(t)) = 0, \quad 1 < \alpha \leq 2, \quad 0 < t < 1,$$

according to some boundary conditions.

Using the theory of the fixed point index, the author in [11], presented the existence of the positive solution for the following system:

$$\begin{cases} D_{0^+}^{\alpha} k(t) + j(t, k(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ k(0) = 0, \quad \beta k(\eta) = k(1). \end{cases}$$

Recently, the upper solution method and lower solution method have been the aim of many papers; see for example the book [16] and the recent papers [17–30]. The main idea of this method is to study some modified problem and, then, give the existence results for the principal problem.

Motivated by the above works, in this article, we will present a new method to study the given problem, that is we combine the lower and upper solution method with the fixed point theorem method in order to prove the existence and uniqueness of the positive solution. Let us assume the following:

Hypothesis (H1). *The function j is nonnegative and continuous on $[0,1] \times [0,+\infty)$.*

Hypothesis (H2). *For each $t \in [0,1]$, the function $j(t,.)$ is bounded and increasing on $[0,+\infty)$.*

Hypothesis (H3). *There exists a function $a : [0,1] \to [0,\infty)$ such that the function j satisfies:*

$$|j(s,x) - j(s,y)| \leq a(s)|x-y|, \; \forall\, s \in [0,1], \; \forall\, x,y \geq 0. \tag{2}$$

The main theorems of this paper are summarized as follows.

Theorem 1. *Under Hypotheses (H_1)–(H_2). If $\beta - 2\gamma\eta^{\alpha-1} > 0$, then Equation (1) admits a positive solution.*

Theorem 2. *Under hypothesis (H_3), if $\beta - 2\gamma\eta^{\alpha-1} > 0$ and if:*

$$\int_0^1 [1 + \frac{\beta}{(\beta - \gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}]\delta^{\alpha-1} a(\delta) d\delta < \Gamma(\alpha), \tag{3}$$

then Equation (1) admits a unique positive solution.

2. Preliminaries

In this section, we collect some basic results and notations that will be used in the forthcoming sections.

We denote by $L(0,1)$ the set of all integrable functions on $(0,1)$ and by $C(0,1)$ the set of functions that are continuous on $(0,1)$.

Lemma 1 ([31]). *Let $\alpha > 0$, $N = [\alpha] + 1$. Assume that the function k is in $C(0,1) \cap L(0,1)$. Then, the following equation:*

$$D_{0^+}^{\alpha} k(t) = 0,$$

admits a unique solution. Moreover, this solution is given by:

$$k(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_N t^{\alpha-N},$$

for some $C_i \in \mathbb{R}$, where $i = 1,2,...,N$.

Lemma 2 ([31]). *Let $\alpha > 0$ and $N = [\alpha] + 1$. Assume that either k and $D_{0^+}^{\alpha} k$ are in $C(0,1) \cap L(0,1)$. Then, there exists $C_i \in \mathbb{R}$, for $i = 1,2,...,N$, such that:*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} k(t) = k(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2} - \ldots - C_N t^{\alpha-N}.$$

Now, we give the Green function associated with the problem (1).

Theorem 3. *If $1 < \alpha < 2$ and h is continuous in $[0,1]$, then equation:*

$$D^{\alpha}_{0^+}k(t) + h(t) = 0, \quad 0 < t < 1, \tag{4}$$

with the following conditions:

$$k(0) = 0, \quad \beta k(1) - \gamma k(\eta) = 0, \quad \eta \in [0,1], \tag{5}$$

admits a unique solution, which is given by:

$$k(t) = \int_0^1 G(t,\delta)h(\delta)d\delta,$$

with $G(t,\delta)$ being the Green function defined by:

$$\begin{aligned}&\Gamma(\alpha)G(t,\delta)\\ &= (t-\delta)^{\alpha-1}\chi_{[0,t]}(\delta) + \frac{t^{\alpha-1}}{\beta-\gamma\eta^{\alpha-1}}\Big[\beta(1-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}\chi_{[0,\eta]}(\delta)\Big],\end{aligned} \tag{6}$$

where χ_A is the function defined by:

$$\chi_A(x) = \begin{cases} 1 & if \ x \in A, \\ 0 & if \ x \notin A. \end{cases}$$

Proof. From Equation (4) and using Lemma 2, there exist two real numbers C_1 and C_2 such that:

$$k(t) = -I^{\alpha}_{0^+}h(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2}.$$

It follows that:

$$k(t) = \int_0^t \frac{(t-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta)d\delta + C_1t^{\alpha-1} + C_2t^{\alpha-2}. \tag{7}$$

Since $k(0) = 0$, then $C_2 = 0$.

On the other hand:

$$\begin{cases} k(1) = \int_0^1 \frac{(1-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta)d\delta + C_1, \\ k(\eta) = \int_0^\eta \frac{(\eta-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta)d\delta + C_1\eta^{\alpha-1}. \end{cases}$$

As $\beta k(1) - \gamma k(\eta) = 0$, then we have:

$$C_1 = \frac{\beta\int_0^1 \frac{(1-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta) - \gamma\int_0^\eta \frac{(\eta-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta)d\delta}{\beta-\gamma\eta^{\alpha-1}}. \tag{8}$$

By substituting the values of C_1 and C_2 into Equation (7), we get:

$$\begin{aligned}&k(t)\\ &= \int_0^t \frac{(t-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta)d\delta + \frac{1}{\beta-\gamma\eta^{\alpha-1}}\Big[\beta\int_0^1 \frac{(1-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta)d\delta - \gamma\int_0^\eta \frac{(\eta-\delta)^{\alpha-1}}{\Gamma(\alpha)}h(\delta)d\delta\Big]t^{\alpha-1}\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1\left[(t-\delta)^{\alpha-1}\chi_{[0,t]}(\delta) + \frac{1}{\beta-\gamma\eta^{\alpha-1}}\Big\{\beta(1-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}\chi_{[0,\eta]}(\delta)\Big\}t^{\alpha-1}\right]h(\delta)d\delta.\end{aligned}$$

That is:

$$k(t) = \int_0^t G(t,\delta)h(\delta)d\delta.$$

It follows that for all real numbers $t,\delta \in [0,1]$, we have:

$$G(t,\delta) = \frac{(t-\delta)^{\alpha-1}}{\Gamma(\alpha)}\chi_{[0,t]}(\delta) + \frac{1}{\Gamma(\alpha)(\beta-\gamma\eta^{\alpha-1})}\left[\beta(1-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}\chi_{[0,\eta]}(\delta)\right]t^{\alpha-1}.$$

□

Proposition 1. *Let G be the function given by Equation (6), then we have the following properties:*

(i) *Put* $q(\delta) = \frac{2\beta-\gamma\eta^{\alpha-1}}{\Gamma(\alpha)(\beta-\gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}$, *then we have:*

$$G(t,\delta) \leq q(\delta), \forall\ t,\delta \in [0,1].$$

(ii) *Put* $p(t) = \frac{\beta}{\beta-\gamma\eta^{\alpha-1}}t^{\alpha-1}$, *then we obtain:*

$$G(t,\delta) \geq q(\delta)p(t),\ \forall\ t,\delta \in [0,1].$$

Proof. (i) Firstly, we remark that $G(1,\delta)$ is given by:

$$G(1,\delta) = \frac{1}{\Gamma(\alpha)}\begin{cases} (1-\delta)^{\alpha-1} + \frac{1}{\beta-\gamma\eta^{\alpha-1}}\left\{\beta(1-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}\right\} & \text{, if } 0 \leq \delta \leq \eta \\ (1-\delta)^{\alpha-1} + \frac{1}{\beta-\gamma\eta^{\alpha-1}}\beta(1-\delta)^{\alpha-1} & \text{, if } \eta \leq \delta \leq 1. \end{cases}$$

On the other hand, for all $\delta \in [0,1]$, the function $t \longmapsto G(t,\delta)$ is increasing, so, for any $t,\delta \in [0,1]$, we have:

$$G(t,\delta) \leq G(1,\delta).$$

As $\gamma > 0$, it is easy to see that:

$$\begin{aligned} \Gamma(\alpha)G(t,\delta) &\leq (1-\delta)^{\alpha-1} + \frac{1}{\beta-\gamma\eta^{\alpha-1}}\beta(1-\delta)^{\alpha-1} \\ &= \frac{2\beta-\gamma\eta^{\alpha-1}}{\beta-\gamma\eta^{\alpha-1}}(1-\delta)^{\alpha-1}, \quad \forall t,\delta \in [0,1]. \end{aligned}$$

That is, if $t,\delta \in [0,1]$, then we have:

$$G(t,\delta) \leq q(\delta).$$

(ii) If $0 \leq \eta \leq t < 1$, then using (6) and the fact that $\beta - 2\gamma\eta^{\alpha-1} > 0$, we obtain:

$$
\begin{aligned}
&G(t,\delta)\\
&= q(\delta)\begin{cases} \frac{\beta-\gamma\eta^{\alpha-1}}{2\beta-\gamma\eta^{\alpha-1}}\left(\frac{t-\delta}{1-\delta}\right)^{\alpha-1} + \frac{1}{2\beta-\gamma\eta^{\alpha-1}}\left\{\beta - \gamma(\frac{\eta-\delta}{1-\delta})^{\alpha-1}\right\}t^{\alpha-1}, & \text{if } 0 \leq \delta \leq \eta \\ \frac{\beta-\gamma\eta^{\alpha-1}}{2\beta-\gamma\eta^{\alpha-1}}\left(\frac{t-\delta}{1-\delta}\right)^{\alpha-1} + \frac{\beta}{2\beta-\gamma\eta^{\alpha-1}}t^{\alpha-1}, & \text{if } \eta \leq \delta \leq t \\ \frac{\beta}{2\beta-\gamma\eta^{\alpha-1}}t^{\alpha-1}, & \text{if } t \leq \delta < 1 \end{cases}\\
&\geq q(\delta)\begin{cases} p(t) + \frac{\beta-\gamma\eta^{\alpha-1}}{2\beta-\gamma\eta^{\alpha-1}}\left(\frac{t-\delta}{1-\delta}\right)^{\alpha-1} - \frac{\gamma}{2\beta-\gamma\eta^{\alpha-1}}(\frac{\eta-\delta}{1-\delta})^{\alpha-1}t^{\alpha-1}, & \text{if } 0 \leq \delta \leq \eta \\ p(t) + \frac{\beta-\gamma\eta^{\alpha-1}}{2\beta-\gamma\eta^{\alpha-1}}\left(\frac{t-\delta}{1-\delta}\right)^{\alpha-1}, & \text{if } \eta \leq \delta \leq t \\ p(t), & \text{if } t \leq \delta < 1, \end{cases}\\
&\geq q(\delta)\begin{cases} p(t) + \dfrac{(\beta-\gamma\eta^{\alpha-1})(t-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}t^{\alpha-1}}{2\beta-\gamma\eta^{\alpha-1}}, & \text{if } 0 \leq \delta \leq \eta \\ p(t), & \text{if } \eta \leq \delta < 1, \end{cases}\\
&\geq q(\delta)\begin{cases} p(t) + \frac{1}{2\beta-\gamma\eta^{\alpha-1}}\left[(\beta-\gamma\eta^{\alpha-1})t^{\alpha-1} - \gamma\eta^{\alpha-1}t^{\alpha-1}\right], & \text{if } 0 \leq \delta \leq \eta \\ p(t), & \text{if } \eta \leq \delta < 1, \end{cases}\\
&\geq q(\delta)\begin{cases} p(t) + \frac{1}{2\beta-\gamma\eta^{\alpha-1}}(\beta-2\gamma\eta^{\alpha-1})t^{\alpha-1}, & \text{if } 0 \leq \delta \leq \eta \\ p(t), & \text{if } \eta \leq \delta < 1, \end{cases}\\
&\geq p(t)q(\delta),
\end{aligned}
$$

where $p(t) = \frac{\beta}{2\beta-\gamma\eta^{\alpha-1}}t^{\alpha-1}$. The proof of Proposition 1 is now completed. □

3. Proof of the Main Results

This section is devoted to proving our main results. To this aim, we will apply the following lemma.

Lemma 3 (See [32]). *Let E be a semi-order Banach space and P be a cone in E. Let $D \subset P$ and a nondecreasing operator $T : D \to E$. Assume that the equation $x - T(x) = 0$ admits a lower solution $x_0 \in D$ and an upper solution $y_0 \in D$, with $x_0 \leq y_0$. Assume that if $x_0 \leq x \leq y_0$, then $x \in D$. If one of the following statements holds:*

(i) *P is normal, and T is compact continuous.*
(ii) *P is regular, and T is continuous.*
(ii) *E is reflexive, P normal, and T continuous or weak continuous.*

Then, the equation

$$T(x) = x,$$

admits a maximum solution x^* and admits a minimum solution y^* such that $x_0 \leq x^* \leq y^* \leq y_0$.

Note that a function v (resp. A function w) is called the lower solution (resp. upper solution) of operator T if:

$$v(t) \leq Tv(t), (\text{ resp. } w(t) \geq Tw(t)).$$

Let $E = C[0,1]$, equipped with the supremum norm. Put $P = \{k \in E\; k(t) \geq 0; 0 \leq t \leq 1\}$, which is a cone in E. Define T on P by:

$$T(k)(t) = \int_0^1 G(t,\delta)j(\delta,k(\delta))d\delta,$$

so it is not difficult to see that k is a solution for Equation (1) if and only if $T(k) = k$.

Proof of Theorem 1. We divide the proof into four steps.

Step 1: We will prove that T maps P into itself and that it is completely continuous.

First, since G and j are nonnegative and continuous, it is easy to see that T maps P into itself and that it is continuous. Let Ω be a bounded subset of P, which is to say the existence of $M > 0$ with:

$$\|k\| \leq M, \forall \ k \in \Omega.$$

Put:

$$L = \max_{0\leq t\leq 1, k\in\Omega} |h(t,k)|.$$

Then, for all k in Ω, we get:

$$|Tk(t)| \leq \int_0^1 G(t,\delta)|h(\delta,k(\delta))|d\delta \leq L\int_0^1 G(t,\delta)d\delta.$$

That is, $T(\Omega)$ is a bounded subset of P.
Now, for any $k \in \Omega$ and $0 \leq t_1 < t_2 \leq 1$, we have:

$$\begin{aligned}
&|Tk(t_2) - Tk(t_1)| = |\int_0^1 G(t_2,\delta)h(\delta,k(\delta))d\delta - \int_0^1 G(t_1,\delta)h(\delta,k(\delta))d\delta| \\
&= |\int_0^1 (G(t_2,\delta) - G(t_1,\delta))h(\delta,k(\delta))d\delta| \\
&\leq \int_0^1 |G(t_2,\delta) - G(t_1,\delta)||h(\delta,k(\delta))|d\delta \\
&\leq L\int_0^1 |G(t_2,\delta) - G(t_1,\delta)|d\delta \\
&\leq L\left(\int_0^{t_1} |G(t_2,\delta) - G(t_1,\delta)|d\delta + \int_{t_1}^{t_2} |G(t_2,\delta) - G(t_1,\delta)|d\delta + \int_{t_2}^1 |G(t_2,\delta) - G(t_1,\delta)|d\delta\right) \\
&\leq L\left(I_1 + I_2 + I_3\right)
\end{aligned}$$

where:

$$\begin{aligned}
I_1 &= \int_0^{t_1} |G(t_2,\delta) - G(t_1,\delta)|d\delta \\
&\leq \int_0^{t_1} |(t_2-\delta)^{\alpha-1}\chi_{[0,t_2]}(\delta) - (t_1-\delta)^{\alpha-1}\chi_{[0,t_1]}(\delta)|d\delta \\
&+ \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})}{\beta - \gamma\eta^{\alpha-1}}\int_0^{t_1} |\beta(1-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}\chi_{[0,\eta]}(\delta)|d\delta \\
&= \begin{cases} \frac{1}{\alpha}\left(t_2^\alpha - (t_2-t_1)^\alpha - t_1^\alpha\right) + \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{\beta-\gamma\eta^{\alpha-1}}\left(\frac{\beta}{\alpha}(1-(1-t_1)^\alpha) + \frac{\gamma\eta^\alpha}{\alpha}\right), & \text{if } \eta < t_1 \\ \frac{t_2^\alpha - t_1^\alpha}{\alpha} + \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{\beta-\gamma\eta^{\alpha-1}}\left(\frac{\beta}{\alpha}(1-(1-t_1)^\alpha) + \gamma\frac{\eta^\alpha - (\eta-t_1)^\alpha}{\alpha}\right), & \text{if } \eta > t_1 \end{cases}
\end{aligned}$$

$$
\begin{aligned}
I_2 &= \int_{t_1}^{t_2} |G(t_2,\delta) - G(t_1,\delta)|d\delta \\
&\leq \int_{t_1}^{t_2} |(t_2-\delta)^{\alpha-1}\chi_{[0,t_2]}(\delta) - (t_1-\delta)^{\alpha-1}\chi_{[0,t_1]}(\delta)|d\delta \\
&+ \frac{(t_2^{\alpha} - (t_2-t_1)^{\alpha})}{\beta - \gamma\eta^{\alpha-1}} \int_{t_1}^{t_2} |\beta(1-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}\chi_{[0,\eta]}(\delta)|d\delta \\
&= \begin{cases}
\frac{1}{\alpha}(t_2-t_1)^{\alpha} + \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{\beta-\gamma\eta^{\alpha-1}} \frac{\beta}{\alpha}\left((1-t_1)^{\alpha} - (1-t_2)^{\alpha}\right), & \text{if } 0 \leq \eta < t_1 \\
\frac{t_2^{\alpha}-t_1^{\alpha}}{\alpha} + + \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{\beta-\gamma\eta^{\alpha-1}} \left\{ \frac{\beta}{\alpha}\left((1-t_1)^{\alpha} - (1-t_2)^{\alpha}\right) - \frac{\gamma}{\alpha}(t_1-\eta)^{\alpha} \right\}, & \text{if } t_1 \leq \eta \leq t_2 \\
\frac{t_2^{\alpha}-t_1^{\alpha}}{\alpha} + + \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{\beta-\gamma\eta^{\alpha-1}} \left\{ \frac{\beta}{\alpha}\left((1-t_1)^{\alpha} - (1-t_2)^{\alpha}\right) + \frac{\gamma}{\alpha}[(t_2-\eta)^{\alpha} - (t_1-\eta)^{\alpha}] \right\}, & \text{if } t_2 \leq \eta \leq 1
\end{cases},
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_{t_2}^{1} |G(t_2,\delta) - G(t_1,\delta)|d\delta \\
&\leq \int_{t_1}^{t_2} |(t_2-\delta)^{\alpha-1}\chi_{[0,t_2]}(\delta) - (t_1-\delta)^{\alpha-1}\chi_{[0,t_1]}(\delta)|d\delta \\
&\leq \int_{t_1}^{t_2} |(t_2-\delta)^{\alpha-1}\chi_{[0,t_2]}(\delta) - (t_1-\delta)^{\alpha-1}\chi_{[0,t_1]}(\delta)|d\delta \\
&+ \frac{(t_2^{\alpha} - (t_2-t_1)^{\alpha})}{\beta - \gamma\eta^{\alpha-1}} \int_{t_1}^{t_2} |\beta(1-\delta)^{\alpha-1} - \gamma(\eta-\delta)^{\alpha-1}\chi_{[0,\eta]}(\delta)|d\delta \\
&= \begin{cases}
\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\beta-\gamma\eta^{\alpha-1}} \frac{\beta}{\alpha}(1-t_2)^{\alpha}, & \text{if } \eta < t_2 \\
\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\beta-\gamma\eta^{\alpha-1}} \left\{ \frac{\beta}{\alpha}(1-t_2)^{\alpha} - \frac{\gamma}{\alpha}(\eta-t_2)^{\alpha} \right\}, & \text{if } t_2 < 1.
\end{cases}
\end{aligned}
$$

Then, we obtain that:

$$
\begin{aligned}
|Tk(t_2) - Tk(t_1)| &\leq L(I_1 + I_2 + I_3) \\
&\leq L\left(\frac{t_2^{\alpha} - t_1^{\alpha}}{\alpha} + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\beta - \gamma\eta^{\alpha-1}} \left(\frac{\beta - \gamma\eta^{\alpha}}{\alpha}\right) \right).
\end{aligned}
$$

Since t^{α} and $t^{\alpha-1}$ are uniformly continuous when $t \in [0,1]$ and $1 < \alpha < 2$, it is easy to prove that $T(\Omega)$ is equicontinuous. From the Arzela–Ascoli theorem (see [33], we deduce that $\overline{T(\Omega)}$ is a compact subset. That is, $T : P \to P$ is a completely continuous operator.

Step 2: T is an increasing operator.

Let $0 \leq t \leq 1$. Since the function $\delta \longmapsto j(t,\delta)$ is nondecreasing, then there exists $a > 0$, such that the function $[0,a] \ni \delta \longmapsto j(t,\delta)$ is strictly increasing. It follows that for $k_1 \leq k_2$, we have:

$$
Tk_1(t) = \int_0^1 G(t,\delta)h(\delta,k_1(\delta))d\delta \leq \int_0^1 G(t,\delta)h(\delta,k_2(\delta))d\delta = Tk_2(t).
$$

Step 3: For each $t \in [0,1]$ and from (H_2), there exists $M > 0$ with $0 < j(t,k(t)) < M$. It follows by applying Theorem 3 that equation:

$$
\begin{cases}
D_{0^+}^{\alpha} w(t) + M = 0, & 0 < t < 1,\ 1 < \alpha \leq 2, \\
w(0) = 0,\ \beta w(1) - \gamma w(\eta) = 0 & ,\ \eta \in [0,1].
\end{cases}
$$

has a solution w. Moreover, this solution satisfies:

$$w(t) = \int_0^1 G(t,\delta)Md\delta \geq \int_0^1 G(t,\delta)j(t,w(\delta))d\delta = Tw(t).$$

That is, the operator T admits w as an upper solution.

On the other hand, the operator T admits the zero function as a lower solution; moreover:

$$0 \leq w(t) \ \ \forall\, 0 \leq t \leq 1.$$

Step 4: Since P is a normal cone, Lemma 3 implies that T admits a fixed point $k \in \langle 0, w(t)\rangle$. Therefore, Equation (1) admits a positive solution. □

Proof of Theorem 2. To prove Theorem 2, we begin to prove that T has a fixed point. We remark that if for n large enough, T^n is a contraction operator, then T has a unique fixed point. Indeed, assume that for n large enough, T^n is a contraction operator, and fix $x \in E$. Since T is an increasing operator, which is uniformly bounded, then the sequence $\{T^m x\}_{m\in\mathbb{N}}$ is convergent, that is there is $p \in E$ such that $\lim\limits_{m\to\infty} T^m x = p$. Since T is continuous, we get:

$$Tp = T\lim_{m\to\infty} T^m p = \lim_{m\to\infty} T^{m+1} = p,$$

On the other hand, if p is a fixed point for the operator T, then it is also a fixed point for the operator T^n, so we obtain the uniqueness of p.

Now, let us prove that for n large enough, the operator T^n is a contraction. Let $k, v \in P$, then we have:

$$\begin{aligned}
|Tk(t) - Tv(t)| &= \int_0^1 G(t,\delta)|j(\delta,k(\delta)) - j(\delta,v(\delta))|d\delta \\
&\leq \int_0^1 G(t,\delta)a(\delta)|k(\delta) - v(\delta)|d\delta \\
&\leq \frac{\|k-v\|}{\Gamma(\alpha)} \int_0^1 [(t-\delta)^{\alpha-1} + \frac{t^{\alpha-1}\beta}{(\beta-\gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}]a(\delta)d\delta \\
&\leq \frac{\|k-v\|t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 [1 + \frac{\beta}{(\beta-\gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}]a(\delta)d\delta \\
&\leq \frac{\|k-v\|t^{\alpha-1}}{\Gamma(\alpha)}K,
\end{aligned}$$

where $K = \int_0^1 [1 + \frac{\beta}{(\beta-\gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}]a(\delta)d\delta$.

Similarly, we have:

$$\begin{aligned}
|T^2k(t) - T^2v(t)| &= \int_0^1 G(t,\delta)|j(\delta,Tk(\delta)) - j(\delta,Tv(\delta))|d\delta \\
&\leq \int_0^1 G(t,\delta)a(\delta)|Tk(\delta) - Tv(\delta)|d\delta \\
&\leq \int_0^1 G(t,\delta)a(\delta)\frac{\|k-v\|\delta^{\alpha-1}}{\Gamma(\alpha)}Kd\delta \\
&\leq \frac{\|k-v\|t^{\alpha-1}}{\Gamma(\alpha)^2} \int_0^1 [1 + \frac{\beta}{(\beta-\gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}]\delta^{\alpha-1}a(\delta)d\delta \\
&\leq \frac{\|k-v\|t^{\alpha-1}}{\Gamma(\alpha)^2}KH,
\end{aligned}$$

where $H = \int_0^1 [1 + \frac{\beta}{(\beta - \gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}]\delta^{\alpha-1}a(\delta)d\delta$.

By mathematical induction, it follows that:

$$|T^n k(t) - T^n v(t)| \leq \frac{\|k-v\| t^{\alpha-1}}{\Gamma(\alpha)^n} K H^{n-1}.$$

By using (3), we get:

$$\frac{H}{\Gamma(\alpha)} < 1.$$

Then, for n large enough, it follows that:

$$\frac{KH^{n-1}}{\Gamma(\alpha)^n} = \frac{K}{\Gamma(\alpha)}\left(\frac{H}{\Gamma(\alpha)}\right)^{n-1} < 1.$$

Hence, it holds that:

$$|T^n k(t) - T^n v(t)| < \|k-v\| t^{\alpha-1} \leq \|k-v\|,$$

and this completes the proof. □

4. Examples

In this section, some examples are presented in order to illustrate the usefulness of our main results.

Example 1. *Consider the system:*

$$\begin{cases} D_{0^+}^{\alpha} k(t) = \sqrt{t} e^{-k(t)}, \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ k(0) = 0, \quad \beta k(1) - \gamma k(\eta) = 0, \quad \eta \in [0,1] \end{cases}, \tag{9}$$

where $\beta, \gamma > 0$, $\beta - 2\gamma\eta^{\alpha-1} > 0$.

Note that since for any $t \in [0,1]$, we have $0 < \sqrt{t}e^{-k(t)} < \sqrt{t}$, which implies that Conditions (H_1) and (H_2) hold. On the other hand, there is an equivalence between the solution of Problem (9) and the fixed point of the operator T given by:

$$Tk(t) = \int_0^1 G(t,\delta)\sqrt{\delta} e^{-k(\delta)} d\delta.$$

Take $w(t) = \int_0^1 G(t,\delta)\sqrt{\delta} d\delta$ and $v(t) \equiv 0$, then:

$$w(t) \geq \int_0^1 G(t,\delta)\sqrt{\delta} e^{-w(\delta)} d\delta = Tw(t),$$

which implies that the operator T admits the function w as an upper solution. Moreover, it is obvious that the zero function is a lower solution for T. Hence, from Theorem 1, we conclude that Problem (9) admits a positive solution.

Example 2. *In the second example, we study the following problem:*

$$\begin{cases} D_{0^+}^{3/2} k(t) = \frac{\sin t}{(1+t^2)} \arctan(1 + k(t)), \quad 0 < t < 1, \\ k(0) = 0, \quad \beta k(1) - \gamma k(\eta) = 0, \quad \eta \in [0,1] \end{cases}, \tag{10}$$

where $\beta, \gamma > 0$, $\beta - 2\gamma\sqrt{\eta} > 0$.

Note that $\arctan(1 + k(t)) < \frac{\pi}{2}$ for each $t \in [0,1]$, then $j(t,k(t)) = \frac{\sin t}{(1+t^2)} \arctan(1 + k(t))$ is an increasing and bounded function on k. Therefore, we can easily prove that Conditions (H_1) and (H_2) are

satisfied.
Take $w(t) = \int_0^1 G(t,\delta)\frac{\sin\delta}{(1+\delta^2)}d\delta$ *and* $v(t) \equiv 0$*, then:*

$$w(t) \geq \int_0^1 G(t,\delta)\sqrt{\delta}\frac{\sin\delta}{(1+\delta^2)}\arctan(1+k(\delta))d\delta = Tw(t),$$

which implies that T *admits the function* w *as an upper solution and the zero function as a lower solution. Thus, from Theorem 1, we obtain a positive solution to Problem (9).*

Example 3. *In this example, we take* $\beta = 1$*,* $\gamma = 0$*, and* $\eta \in [0,1]$*, and we consider the following problem:*

$$\begin{cases} D^{\frac{3}{2}}y(t) = \lambda y(t) + f(t), \\ y(0) = 0, \quad y(1) = 0, \end{cases} \tag{11}$$

where f *is a nonnegative function. It is clear that we have:*

$$j(t,x) = \lambda x + f(t),$$

and $\alpha = \frac{3}{2}$*. Moreover, for all* $t \in [0,1]$*, one has:*

$$|j(t,x_1) - j(t,x_2)| \leq \lambda|x_1 - x_2|, \text{ that is } a(t) = \lambda,$$

and:

$$\begin{aligned} \int_0^1 [1 + \frac{\beta}{(\beta - \gamma\eta^{\alpha-1})}(1-\delta)^{\alpha-1}]\delta^{\alpha-1}a(\delta)d\delta &= \lambda\int_0^1 \left(1 + (1-\delta)^{\alpha-1}\right)\delta^{\alpha-1}d\delta \\ &= \lambda(B(1,\alpha) + B(\alpha,\alpha)) \\ &= \lambda\Gamma(\alpha)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\right) \\ &= \lambda\Gamma(\frac{3}{2})\left(\frac{1}{\Gamma(\frac{5}{2})} + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)}\right) < \Gamma(\frac{3}{2}), \end{aligned}$$

for all $0 < \lambda < \frac{12\sqrt{\pi}}{16+3\pi}$*, where* $B(.,.)$ *is the beta function.*

Finally, all conditions of Theorem 2 are satisfied. Therefore, for $0 < \lambda < \frac{12\sqrt{\pi}}{16+3\pi}$*, Problem (11) admits a unique solution.*

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References

1. Ben Ali, K.; Ghanmi, A.; Kefi, K. Existence of solutions for fractional differential equations with dirichlet boundary conditions. *Electr. J. Differ. Equ.* **2016**, *2016*, 1–11.
2. Cabada, A.; Wang, G. Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. *J. Math. Anal. Appl.* **2012**, *389*, 403–411. [CrossRef]
3. Jafari, H.; Gejji, V.D. Positive solutions of nonlinear fractional boundary value problems using adomian decomposition method. *Appl. Math. Comput.* **2006**, *180*, 700–706. [CrossRef]
4. Liang, S.H.; Zhang, J.H. Positive solutions for boundary value problems of nonlinear fractional differential equation. *Nonlinear Anal.* **2009**, *71*, 5545–5550. [CrossRef]

5. Miller, K.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley and Sons: New York, NY, USA, 1993.
6. Nieto, J.J. Maximum principles for fractional differential equations derived from Mittag-Leffler functions. *Appl. Math. Lett.* **2010**, *23*, 1248–1251. [CrossRef]
7. Podlubny, I. Fractional Differential Equations. In *Mathematics in Science and Engineering*; Academic Press: New York, NY, USA, 1999.
8. Wei, Z. Dong, W.; Che, J. Periodic boundary value problems for fractional differential equations involving a Riemann–Liouville fractional derivative. *Nonlinear Anal.* **2010**, *73*, 3232–3238. [CrossRef]
9. Ahmad, B.; Nieto, J.J. Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions. *Bound Value Probl.* **2009**, *2009*, 708576.
10. Băleanu, D.; Octavian, G.M.; O'Regan, D. On a fractional differential equation with infinitely many solutions. *Adv. Diff. Equ.* **2012**, *2012*, 145. [CrossRef]
11. Bai, Z. On positive solutions of a nonlocal fractional boundary value problem. *Nonlinear Anal.* **2010**, *72*, 916–924. [CrossRef]
12. Bai, Z.; Lü, H.S. Positive solutions of boundary value problems of nonlinear fractional differential equation. *J. Math. Anal. Appl.* **2005**, *311*, 495–505. [CrossRef]
13. Ghanmi, A.; Kratou, M.; Saoudi, K. A Multiplicity Results for a Singular Problem Involving a Riemann–Liouville Fractional Derivative. *FILOMAT* **2018**, *32*, 653–669. [CrossRef]
14. Saoudi, K.; Agarwal, P.; Kumam, P.; Ghanmi, A.; Thounthong, P. The Nehari manifold for a boundary value problem involving Riemann–Liouville fractional derivative. *Adv. Diff. Equ.* **2018**, *2018*, 263. [CrossRef]
15. Ghanmi, A.; Horrigue, S. Existence Results for Nonlinear Boundary Value Problems. *FILOMAT* **2018**, *32*, 609–618. [CrossRef]
16. Lakshmikantham, V.; Leela, S.; Vasundhara, J. *Theory of Fractional Dynamic Systems*; Cambridge Academic Publishers: Cambridge, UK, 2009.
17. Wang, G.; Pei, K.; Agarwal, R.P.; Zhang, L.; Ahmad, B. Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line. *J. Comput. Appl. Math.* **2018**, *343*, 230–239. [CrossRef]
18. Wang, G.; Ren, X.; Bai, Z.; Hou, W. Radial symmetry of standing waves for nonlinear fractional Hardy-Schrödinger equation. *Appl. Math. Lett.* **2019**, *96*, 131–137. [CrossRef]
19. Pei, K.; Wang, G.; Sun, Y. Successive iterations and positive extremal solutions for a Hadamard type fractional integro-differential equations on infinite domain. *Appl. Math. Comput.* **2017**, *312*, 158–168. [CrossRef]
20. Bai, Z.; Zhang, S.; Sun, S.; Yin, C. Monotone iterative method for a class of fractional differential equations. *Electron. J. Differ. Equ.* **2016**, *2016*, 1–8.
21. Ding, Y.; Wei, Z.; Xu, J.; O'Regan, D. Extremal solutions for nonlinear fractional boundary value problems with p-Laplacian. *J. Comput. Appl. Math.* **2015**, *288*, 151–158. [CrossRef]
22. Wang, G. Twin iterative positive solutions of fractional q-difference Schrödinger equations. *Appl. Math. Lett.* **2018**, *76*, 103–109. [CrossRef]
23. Wang, G.; Sudsutad, W.; Zhang, L.; Tariboon, J. Monotone iterative technique for a nonlinear fractional q-difference equation of Caputo type. *Adv. Differ. Equ.* **2016** *211*. [CrossRef]
24. Hu, C.; Liu, B.; Xie, S. Monotone iterative solutions for nonlinear boundary value problems of fractional differential equation with deviating arguments. *Appl. Math. Comput.* **2013**, *222*, 72–81. [CrossRef]
25. Liu, Z.; Sun, J.; Szanto, I. Monotone iterative technique for Riemann–Liouville fractional integro- differential equations with advanced arguments. *Results Math.* **2013**, *63*, 1277–1287. [CrossRef]
26. Zhang, X.; Liu, L.; Wu, Y.; Lu, Y. The iterative solutions of nonlinear fractional differential equations. *Appl. Math. Comput.* **2013**, *219*, 4680–4691. [CrossRef]
27. Zhang, X.; Liu, L.; Wu, Y. The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium. *Appl. Math. Lett.* **2014**, *37*, 26–33. [CrossRef]
28. Zhang, L.; Ahmad, B.; Wang, G.; Agarwal, R.P. Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. *J. Comput. Appl. Math.* **2013**, *249*, 51–56. [CrossRef]
29. Zhang, L.; Ahmad, B.; Wang, G. Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half-line. *Bull. Aust. Math. Soc.* **2015**, *91*, 116–128. [CrossRef]
30. Wang, G. Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval. *Appl. Math. Lett.* **2015**, *47*, 1–7. [CrossRef]

31. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations. In *North-Holland Mathematics Studies*; Elsevier Science BV: Amsterdam, The Netherlands, 2006; Volume 204.
32. Zhong, C.; Fan, X.; Chen, W. *Nonlinear Functional Analysis and Its Application*; Lan Zhou University Press: Lanzhou, China, 1998.
33. Rudin, W. *Principles of Mathematical Analysis*; McGraw-Hill New York: 1976; ISBN 978-0-07-054235-8.