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An Iterative Algorithm for the Nonlinear MC² Model with Variational Inequality Method

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Abstract: The multiple criteria and multiple constraint level (MC²) model is a useful tool to deal with the decision programming problems, which concern multiple decision makers and uncertain resource constraint levels. In this paper, by regarding the nonlinear MC² problems as a class of mixed implicit variational inequalities, we develop an iterative algorithm to solve the nonlinear MC² problems through the resolvent operator technique. The convergence of the generated iterative sequence is analyzed and discussed by a calculation example, and the stability of Algorithm 1 is also verified by error propagation. By comparing with two other MC²-algorithms, Algorithm 1 performs well in terms of number of iterations and computation complexity.

Keywords: nonlinear MC² models; implicit variational inequality; resolvent operator technique; iterative algorithm

1. Introduction

In practical applications, the problem involving multi criteria decision-making has become a research hotspot. The design choices based on multi-criteria data acquisition schemes to determine the specific parameters of the attack were discussed, and a new attack type was proposed in [1]. The optimal mapping of hybrid energy systems based on wind and PV (photovoltaic system) was presented by [2]. In addition, the mixed energy system was also obtained by using HOMER (Hybrid Optimization Model for Multiple Energy Resources) software (National Renewable Energy Laboratory (NREL), CO, Boulder, USA) and the TOPSIS multi criteria algorithm. In [3], a nonlinear programming (NP) model based on the technique for order preference by similarity to ideal solution (TOPSIS) was developed to solve decision-making problems. Suzdaltsev et al. [4] developed the genetic, ant colony and bee algorithms for solving the printed circuit board (PCB) design multi criteria optimization problems.

As an extension of linear programming (LP), the multi criteria and multi constraint level linear programming (MC²LP) is a useful tool to handle the decision problems with multiple decision makers and multiple resource constraint levels [5], which can be seen in many economic situations [6]. The concept of MC²LP is attractive to practitioners and has been widely applied in many fields such as transportation [7], data mining [8,9], finance [10], telecommunication management [11], management information systems [12,13], and production planning [14,15]. Specifically, the MC² branch-and-partition algorithm and the MC² branch-and-bound algorithm were presented to solve MC² integer linear programs in [16]. Chen et al. [17] illustrated that MC²-simplex method was generated by the remarkable LP simplex method, and MC²-interior point method was driven by MC-interior point method, as well as introduced the current status and application areas of MC²LP.

A new MC²LP model was proposed based on the structure of MC²LP to correct two types of errors in [18]. Nonlinear programming as an important branch of operations research is a mathematical programming with nonlinear constraints or objective functions, which is explained in a mathematical terms, that is,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) = 0 \quad \forall i \in \mathbf{I}_1, \\ & g_i(x) \leq 0 \quad \forall i \in \mathbf{I}_2, \end{aligned}$$

where each $g_i(x)$ is a mapping from $\mathbf{R}^n$ to $\mathbf{R}$. Traditional methods for solving nonlinear programming include the steepest descent algorithm, Newton method, feasible direction method, function approximation method and trust region method. Aside from those methods, the enhanced Lagrange method is to solve the problem by replacing the original constraint problem with a series of unconstrained sub-problems, [19] proposed an algorithm for the infeasible constrained nonlinear programming problem based on the large-scale augmented Lagrangian function, and analyzed the global convergence considering the possibility of not being feasible. Sequential quadratic programming (SQP) generates steps by solving quadratic subproblems, which can be applied to small and large problems, as well as problems with important nonlinearity [20]. Algorithms for feasible SQP (Sequential Quadratic Programming) were designed by Craig and André [21] to solve optimization problems with nonlinear constraints. In [22], the original problem was reduced to a bounded-constrained nonlinear optimization problem, with reduced gradient algorithms instead of the penalty method. Regardless of the maturity of MC²LP theory and the further study of general nonlinear programming problems, we should notice that there is not yet very much progress in the research of nonlinear MC² models. In this paper, we will introduce a novel method for MC²NLP problems.

It is well known that the variational inequality theory is a very powerful tool to study the problems arising in nonlinear programming. Mathematical conditions, including a constraint qualification and convexity of the feasible set were shown by Toyasaki et al. [23], which allowed for characterizing the economic problem by using a variational inequality formulation. An iterative algorithm was suggested by the resolvent operator technique to compute approximate solutions of the system of nonlinear set valued variational inclusion [24]. The affine variational inequality problems and the polynomial complementary problems were discussed in [25]; here, it is the extension of the results in [24]. Then, the authors applied their results to discuss the existence of the solutions of weakly homogeneous nonlinear equations, the domains of which are closed convex cones. Motivated and inspired by [26,27], the purpose of this paper is to develop a new iterative algorithm for MC²NLP problems by employing the theory of variational inequalities and the resolvent operator technique. Considering the accuracy of solution for MC²NLP problems, the convergence and stability of the new algorithm are discussed in this paper. The result of this paper is the generalization of Theorem 2A.8 (Lagrange multiplier rule) in [28].

2. Preliminaries

The MC²LP model could be formulated in the following form

$$\begin{aligned} \min \quad & \lambda^T Cx \\ \text{s.t.} \quad & Ax \in \text{Conv}\{D\}, \\ & x \geq 0, \end{aligned}$$

where $x \in \mathbf{R}^n$ is the decision vector and $C \in \mathbf{R}^{q \times n}$ is the criteria matrix. $\lambda \in \mathbf{R}^q$ is the vector of weight parameters. $A \in \mathbf{R}^{m \times n}$ is the resource consumption matrix and $D \in \mathbf{R}^{m \times p}$ stands for the multiple resource availability levels. Each column of D indicates a constraint level. The constraint “ $\text{Conv}\{D\}$ ” denotes that x is feasible if Ax is contained in the convex set generated by the column vectors of D ,

and we always denote $\text{Conv}\{D\}$ by D in the following content. Then, this model could be simply extended to the nonlinear case:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \in D, \\ & x \geq 0, \end{aligned} \quad (1)$$

where $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is a nonlinear function and $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a nonlinear mapping. As D is a nonempty, closed and convex set, if both f and g are continuously differentiable, there exists the Lagrange multiplier rule the problem (1), which is listed in the book of Dontchev and Rockafellar [28].

Theorem 1. *Let the following constraint qualification condition be fulfilled: for a given $x \in C := \{x \in \mathbf{R}^n : g(x) \in D\}$, there is no $y \in N_D(g(x))$, $y \neq 0$, such that $-y \nabla g(x) \in N_{\mathbf{R}^n}(x) \equiv 0$. If f has a local minimum relative to C , then there exists $y \in N_D(g(x))$ such that*

$$-[\nabla f(x) + y \nabla g(x)] = 0.$$

Here, $\nabla g(x)$ is the gradient of g at x , and $N_D(x) = \{v \in \mathbf{R}^m : \langle v, x' - x \rangle \leq 0, \forall x' \in D\}$ is the normal cone to D .

Remark 1. *We should notice that the condition above is necessary and the differentiability assumptions on f and g are not always satisfied, so the Lagrange multiplier rule may not always hold. Thus we need some more generalized method.*

First, we should notice that problem (1) is a special case of the following problem when f is differentiable: finding the solution $u \in \mathbf{R}^n$ of the inequality such that

$$\langle \nabla f(u), v - u \rangle \geq 0, \quad \forall v \in C := \{x \in \mathbf{R}^n : g(x) \in D\},$$

which means that any v in the feasible set is always “in the front direction of” u (in the direction that makes the value of f increase). Here, $\langle \cdot, \cdot \rangle$ is the scalar product.

Then, let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a continuous linear operator (such as the gradient or the Gateaux derivative of f : ∇f , Df etc.) and $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a single-valued mapping. We can see that problem (1) is generalized to the implicit variational inequality problem: for the given $D \subset \mathbf{R}^m$, find $u \in \mathbf{R}^n$ such that

$$\begin{aligned} g(u) &\in D, \\ \langle Tu, v - u \rangle &\geq 0, \quad \forall v \in C := \{x \in \mathbf{R}^n : g(x) \in D\}. \end{aligned} \quad (2)$$

Here, we can transform problem (2) to a more general case as finding $u \in \mathbf{R}^n$ such that

$$\begin{aligned} g(u) &\in \text{dom}(\partial\varphi), \text{ and for all } g(v) \in \text{dom}(\partial\varphi) \\ \langle Tu, v - u \rangle &\geq \varphi(g(u)) - \varphi(g(v)), \end{aligned} \quad (3)$$

where $\partial\varphi$ denotes the subdifferential of φ . If $\varphi = \delta_D$, where δ_D denotes the indicator function of a nonempty closed convex set D , then problem (3) is equivalent to problem (2).

Definition 1. *A mapping $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is said to be Lipschitz continuous if there exists a constant $l > 0$ such that*

$$\|T(u) - T(v)\| \leq l\|u - v\|, \quad \forall u, v \in \mathbf{R}^n.$$

Definition 2. A mapping $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is said to be monotone if

$$\langle T(u) - T(v), u - v \rangle \geq 0, \quad \forall u, v \in \mathbf{R}^n.$$

Definition 3. Let $\varphi : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. For some constant $\rho > 0$, the resolvent operator $J_\varphi^\rho : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined by

$$J_\varphi^\rho(v) = (I + \rho \partial \varphi)^{-1}(v), \quad \forall v \in \mathbf{R}^n.$$

By the definition of J_φ^ρ , $u = J_\varphi^\rho(z)$ means that $\frac{1}{\rho}(z - u) \in \partial \varphi(u)$. Combining with the definition of subdifferential $\partial \varphi$, we can see $\langle \frac{1}{\rho}(z - u), v - u \rangle \leq \varphi(v) - \varphi(u), \forall v \in \mathbf{R}^n$, then we have the following lemma.

Lemma 1. For a given $u \in \mathbf{R}^n$ and a point [29] $z \in \mathbf{R}^n$, $u = J_\varphi^\rho(z)$ if and only if

$$\langle u - z, v - u \rangle \geq \rho \varphi(u) - \rho \varphi(v), \quad \forall v \in \mathbf{R}^n.$$

3. Main Results

In this section, an iterative algorithm for problem (3) will be presented, and the related properties will be discussed.

Let $\rho > 0$ be a constant, $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ a single-valued mapping, and $\varphi : \mathbf{R}^m \rightarrow \mathbf{R} \cup \{+\infty\}$ a proper monotone convex lower semicontinuous function. Suppose $g = (g_1, g_2, \dots, g_m) : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a monotone and continuous mapping, and each component g_i of g is monotone continuous convex function. Therefore, the compound function $\varphi \circ g$ is also proper convex lower semicontinuous.

Set

$$\begin{aligned} e(u, \rho) &= u - J_{\varphi \circ g}^\rho(u - \rho Tu) \\ d(u, \rho) &= e(u, \rho) - \rho T e(u, \rho) \\ &= e(u, \rho) - \rho(Tu - T(J_{\varphi \circ g}^\rho(u - \rho Tu))). \end{aligned} \quad (4)$$

Then, we have the following proposition.

Proposition 1. The problem (3) has a solution $u \in \mathbf{R}^n$ if and only if $e(u, \rho) = 0$.

Proof of Proposition 1. If $e(u, \rho) = 0$, then

$$u - J_{\varphi \circ g}^\rho(u - \rho Tu) = 0.$$

By Lemma 1, for any $v \in \mathbf{R}^n$,

$$\langle u - (u - \rho Tu), v - u \rangle \geq \rho \varphi(g(u)) - \rho \varphi(g(v)).$$

Thus, the sufficiency is proved. From the above routine, the necessity is obvious. $\square$

We can give the algorithm for problem (3) as follows:

Algorithm 1. For any $u_0 \in \mathbf{R}^n$, define the iterative sequence $\{u_k\}$ as

$$u_{k+1} = u_k - \frac{1}{\rho} d(u_k, \rho), \quad k = 0, 1, 2, \dots$$

To verify the convergence of $\{u_k\}$ in Algorithm 1, we first introduce the following lemma.

Lemma 2. Let T , φ and g satisfy the conditions presented at the beginning of this section. Moreover, suppose that T is monotone; then, for any solution u^* of problem (3), the following inequality holds:

$$\langle u - u^*, d(u, \rho) \rangle \geq \langle e(u, \rho), d(u, \rho) \rangle.$$

Proof of Lemma 2. Let u^* be the solution of problem (3); then, for any $u \in \mathbf{R}^n$,

$$\langle Tu^*, J_{\varphi \circ g}^\rho(u - \rho Tu) - u^* \rangle \geq \varphi \circ g(u^*) - \varphi \circ g(J_{\varphi \circ g}^\rho(u - \rho Tu)). \quad (5)$$

Moreover, by Lemma 1, we have $\forall u \in \mathbf{R}^n$,

$$\langle u - \rho Tu - J_{\varphi \circ g}^\rho(u - \rho Tu), J_{\varphi \circ g}^\rho(u - \rho Tu) - u^* \rangle \geq \rho \varphi \circ g(J_{\varphi \circ g}^\rho(u - \rho Tu)) - \rho \varphi \circ g(u^*),$$

that is,

$$\langle \frac{1}{\rho} e(u, \rho) - Tu, J_{\varphi \circ g}^\rho(u - \rho Tu) - u^* \rangle \geq \varphi \circ g(J_{\varphi \circ g}^\rho(u - \rho Tu)) - \varphi \circ g(u^*). \quad (6)$$

By the monotonicity of T , we also have

$$\langle T(J_{\varphi \circ g}^\rho(u - \rho Tu)) - Tu^*, J_{\varphi \circ g}^\rho(u - \rho Tu) - u^* \rangle \geq 0. \quad (7)$$

Adding (5), (6) and (7), then

$$\langle T(J_{\varphi \circ g}^\rho(u - \rho Tu)) - Tu + \frac{1}{\rho} e(u, \rho), J_{\varphi \circ g}^\rho(u - \rho Tu) - u^* \rangle \geq 0.$$

i.e.,

$$\begin{aligned} & \langle d(u, \rho), ((J_{\varphi \circ g}^\rho(u - \rho Tu) - u)) + u - u^* \rangle \\ &= \langle d(u, \rho), -e(u, \rho) + u - u^* \rangle \geq 0. \end{aligned}$$

This completes the proof. $\square$

By Lemma 2, we can get the following convergence theorem.

Theorem 2. Let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a monotone and Lipschitz continuous mapping with the Lipschitz constant $l > 0$ and $\{u_k\}$ be the iterative sequence generated by Algorithm 1; then, for the solution u^* of problem (3), the following inequality holds:

$$\|u_{k+1} - u^*\| \leq \|u_k - u^*\|^2 - (\frac{2}{\rho} - \frac{1}{\rho^2} - l^2 - 2l) \|e(u_k, \rho)\|^2. \quad (8)$$

Proof of Theorem 2. It follows from Algorithm 1 that

$$\begin{aligned} \|u_{k+1} - u^*\| &= \|u_k - u^* - \frac{1}{\rho} d(u_k, \rho)\|^2 \\ &= \|u_k - u^*\|^2 + \frac{1}{\rho^2} \|d(u_k, \rho)\|^2 - \frac{2}{\rho} \langle u_k - u^*, d(u_k, \rho) \rangle. \end{aligned} \quad (9)$$

Since T is monotone and Lipschitz continuous,

$$\begin{aligned}
 & \|d(u_k, \rho)\|^2 \\
 = & \|e(u_k, \rho) - \rho(Tu_k - T(J_{\varphi \circ g}^\rho(u_k - \rho Tu_k)))\|^2 \\
 = & \|e(u_k, \rho)\|^2 + \rho^2 \|Tu_k - T(J_{\varphi \circ g}^\rho(u_k - \rho Tu_k))\|^2 - 2\rho \langle e(u_k, \rho), Tu_k - T(J_{\varphi \circ g}^\rho(u_k - \rho Tu_k)) \rangle \\
 \leq & \|e(u_k, \rho)\|^2 + \rho^2 l^2 \|u_k - J_{\varphi \circ g}^\rho(u_k - \rho Tu_k)\|^2 \\
 \leq & (1 + l^2 \rho^2) \|e(u_k, \rho)\|^2.
 \end{aligned} \tag{10}$$

By Lemma 2, we have

$$\langle u_k - u^*, d(u_k, \rho) \rangle \geq \langle e(u_k, \rho), d(u_k, \rho) \rangle.$$

Moreover,

$$\begin{aligned}
 \langle e(u_k, \rho), d(u_k, \rho) \rangle &= \langle e(u_k, \rho), e(u_k, \rho) - \rho(Tu_k - T(J_{\varphi \circ g}^\rho(u_k - \rho Tu_k))) \rangle \\
 &= \|e(u_k, \rho)\|^2 - \rho \langle e(u_k, \rho), Tu_k - T(J_{\varphi \circ g}^\rho(u_k - \rho Tu_k)) \rangle \\
 &= \|e(u_k, \rho)\|^2 - \rho \langle u_k - J_{\varphi \circ g}^\rho(u_k - \rho Tu_k), Tu_k - T(J_{\varphi \circ g}^\rho(u_k - \rho Tu_k)) \rangle \\
 &\geq \|e(u_k, \rho)\|^2 - \rho \|u_k - J_{\varphi \circ g}^\rho(u_k - \rho Tu_k)\| \|Tu_k - T(J_{\varphi \circ g}^\rho(u_k - \rho Tu_k))\| \\
 &\geq \|e(u_k, \rho)\|^2 - \rho l \|u_k - J_{\varphi \circ g}^\rho(u_k - \rho Tu_k)\|^2 \\
 &= \|e(u_k, \rho)\|^2 - \rho l \|e(u_k, \rho)\|^2 = (1 - \rho l) \|e(u_k, \rho)\|^2,
 \end{aligned} \tag{11}$$

which follows Labels (9)–(11) that

$$\|u_{k+1} - u^*\| \leq \|u_k - u^*\|^2 - \left(\frac{2}{\rho} - \frac{1}{\rho^2} - l^2 - 2l\right) \|e(u_k, \rho)\|^2.$$

□

When $l < 1$ and $\rho > \frac{1}{1 + \sqrt{2 - (l+1)^2}}$, we know that $\{u_k\}$ is bounded and $e(u_k, \rho) \rightarrow 0$, which means that $\{u_k\}$ converges to a solution of problem (3). Thus, we get the corollary of Theorem 2.

Corollary 1. Let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a monotone and Lipschitz continuous mapping with the Lipschitz constant $1 > l > 0$ and $\{u_k\}$ be the iterative sequence generated by Algorithm 1. When $\rho > \frac{1}{1 + \sqrt{2 - (l+1)^2}}$, the iterative sequence $\{u_k\}$ generated by Algorithm 1 converges to a solution of problem (3).

To verify the above results, next we will construct a simple example.

Example 1. Let $u = (x_1, x_2, \dots, x_n)^T$ be a dimension variable, $f(u) = \frac{1}{6}(x_1^2 + x_2^2 + \dots + x_n^2)$, $g_1(u) = x_1$, $g_2(u) = x_2, \dots, g_n(u) = x_n$ and $\varphi(g_1, g_2, \dots, g_n) = \sum_{i=1}^n g_i$, thus $\varphi \circ g(u) = \sum_{i=1}^n x_i$, which satisfy all the conditions of Theorem 2. Here, one can easily see that the minima of f is $(0, 0, \dots, 0)^T$. Then, we will use Algorithm 1 to construct a convergent sequence.

Let T be the operator ∇f , then $T(u) = \frac{1}{3}(x_1, x_2, \dots, x_n)^T$, which is Lipschitzian and the Lipschitz constant $l = \frac{1}{3} < 1$. For some given ρ , we can see that

$$J_{\varphi \circ g}^\rho(u) = (I + \rho \partial \varphi \circ g)^{-1}(u) = \frac{1}{1 + \rho} u,$$

thus

$$J_{\varphi \circ g}^\rho(u - \rho Tu) = \frac{1}{1 + \rho} (u - \frac{1}{3} \rho u) = \frac{1 - \frac{1}{3} \rho}{1 + \rho} u.$$

Then, by the algorithm, we have that

$$\begin{aligned} e(u, \rho) &= \frac{\frac{4}{3}\rho}{1+\rho}u \\ d(u, \rho) &= e(u, \rho) - \rho Te(u, \rho) = \frac{\frac{4}{3}\rho}{1+\rho}u - \rho \frac{1}{3} \frac{\frac{4}{3}\rho}{1+\rho}u \\ &= \frac{(\frac{4}{3}\rho - \frac{4}{9}\rho^2)}{1+\rho}u. \end{aligned}$$

Thus, the sequence $\{u_k\}$ is constructed as

$$u_{k+1} = u_k - \frac{1}{\rho}d(u_k, \rho) = \frac{(\frac{13}{9}\rho - \frac{4}{3})}{1+\rho}u_k.$$

When $|\frac{(\frac{13}{9}\rho - \frac{4}{3})}{1+\rho}| < 1$, the sequence $\{u_k\}$ is always convergent to the minima $(0, 0)^T$ of f . The smaller ρ is, the higher convergence speed will be. By the condition $\rho > \frac{1}{1+\sqrt{2-(1+1)^2}}$, we can find the smallest ρ to get the highest speed of the algorithm.

Remark 2. Problem (2) is the special case of problem (3) where $\varphi = \delta_D$, thus we can also get an iterative sequence $\{u_k\}$ converging to a solution of problem (2) by Algorithm 1.

Then, we consider the stability of Algorithm 1.

Theorem 3. Let $T : R^n \rightarrow R^n$ be a monotone and Lipschitz continuous mapping with the Lipschitz constant $1 > l > 0$ and $\{u_k\}$ be the iterative sequence generated by Algorithm 1 for problem (3), the stability of Algorithm 1 can be guaranteed.

Proof of Theorem 3. We first prove that the compound resolvent operator $J_{\varphi \circ g}^\rho$ is Lipschitz continuous. Let u, v be any given points in R^n ; it follows from Definition 3 that

$$J_{\varphi \circ g}^\rho(u) = (I + \rho \partial \varphi \circ g)^{-1}(u) \text{ and } J_{\varphi \circ g}^\rho(v) = (I + \rho \partial \varphi \circ g)^{-1}(v),$$

which implies that

$$\frac{1}{\rho} [u - I(J_{\varphi \circ g}^\rho(u))] \in \partial(J_{\varphi \circ g}^\rho(u)) \text{ and } \frac{1}{\rho} [v - I(J_{\varphi \circ g}^\rho(v))] \in \partial(J_{\varphi \circ g}^\rho(v)).$$

Then,

$$\begin{aligned} & \frac{1}{\rho} \langle [u - I(J_{\varphi \circ g}^\rho(u))] - [v - I(J_{\varphi \circ g}^\rho(v))] , J_{\varphi \circ g}^\rho(u) - J_{\varphi \circ g}^\rho(v) \rangle \\ &= \frac{1}{\rho} \langle u - v - [I(J_{\varphi \circ g}^\rho(u)) - I(J_{\varphi \circ g}^\rho(v))] , J_{\varphi \circ g}^\rho(u) - J_{\varphi \circ g}^\rho(v) \rangle \geq 0. \end{aligned}$$

It follows that

$$\begin{aligned} \|u - v\| \cdot \|J_{\varphi \circ g}^\rho(u) - J_{\varphi \circ g}^\rho(v)\| &\geq \langle u - v , J_{\varphi \circ g}^\rho(u) - J_{\varphi \circ g}^\rho(v) \rangle \\ &\geq \langle I(J_{\varphi \circ g}^\rho(u)) - I(J_{\varphi \circ g}^\rho(v)) , J_{\varphi \circ g}^\rho(u) - J_{\varphi \circ g}^\rho(v) \rangle \\ &= \|J_{\varphi \circ g}^\rho(u) - J_{\varphi \circ g}^\rho(v)\|^2 \end{aligned}$$

so

$$\|J_{\varphi \circ g}^{\rho}(u) - J_{\varphi \circ g}^{\rho}(v)\| \leq \|u - v\|.$$

Let u_k be the iteration value and $\hat{u}_k$ be the accurate value. Then, we have

$$u_{k+1} = u_k - \frac{1}{\rho} d(u_k, \rho) \text{ and } \hat{u}_{k+1} = \hat{u}_k - \frac{1}{\rho} d(\hat{u}_k, \rho)$$

Supposing that ε_k is the error between the iteration value and the exact value of the k th iteration, there holds

$$\varepsilon_{k+1} = \varepsilon_k - \frac{1}{\rho} \left| d(u_k, \rho) - d(\hat{u}_k, \rho) \right|. \quad (12)$$

From Equation (4), we have

$$\begin{aligned} & d(u_k, \rho) - d(\hat{u}_k, \rho) \\ &= e(u_k, \rho) - \rho \left(Tu_k - T(J_{\varphi \circ g}^{\rho}(u_k - \rho Tu_k)) \right) - \left[e(\hat{u}_k, \rho) - \rho \left(T\hat{u}_k - T(J_{\varphi \circ g}^{\rho}(\hat{u}_k - \rho T\hat{u}_k)) \right) \right] \\ &= e(u_k, \rho) - e(\hat{u}_k, \rho) - \rho \left[Tu_k - T\hat{u}_k - \left(T(J_{\varphi \circ g}^{\rho}(u_k - \rho Tu_k)) - T(J_{\varphi \circ g}^{\rho}(\hat{u}_k - \rho T\hat{u}_k)) \right) \right]. \end{aligned}$$

Here,

$$\begin{aligned} e(u_k, \rho) - e(\hat{u}_k, \rho) &= \left(u_k - J_{\varphi \circ g}^{\rho}(u_k - \rho Tu_k) \right) - \left(\hat{u}_k - J_{\varphi \circ g}^{\rho}(\hat{u}_k - \rho T\hat{u}_k) \right) \\ &= u_k - \hat{u}_k - \left(J_{\varphi \circ g}^{\rho}(u_k - \rho Tu_k) - J_{\varphi \circ g}^{\rho}(\hat{u}_k - \rho T\hat{u}_k) \right) \\ &\geq \rho l \varepsilon_k \end{aligned}$$

$$\begin{aligned} & Tu_k - T\hat{u}_k - \left(T(J_{\varphi \circ g}^{\rho}(u_k - \rho Tu_k)) - T(J_{\varphi \circ g}^{\rho}(\hat{u}_k - \rho T\hat{u}_k)) \right) \\ &\leq l \varepsilon_k - l \| J_{\varphi \circ g}^{\rho}(u_k - \rho Tu_k) - J_{\varphi \circ g}^{\rho}(\hat{u}_k - \rho T\hat{u}_k) \| \\ &\leq l \varepsilon_k - l \| (u_k - \rho Tu_k) - (\hat{u}_k - \rho T\hat{u}_k) \| \\ &\leq \rho l^2 \varepsilon_k \end{aligned}$$

so

$$d(u_k, \rho) - d(\hat{u}_k, \rho) \geq \rho l \varepsilon_k - \rho^2 l^2 \varepsilon_k. \quad (13)$$

According to (12) and (17), we can get

$$\begin{aligned} \varepsilon_{k+1} &\leq \varepsilon_k - \frac{1}{\rho} \left| \rho l \varepsilon_k - \rho^2 l^2 \varepsilon_k \right| \\ &= \left| 1 - l + \rho l^2 \right| \varepsilon_k \\ &= \left| 1 - l + \rho l^2 \right|^k \varepsilon_1. \end{aligned}$$

When $0 \leq l \leq 1$ and $\rho \leq \frac{1}{l}$, we know that the effect of error ε_k on the latter is attenuated, thus the stability of Algorithm 1 is guaranteed. $\square$

4. Application and Comparison

It is well known that linear problems can be seen as special cases of nonlinear problems, so the following application of the oilfield production distribution optimization can be showed based on our results.

The historical data of eight sub-measures and the corresponding influencing factors of a oilfield which were in the mid-later stage from 2000 to 2006 are presented in Table 1. These historical data reflect to some extent the law of dynamic change of its development [30].

Table 1. The historical data.

Parameter	2000	2001	2002	2003	2004	2005	2006
oil production/ 10^4 t (fracturing)	6.348	7.211	3.203	2.186	3.012	4.386	8.711
number of wells (fracturing)	55	60	44	41	33	53	91
cost/ 10^4 yuan (fracturing)	3366	4671	2144	1472	1968	3038	5364
oil production/ 10^4 t (acidification)	3.65	2.26	4.59	4.24	4.02	4.08	5.34
number of wells (acidification)	54	61	83	74	61	72	85
cost/ 10^4 yuan (acidification)	2847	1599	3374	3282	3039	3224	4013
oil production/ 10^4 t (fill holes)	69.58	71.06	86.82	72.03	71.05	67.87	53.65
number of wells (fill holes)	883	893	961	812	811	920	735
cost/ 10^4 yuan (fill holes)	15,277	15,352	18,647	15,644	18,311	11,128	10,977
oil production/ 10^4 t (turn extraction)	4.23	4.12	3.83	2.47	2.03	2.57	2.39
number of wells (turn extraction)	56	46	54	47	44	47	42
cost/ 10^4 yuan (turn extraction)	2965	3050	2886	1522	1192	1720	1623
oil production/ 10^4 t (large pumps)	5.73	9.83	8.69	5.42	8.00	5.98	5.80
number of wells (large pumps)	120	147	144	116	129	114	87
cost/ 10^4 yuan (large pumps)	4975	5222	4582	2852	5209	3304	3145
oil production/ 10^4 t (water plugging)	11.02	10.52	11.03	11.48	11.00	10.51	17.11
number of wells (water plugging)	262	253	255	243	239	216	303
cost/ 10^4 yuan (water plugging)	6958	6717	7046	7394	7182	5904	8760
oil production/ 10^4 t (overhaul)	4.97	5.63	5.58	2.08	3.00	2.27	1.73
number of wells (overhaul)	65	73	69	32	44	40	35
cost/ 10^4 yuan (overhaul)	3826	4036	3666	1325	2233	1454	1183
oil production/ 10^4 t (other measures)	8.85	10.13	6.23	5.73	4.00	8.29	7.24
number of wells (other measures)	176	184	132	119	107	187	146
cost/ 10^4 yuan (other measures)	7215	7559	4402	4658	2515	5877	5632

The measure cost c_i per unit of output and the coefficient of resource consumption a_{3i} per unit of output in the planning year are as follows:

$$\begin{aligned}
 c_i &= \frac{\text{Cost for the } i\text{th measure in the planning year}(10^4 \text{ yuan})}{\text{Oil production of the } i\text{th measure in the planning year}(10^4 \text{ t})} \\
 &\approx \frac{\text{Total cost for the } i\text{th measure before the planning year}(10^4 \text{ yuan})}{\text{Total oil production of the } i\text{th measure before the planning year}(10^4 \text{ t})}, \quad (14)
 \end{aligned}$$

$$a_{3i} \approx \frac{\text{Total workload of the } i\text{th measure before the planning year}(10^4 \text{ yuan})}{\text{Total oil production of the } i\text{th measure before the planning year}(10^4 \text{ t})}. \quad (15)$$

Suppose that the forecast annual oil price here is 2760×104 yuan/ 10^4 t, the measure cost c_i per unit of output (Table 2) and the resource consumption coefficient a_{3i} per unit of output (Table 3) in the planning year can be calculated from Table 1 and Formulas (14) and (15).

Table 2. Objective coefficient (yuan/t).

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
2119.74	2003.78	2546.59	2081.86	2160.20	2148.73	2067.41	2017.49

Table 3. Unit resource consumption coefficient (yuan/t).

a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	a_{37}	a_{38}
640.3	756.2	213.4	678.1	599.8	611.3	692.6	742.5

According to the overall historical oil production, investment and workload from 2000 to 2006, as well as the country's demand for crude oil and the technical level of oilfields, the group headquarters and the oilfield branch determined the resource constraint levels for 2007 in Table 4.

Table 4. Resource constraint levels (10^4) yuan.

d_{11}	d_{12}	d_{21}	d_{22}	d_{31}	d_{32}
90	140	35,000	49,000	1450	1760

Here, d_{j_1} ($j = 1, 2, 3$) represents the corresponding constraints as high-level constraints; d_{j_2} ($j = 1, 2, 3$) represents the corresponding constraints as low-level constraints. The new model of the output distribution of the oilfield in 2007 is

$$\begin{aligned} \max \quad & \mathbf{CX}, \\ \text{s.t.} \quad & \mathbf{AX} \leq \mathbf{b}, \\ & \mathbf{X} \geq 0. \end{aligned} \quad (16)$$

Here,

$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2119.74 & 2003.78 & 2546.59 & 2081.86 & 2160.20 & 2148.73 & 2067.41 & 2017.49 \end{pmatrix}, \\ \mathbf{X} &= (x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8)^T, \\ \mathbf{A} &= \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 640.3 & 756.2 & 213.4 & 678.1 & 599.8 & 611.3 & 692.6 & 742.5 \end{pmatrix}, \\ \mathbf{b} &= \begin{pmatrix} -90 & 35,000 & -1450 \\ -140 & 49,000 & -1760 \end{pmatrix}^T, \\ x_i &\geq 0, \ (i = 1, 2, \dots, 8). \end{aligned}$$

The corresponding relaxation form of this problem is

$$\begin{aligned} \max \quad & \lambda \mathbf{CX}, \\ \text{s.t.} \quad & \mathbf{AX} \leq \mathbf{b}\gamma, \\ & \mathbf{X} \geq 0. \end{aligned} \quad (17)$$

Here, $\lambda = (\lambda_1 \ \lambda_2)^T$, $\gamma = (\gamma_1 \ \gamma_2)^T$.

By Algorithm 1, we have

$$\begin{aligned} T(u) &= \nabla f \\ &= (\lambda_1 - 2119.74\lambda_2, \lambda_1 - 2003.78\lambda_2, \lambda_1 - 2546.59\lambda_2, \lambda_1 - 2081.86\lambda_2, \\ &\quad \lambda_1 - 2160.20\lambda_2, \lambda_1 - 2148.73\lambda_2, \lambda_1 - 2067.41\lambda_2, \lambda_1 - 2017.49\lambda_2). \end{aligned}$$

Here, we set $\mathcal{A} = A^T A$, $\mathcal{B} = A^T b$ to make problem (18) easier to solve. Then,

$$\begin{aligned} J_{\varphi \circ g}^{\rho}(u_k - Tu_k) &= (I + \rho \partial \varphi \circ g)^{-1}(u_k - Tu_k) \\ &= (I + \mathcal{A})^{-1}(u_k - Tu_k), \end{aligned} \quad (18)$$

$$\begin{aligned} e(u_k, \rho) &= u_k - J_{\varphi \circ g}^{\rho}(u_k - Tu_k) \\ &= u_k - (I + \mathcal{A})^{-1}(u_k - Tu_k), \end{aligned} \quad (19)$$

$$\begin{aligned} d(u_k, \rho) &= e(u_k, \rho) - \rho[Tu_k - T(J_{\varphi \circ g}^{\rho}(u_k - Tu_k))] \\ &= u_k - [(I + \mathcal{A})^{-1}(u_k - Tu_k)] - \rho[Tu_k - T(J_{\varphi \circ g}^{\rho}(u_k - Tu_k))]. \end{aligned} \quad (20)$$

Take $\lambda_1 = \lambda_2 = 0.5$ and $\gamma_1 = 0.4$, $\gamma_2 = 0.6$, the measure production and the total production in planning year can be calculated with Formulas (18)–(20). Table 5 shows the results of measure production and total production, and Figure 1 shows the convergence of the solution for each measure.

Table 5. Result of measure production and total production (10^4 t).

Measure	Fracturing	Acidification	Fill Holes	Turn Extraction	Larg Pumps
Production	5.9625	0.4221	81.4449	2.3421	5.4038
Measure	Card Water Plugging	Overhaul	Other Measures	Total	
Production	13.8675	2.8115	6.1651	118.4195

Figure 1. Convergence result of Algorithm 1.

By comparing with MC^2 -simplex method and MC^2 -interior point method, some differences can be found in number of iterations and computation complexity.

Table 6 has shown that the computation complexity of the three algorithms differs slightly in this problem. This is because the decision variables, objective functions and constraint levels involved in this problem are relatively small. However, for large-scale problems, the obvious differences in number of iterations and computation complexity could be discovered. Although the number of iterations

of Algorithm 1 and MC^2 -simplex method are the same, we still believe that our method would be efficient if the model turns to a nonlinear case.

Table 6. Comparison study of the speed of three algorithms.

Comparison Factors	Algorithm 1	MC^2 -Simplex Method	MC^2 -Interior Point Method
Number of iterations	2	2	6
Computation complexity (s)	0.003291	0.002884	0.010653

5. Conclusions

This paper proposed Algorithm 1 for MC^2 NLP problems, and its novelty lies in the application of the theory of variational inequalities and the resolvent operator technique. The convergence of the generated iterative sequence was analyzed by Example 1, and the stability of Algorithm 1 was also verified by error propagation. During the considering and discussion, we found that Lipschitz constant l and the parameter ρ are important factors for Algorithm 1 to maintain convergence and stability. In Section 4, Algorithm 1 was utilized to solve the application problem (17); the comparison with two other algorithms showed that this method performs efficiently for the nonlinear problems.

The multi objective multi criteria programming is an important research topic in operations research and management science, not only because of the multi criteria and multi constraint level nature of most real-world decision problems, but also because there are still many open questions in this area. Algorithm 1 as a new tool to solve multi criteria and multi objective problems in decision systems provides algorithm support for decision makers; the basic structural ideas of this algorithm can be extended to practical linear and nonlinear problems in the future, such as project evaluation, program decision-making, engineering, industrial sector development sequencing and rational allocation of resources, etc., so as to consume fewer resources. In addition, the algorithm proposed will be improved in actual application.

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