mathematics

MDPI

Article

On the Hyers-Ulam-Rassias Stability of a General Quintic Functional Equation and a General Sextic Functional Equation

Yang-Hi Lee

Department of Mathematics Education, Gongju National University of Education, Gongju 32553, Korea; yanghi2@hanmail.net

Received: 14 May 2019; Accepted: 31 May 2019; Published: 4 June 2019

Abstract: The general quintic functional equation and the general sextic functional equation are generalizations of many functional equations such as the additive function equation and the quadratic function equation. In this paper, we investigate Hyers–Ulam–Rassias stability of the general quintic functional equation and the general sextic functional equation.

Keywords: the stability of a functional equation; general quintic functional equation; a general quintic mapping; general sextic functional equation; a general sextic mapping

MSC: 39B82; 39B52

1. Introduction

Let X be a real normed space and Y be a real Banach space. In 1940, Ulam [1] raised the question about the stability of group of homomorphisms, and in the following year, Hyers [2] solved this question about the additive functional equation, which gave a partial answer to Ulam's question. In 1978, Rassias [3] generalized Hyers' result (refer to [4–8] for a more generalized result). Since then, many mathematicians have investigated the stability of different types of functional equations [9,10]. Rassias [3] investigated the stability problem for approximately linear mappings controlled by the unbounded function $\theta(\|x\|^p + \|y\|^p)$ as follow:

Theorem 1. *Let $f : X \to Y$ be a mapping from a real normed vector space X into a Banach space Y satisfying the inequality:*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in X \backslash \{0\}$, where θ and p are constants with $\theta > 0$ and $p < 1$. If $f(tx)$ is continuous in t for each fixed x, then there exists a unique linear mapping $T : X \to Y$ such that:

$$\|f(x) - T(x)\| \le \frac{2\theta\,\|x\|^p}{|2-2^p|},$$

for all $x \in X \backslash \{0\}$.

The functional equation is said to have Hyers–Ulam–Rassias stability when the stability can be proven under the control function $\theta(\|x\|^p + \|y\|^p)$.

A mapping $f : X \to Y$ is called a general quintic mapping if f satisfies the functional equation:

$$\sum_{i=0}^{6} {}_6C_i(-1)^{6-i} f(x + (i-3y)) = 0 \tag{1}$$

which is called a general quintic functional equation. A mapping $f : X \to Y$ is called a general sextic mapping:

$$\sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x+iy) = 0 \tag{2}$$

which is called a general sextic functional equation. For example, the functions $f, g : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \sum_{i=0}^{5} a_i x^i$ and $g(x) = \sum_{i=0}^{6} a_i x^i$, $a_i \in \mathbb{R}$, satisfy the above functional equations. More detailed terms for the concepts of "a general quintic mapping" and "a general sextic mapping" can be found in Baker's paper [11] by the terms "generalized polynomial mapping of degree at most 5" and "generalized polynomial mapping of degree at most 6", respectively. Kim et al. [12] previously studied the stability of a general a general cubic functional equation, and Lee [13–15] studied the stability of a general quadratic functional equation, a general cubic functional equation, and a general quartic functional equation.

In Section 2, we will investigate the Hyers–Ulam–Rassias stability of the general quintic functional equation. In Section 3, we will investigate the Hyers–Ulam–Rassias stability of the general sextic functional equation.

2. Stability of a General Quintic Functional Equation

Throughout this section, for a given mapping $f : X \to Y$, we use the following abbreviations:

$$\begin{aligned}
f_o(x) &:= \frac{f(x)-f(-x)}{2}, \quad f_e(x) := \frac{f(x)+f(-x)}{2}, \\
Df(x,y) &:= \sum_{i=0}^{6} {}_6C_i(-1)^{6-i} f(x+(i-3y)), \\
\Gamma f(x) &:= Df_o(2x,2x) + 6Df_o(3x,x) + 36Df_o(2x,x) + 70Df_o(x,x), \\
\Delta f(x) &:= Df_e(x,x) + 3Df_e(0,x)
\end{aligned}$$

for all $x, y \in X$. By laborious computation, we can get the equalities:

$$\Gamma f(x) = f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x), \tag{3}$$

$$\Delta f(x) = f_e(4x) - 20f_e(2x) + 64f_e(x) \tag{4}$$

for all $x \in X$.

Lemma 1. *Let p be a fixed nonnegative real number such that $p \notin \{1,2,3,4,5\}$. For a given mapping $f : X \to Y$ with $f(0) = 0$, let $J_n f : X \to Y$ be the mappings defined by:*

$$
J_n f(x) := \begin{cases}
-\frac{4^n}{3}\left(f_e\left(\frac{x}{2^n}\right) - 16 f_e\left(\frac{x}{2^{n+1}}\right)\right) + \frac{16^{n+1}}{12}\left(f_e\left(\frac{x}{2^n}\right) - 4 f_e\left(\frac{x}{2^{n+1}}\right)\right) & \\
+\frac{2^n - 20\times 8^n + 64\times 32^n}{45} f_o\left(\frac{x}{2^n}\right) - \frac{40\times 2^n - 680\times 8^n + 640\times 32^n}{45} f_o\left(\frac{x}{2^{n+1}}\right) & \\
+\frac{256\times 2^n - 1280\times 8^n + 1024\times 32^n}{45} f_o\left(\frac{x}{2^{n+2}}\right) & \text{if } 5 < p, \\
-\frac{4^n}{3}\left(f_e\left(\frac{x}{2^n}\right) - 16 f_e\left(\frac{x}{2^{n+1}}\right)\right) + \frac{16^{n+1}}{12}\left(f_e\left(\frac{x}{2^n}\right) - 4 f_e\left(\frac{x}{2^{n+1}}\right)\right) & \\
+\frac{2^n - 5\times 8^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n - 170\times 8^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n - 320\times 8^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \\
+\frac{4}{90\times 32^n}\left(f_o(2^{n+1}x) - 10 f_o(2^n x) + 16 f_o(2^{n-1}x)\right) & \text{if } 4 < p < 5, \\
\frac{4^n}{12}\left(16 f_e(2^{-n}x) - f_e(2^{-n+1}x)\right) - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} & \\
+\frac{2^n - 5\times 8^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n - 170\times 8^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n - 320\times 8^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \\
+\frac{4}{90\times 32^n}\left(f_o(2^{n+1}x) - 10 f_o(2^n x) + 16 f_o(2^{n-1}x)\right) & \text{if } 3 < p < 4, \\
\frac{4^n}{12}\left(16 f_e(2^{-n}x) - f_e(2^{-n+1}x)\right) - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} & \\
+\frac{4 f_o(2^{n+1}x)}{90\times 32^n} - \frac{40 f_o(2^n x)}{90\times 32^n} + \frac{64 f_o(2^{n-1}x)}{90\times 32^n} - \frac{5 f_o(2^{n+1}x)}{90\times 8^n} + \frac{170 f_o(2^n x)}{90\times 8^n} - \frac{320 f_o(2^{n-1}x)}{90\times 8^n} & \\
+\frac{2^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 2 < p < 3, \\
\frac{16 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 4^n} - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} & \\
+\frac{4 f_o(2^{n+1}x)}{90\times 32^n} - \frac{40 f_o(2^n x)}{90\times 32^n} + \frac{64 f_o(2^{n-1}x)}{90\times 32^n} - \frac{5 f_o(2^{n+1}x)}{90\times 8^n} + \frac{170 f_o(2^n x)}{90\times 8^n} - \frac{320 f_o(2^{n-1}x)}{90\times 8^n} & \\
+\frac{2^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 1 < p < 2, \\
\frac{16 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 4^n} - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} & \\
+\frac{f_o(2^{n+2}x)}{720\times 32^n} - \frac{10 f_o(2^{n+1}x)}{720\times 32^n} + \frac{16 f_o(2^n x)}{720\times 32^n} - \frac{5 f_o(2^{n+2}x)}{720\times 8^n} + \frac{170 f_o(2^{n+1}x)}{720\times 8^n} - \frac{320 f_o(2^n x)}{720\times 8^n} & \\
+\frac{f_o(2^{n+2}x) - 40 f_o(2^{n+1}x) + 256 f_o(2^n x)}{180\times 2^n} & \text{if } 0 \le p < 1
\end{cases}
$$

for all $x \in X$ and all nonnegative integers n. Then,
$J_n f(x) - J_{n+1} f(x) =$:

$$\begin{cases} (\frac{4^{2n+1}}{3} - \frac{4^n}{3})\Delta f(\frac{x}{2^{n+2}}) + (\frac{2^n}{45} - \frac{4\times 8^n}{9} + \frac{64\times 32^n}{45})\Gamma f(\frac{x}{2^{n+3}}) & \text{if } 5 < p, \\ (\frac{4^{2n+1}}{3} - \frac{4^n}{3})\Delta f(\frac{x}{2^{n+2}}) + (\frac{2^n}{90} - \frac{8^n}{18})\Gamma f(\frac{x}{2^{n+2}}) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 4 < p < 5, \\ -\frac{4^n}{12}\Delta f(\frac{x}{2^{n+1}}) - \frac{\Delta f(2^n x)}{192\times 16^n} + (\frac{2^n}{90} - \frac{8^n}{18})\Gamma f(\frac{x}{2^{n+2}}) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 3 < p < 4, \\ -\frac{4^n}{12}\Delta f(\frac{x}{2^{n+1}}) - \frac{\Delta f(2^n x)}{192\times 16^n} + \frac{2^n}{90}\Gamma f(\frac{x}{2^{n+2}}) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 2 < p < 3, \\ \frac{\Delta f(2^n x)}{48\times 4^n} - \frac{\Delta f(2^n x)}{192\times 16^n} + \frac{2^n}{90}\Gamma f(\frac{x}{2^{n+2}}) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 1 < p < 2, \\ \frac{\Delta f(2^n x)}{48\times 4^n} - \frac{\Delta f(2^n x)}{192\times 16^n} + \frac{\Gamma f(2^n x)}{180\times 2^{n+1}} + \frac{\Gamma f(2^n x)}{144\times 8^{n+1}} - \frac{\Gamma f(2^n x)}{720\times 32^{n+1}} & \text{if } 0 \le p < 1 \end{cases} \tag{5}$$

for all $x \in X$ and all nonnegative integers n.

Proof. For the case $2 < p < 3$, from the definition of $J_n f$ and the equalities (3), we obtain that:

$$\begin{aligned} & J_n f(x) - J_{n+1} f(x) \\ = & \frac{4^n}{12}\left(16 f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{x}{2^{n-1}}\right)\right) - \frac{4^{n+1}}{12}\left(16 f_e\left(\frac{x}{2^{n+1}}\right) - f_e\left(\frac{x}{2^n}\right)\right) \\ & - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} + \frac{4 f_e(2^{n+1}x) - f_e(2^{n+2}x)}{12\times 16^{n+1}} \\ & + \frac{4 f_o(2^{n+1}x)}{90\times 32^n} - \frac{40 f_o(2^n x)}{90\times 32^n} + \frac{64 f_o(2^{n-1}x)}{90\times 32^n} \\ & - \frac{4 f_o(2^{n+2}x)}{90\times 32^{n+1}} + \frac{40 f_o(2^{n+1}x)}{90\times 32^{n+1}} - \frac{64 f_o(2^n x)}{90\times 32^{n+1}} \\ & - \frac{5 f_o(2^{n+1}x)}{90\times 8^n} + \frac{170 f_o(2^n x)}{90\times 8^n} - \frac{320 f_o(2^{n-1}x)}{90\times 8^n} \\ & + \frac{5 f_o(2^{n+2}x)}{90\times 8^{n+1}} - \frac{170 f_o(2^{n+1}x)}{90\times 8^{n+1}} + \frac{320 f_o(2^n x)}{90\times 8^{n+1}} \\ & + \frac{2^n}{90}\left(f_o(\frac{x}{2^{n-1}}) - 40 f_o(\frac{x}{2^n}) + 256 f_o(\frac{x}{2^{n+1}})\right) \\ & - \frac{2^{n+1}}{90}\left(f_o(\frac{x}{2^n}) - 40 f_o(\frac{x}{2^{n+1}}) + 256 f_o(\frac{x}{2^{n+2}})\right) \\ = & \frac{4^n}{12}\left(-f_e\left(\frac{x}{2^{n-1}}\right) + 20 f_e\left(\frac{x}{2^n}\right) - 64 f_e\left(\frac{x}{2^{n+1}}\right)\right) \\ & - \frac{f_e(2^{n+2}x) - 20 f_e(2^{n+1}x) + 64 f_e(2^n x)}{12\times 16^{n+1}} \\ & - \frac{4 f_o(2^{n+2}x) - 168 f_o(2^{n+1}x) + 1344 f_o(2^n x) - 2048 f_o(2^{n-1}x)}{90\times 32^{n+1}} \\ & + \frac{5 f_o(2^{n+2}x) - 210 f_o(2^{n+1}x) + 1680 f_o(2^n x) - 2560 f_o(2^{n-1}x)}{90\times 8^{n+1}} \\ & + \frac{2^n}{90}\left(f_o(\frac{x}{2^{n-1}}) - 42 f_o(\frac{x}{2^n}) + 336 f_o(\frac{x}{2^{n+1}}) - 512 f_o(\frac{x}{2^{n+2}})\right) \\ = & -\frac{4^n}{12}\Delta f(\frac{x}{2^{n+1}}) - \frac{\Delta f(2^n x)}{192\times 16^n} + \frac{2^n}{90}\Gamma f(\frac{x}{2^{n+2}}) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} \end{aligned}$$

for all $x \in X$ and all nonnegative integers n. Furthermore, we easily show that the equality (5) holds by a similar method for the other cases, either $0 \le p < 1$, or $1 < p < 2$, or $3 < p < 4$, or $4 < p < 5$, or $5 < p$. □

Lemma 2. *If $f : X \to Y$ is a mapping such that:*

$$Df(x,y) = 0$$

for all $x, y \in X$ with $f(0) = 0$, then

$$J_n f(x) = f(x)$$

for all $x \in X$ and all positive integers n.

Proof. If $f : X \to Y$ is a mapping such that:

$$Df(x,y) = 0$$

for all $x, y \in X$ with $f(0) = 0$, then it follows from the definitions of $\Delta f(x)$ and $\Gamma f(x)$ that $\Delta f(x) = 0$ and $\Gamma f(x) = 0$ for all $x \in X$. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1}(J_i f(x) - J_{i+1} f(x))$ and the equality (6), we conclude that:

$$J_n f(x) = f(x)$$

for all $x \in X$ and all positive integers n. □

From Lemma 2, we can prove the following stability theorem.

Theorem 2. *Let $p \neq 1,2,3,4,5$ be a fixed nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that:*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{6}$$

for all $x, y \in X$. Then, there exists a general quintic mapping F such that
$\|f(x) - f(0) - F(x)\| \leq$*:*

$$\begin{cases} \left(\frac{K}{45 \times 2^{2p}(2^p-2)} + \frac{(128+44\times 2^p)K}{45(2^p-32)(2^p-8)2^{2p}} + \frac{5}{2^p(2^p-16)(2^p-4)}\right)\theta\|x\|^p & \text{if } 5 < p, \\ \left(\frac{(2\times 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{5}{2^p(2^p-16)(2^p-4)}\right)\theta\|x\|^p & \text{if } 4 < p < 5, \\ \left(\frac{(2\times 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{5}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p & \text{if } 3 < p < 4, \\ \left(\frac{K}{90\times 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{5}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p & \text{if } 2 < p < 3, \\ \left(\frac{K}{90\times 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{5}{(16-2^p)(4-2^p)}\right)\theta\|x\|^p & \text{if } 1 < p < 2, \\ \left(\frac{K}{180(2-2^p)} + \frac{(38-2^p)K}{180(8-2^p)(32-2^p)} + \frac{5}{(16-2^p)(4-2^p)}\right)\theta\|x\|^p & \text{if } 0 \leq p < 1 \end{cases} \tag{7}$$

for all $x \in X$ and $F(0) = 0$, where $K = 182 + 38 \times 2^p + 6 \times 3^p$.

Proof. If $\tilde{f}$ is the mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then the mapping $\tilde{f}$ satisfies the properties $D\tilde{f}(x,y) = Df(x,y)$ and $\tilde{f}(0) = 0$. By (6) and the definitions of Γf and Δf, we have:

$$\begin{aligned} \|\Gamma\tilde{f}(x)\| =& \big\|Df_o(2x,2x) + 6Df_o(3x,x) + 36Df_o(2x,x) + 70Df_o(x,x)\big\| \\ \leq& \theta(182 + 38\times 2^p + 6\times 3^p)\|x\|^p, \\ \|\Delta\tilde{f}(x)\| =& \big\|Df_e(x,x) + 3Df_e(0,x)\big\| \leq 5\theta\|x\|^p \end{aligned}$$

for all $x \in X$. It follows from (5) and (6) that
$\|J_n\tilde{f}(x) - J_{n+1}\tilde{f}(x)\| \leq$:

$$\begin{cases} (\frac{2^nK}{45\times 2^{(n+3)p}} - \frac{4\times 8^nK}{9\times 2^{(n+3)p}} + \frac{64\times 32^nK}{45\times 2^{(n+3)p}} + \frac{5(4^{2n+1}-4^n)}{3\times 2^{(n+2)p}})\theta\|x\|^p & \text{if } 5 < p, \\ (\frac{(5\times 8^n - 2^n)K}{90\times 2^{(n+2)p}} + \frac{2\times 2^{(n-1)p}K)}{45\times 32^{n+1}} + \frac{5(4^{2n+1}-4^n)}{3\times 2^{(n+2)p}})\theta\|x\|^p & \text{if } 4 < p < 5, \\ (\frac{(5\times 8^n - 2^n)K}{90\times 2^{(n+2)p}} + \frac{2\times 2^{(n-1)p}K)}{45\times 32^{n+1}} + \frac{4^n}{12}\frac{5}{2^{(n+1)p}} + \frac{5\times 2^{np}}{192\times 16^n})\theta\|x\|^p & \text{if } 3 < p < 4, \\ (\frac{2^nK}{90\times 2^{(n+2)p}} + \frac{(5\times 4^{n+1}-4)\times 2^{(n-1)p}K}{90\times 32^{n+1}} + \frac{4^n}{12}\frac{5}{2^{(n+1)p}} + \frac{5\times 2^{np}}{192\times 16^n})\theta\|x\|^p & \text{if } 2 < p < 3, \\ (\frac{2^nK}{90\times 2^{(n+2)p}} + \frac{(5\times 4^{n+1}-4)\times 2^{(n-1)p}K}{90\times 32^{n+1}} + \frac{5(4^{n+1}-1)2^{np}}{192\times 16^n})\theta\|x\|^p & \text{if } 1 < p < 2, \\ (\frac{2^{np}K}{180\times 2^{n+1}} + \frac{2^{np}K}{144\times 8^{n+1}} - \frac{2^{np}K}{720\times 32^{n+1}} + \frac{5(4^{n+1}-1)2^{np}}{192\times 16^n})\theta\|x\|^p & \text{if } 0 \leq p < 1 \end{cases}$$

for all $x \in X$. Together with the equality $J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x) = \sum_{i=n}^{n+m-1}(J_i\tilde{f}(x) - J_{i+1}\tilde{f}(x))$ for all $x \in X$, we obtain that
$\|J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x)\| \leq$:

$$\begin{cases} \sum_{i=n}^{n+m-1}(\frac{(2^i - 20\times 8^i + 64\times 32^i)K}{45\times 2^{(i+3)p}} + \frac{5(4^{2i+1}-4^i)}{3\times 2^{(i+2)p}})\theta\|x\|^p & \text{if } 5 < p, \\ \sum_{i=n}^{n+m-1}(\frac{(5\times 8^i - 2^i)K}{90\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K)}{45\times 32^{i+1}} + \frac{5(4^{2i+1}-4^i)}{3\times 2^{(i+2)p}})\theta\|x\|^p & \text{if } 4 < p < 5, \\ \sum_{i=n}^{n+m-1}(\frac{(5\times 8^i - 2^i)K}{90\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K)}{45\times 32^{i+1}} + \frac{4^i}{12}\frac{5}{2^{(i+1)p}} + \frac{5\times 2^{ip}}{192\times 16^i})\theta\|x\|^p & \text{if } 3 < p < 4, \\ \sum_{i=n}^{n+m-1}(\frac{32\times 64^iK + (5\times 4^{i+1}-4)\times 2^{(2i+1)p}K}{90\times 32^{i+1}\times 2^{(i+2)p}} + \frac{4^i}{12}\frac{5}{2^{(i+1)p}} + \frac{5\times 2^{ip}}{192\times 16^i})\theta\|x\|^p & \text{if } 2 < p < 3, \\ \sum_{i=n}^{n+m-1}(\frac{32\times 64^iK + (5\times 4^{i+1}-4)\times 2^{(2i+1)p}K}{90\times 32^{i+1}\times 2^{(i+2)p}} + \frac{5(4^{i+1}-1)2^{ip}}{192\times 16^i})\theta\|x\|^p & \text{if } 1 < p < 2, \\ \sum_{i=n}^{n+m-1}(\frac{2^{ip}K}{180\times 2^{i+1}} + \frac{2^{ip}K}{144\times 8^{i+1}} - \frac{2^{ip}K}{720\times 32^{i+1}} + \frac{5(4^{i+1}-1)2^{ip}}{192\times 16^i})\theta\|x\|^p & \text{if } 0 \leq p < 1 \end{cases} \tag{8}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from (8) that the sequence $\{J_n\tilde{f}(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J_n\tilde{f}(x)\}$ converges for all $x \in X$. Hence, we can define a mapping $F : X \to Y$ by:

$$F(x) := \lim_{n\to\infty} J_n\tilde{f}(x)$$

for all $x \in X$. Note that $F(0) = 0$ follows from $\tilde{f}(0) = 0$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (8), we get the inequality (7). For the case $2 < p < 3$, from the definition of F, we easily get:

$$
\begin{aligned}
\|DF(x,y)\| = &\lim_{n\to\infty}\Bigg\|\frac{4^n}{12}\left(-Df_e\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)+16Df_e\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right)\\
&+\frac{Df_e\left(2^{n+1}x,2^{n+1}y\right)-4Df_e\left(2^nx,2^ny\right)}{12\times 16^n}\\
&+\frac{4Df_o(2^{n+1}x,2^{n+1}y)}{90\times 32^n}-\frac{40Df_o(2^nx,2^ny)}{90\times 32^n}+\frac{64Df_o(2^{n-1}x,2^{n-1}y)}{90\times 32^n}\\
&-\frac{5[Df_o(2^{n+1}x,2^{n+1}y)-34Df_o(2^nx,2^ny)+64Df_o(2^{n-1}x,2^{n-1}y)]}{90\times 8^n}\\
&+\frac{2^n}{90}\left[Df_o\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)-40Df_o\left(\frac{x}{2^n},\frac{y}{2^n}\right)+256Df_o\left(\frac{x}{2^{n+1}},\frac{y}{2^{n+1}}\right)\right]\Bigg\|\\
\leq &\lim_{n\to\infty}\left(\frac{4^n(2^p+16)}{12\times 2^{np}}+\frac{2^{np}(2^p+4)}{12\times 16^n}+\frac{4(4^p+10\times 2^p+16)2^{(n-1)p}}{90\times 32^n}\right.\\
&\left.+\frac{5(4^p+34\times 2^p+64)2^{(n-1)p}}{90\times 8^n}+\frac{2^n(4^p+40\times 2^p+256)}{90\times 2^{(n+1)p}}\right)\times\theta(\|x\|^p+\|y\|^p)\\
=&0
\end{aligned}
$$

for all $x, y \in X$. Furthermore, we easily show that $DF(x,y) = 0$ by a similar method for the other cases, either $0 \leq p < 1$, or $1 < p < 2$, or $3 < p < 4$, or $4 < p < 5$, or $5 < p$. To prove the uniqueness of F, let $F' : X \to Y$ be another general quintic mapping satisfying (7) and $F'(0) = 0$. By Lemma 2, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $2 < p < 3$, we have:

$$
\begin{aligned}
\|J_n\tilde{f}(x)-F'(x)\| =&\|J_n\tilde{f}(x)-J_nF'(x)\|\\
\leq&\frac{4^{n+2}}{12}\left\|(\tilde{f}_e-F_e)\left(\frac{x}{2^n}\right)\right\|+\frac{4^n}{12}\left\|(\tilde{f}_e-F_e)\left(\frac{2x}{2^n}\right)\right\|+\frac{4\|(\tilde{f}_e-F_e)(2^nx)\|}{12\times 16^n}\\
&+\frac{\|(\tilde{f}_e-F_e)(2^{n+1}x)\|}{12\times 16^n}+\frac{4\|(\tilde{f}_o-F_o)(2^{n+1}x)\|}{90\times 32^n}+\frac{40\|(\tilde{f}_o-F_o)(2^nx)\|}{90\times 32^n}\\
&+\frac{64\|(\tilde{f}_o-F_o)(2^{n-1}x)\|}{90\times 32^n}+\frac{5\|(\tilde{f}_o-F_o)(2^{n+1}x)\|}{90\times 8^n}+\frac{170\|(\tilde{f}_o-F_o)(2^nx)\|}{90\times 8^n}\\
&+\frac{320\|(\tilde{f}_o-F_o)(2^{n-1}x)\|}{90\times 8^n}+\frac{2^n}{90}\left\|(\tilde{f}_o-F_o)\left(\frac{2x}{2^n}\right)\right\|\\
&+\frac{40\times 2^n}{90}\left\|(\tilde{f}_o-F_o)\left(\frac{x}{2^n}\right)\right\|+\frac{256\times 2^n}{90}\left\|(\tilde{f}_o-F_o)\left(\frac{x}{2^{n+1}}\right)\right\|\\
\leq&\left(\frac{4^{n+2}+4^n2^p}{12\times 2^{np}}+\frac{4\times 2^{np}+2^{(n+1)p}}{12\times 16^n}+\frac{4(4^p+10\times 2^p+16)2^{(n-1)p}}{90\times 32^n}\right.\\
&\left.+\frac{5(4^p+34\times 2^p+64)2^{(n-1)p}}{90\times 8^n}+\frac{(4^p+40\times 2^p+256)2^n}{90\times 2^{(n+1)p}}\right)\\
&\times\left(\frac{K}{90\times 2^p(2^p-2)}+\frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p}+\frac{5}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p
\end{aligned}
$$

for all $x \in X$ and all positive integers n. Taking the limit in the above inequality as $n \to \infty$, we obtain the equality $F'(x) = \lim_{n\to\infty} J_n\tilde{f}(x)$ for all $x \in X$, which means that $F(x) = F'(x)$ for all $x \in X$. Furthermore, we easily show that $F(x) = F'(x)$ by a similar method for the other cases, either $0 \leq p < 1$, or $1 < p < 2$, or $3 < p < 4$, or $4 < p < 5$, or $5 < p$. □

When n is a fixed number such that $n \in \{1,2,3,4,5\}$, if $f : \mathbb{R} \to \mathbb{R}$ is a solution of the functional equation $\sum_{i=0}^{n} {}_nC_i(-1)^i f(ix+y) - n!f(x) = 0$ for all $x, y \in \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is a solution of the functional equation $Df(x,y) = 0$ for all $x, y \in \mathbb{R}$.

Therefore, Example 1 in [16] shows that the assumption $p \neq 1,2,3,4,5$ cannot be omitted in Theorem 2.

Example 1. (Example 1 in [16]) *There is a mapping $f : \mathbb{R} \to \mathbb{R}$:*

$$\left| \sum_{i=0}^{n} {}_nC_i(-1)^i f(ix+y) - n!f(x) \right| \leq 4 \times (n+1)!(n+1)^{2n}(|x|^n + |y|^n). \tag{9}$$

for all $x, y \in \mathbb{R}$, but there do not exist a mapping $F : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that $\sum_{i=0}^{n} {}_nC_i(-1)^i F(ix+y) - n!F(x) = 0$ and $|f(x) - F(x)| \leq d|x|^n$ for all $x \in \mathbb{R}$.

3. Stability of a General Sextic Functional Equation

Throughout this section, for a given mapping $f : X \to Y$, we use the following abbreviations:

$$\begin{aligned} Df(x,y) &:= \sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x+iy), \\ \Gamma(x) &:= Df_o(-6x,2x) + 6Df_o(-x,x) + 42Df_o(-2x,x) + 112Df_o(-3x,x), \\ \Delta f(x) &:= Df_e(-6x,2x) + 8Df_e(-x,x) + 56Df_e(-2x,x) + 112Df_e(-3x,x) \end{aligned}$$

for all $x, y \in X$. By laborious computation, we can get the equalities:

$$\Gamma f(x) = f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x), \tag{10}$$
$$\Delta f(x) = f_e(8x) - 84f_e(4x) + 1344f_e(2x) - 4096f_e(x) \tag{11}$$

for all $x \in X$.

The proofs of the following two lemmas are very similar to the proofs of Lemmas 1 and 2, so we omit them and just describe them.

Lemma 3. *Let $p \neq 1, 2, 3, 4, 5, 6$ be a fixed real number. For a given mapping $f : X \to Y$ with $f(0) = 0$, let $J_n f : X \to Y$ be the mappings defined by:*

$$
J_n f(x) := \begin{cases}
\begin{array}{l}
-\frac{4^n - 20 \times 16^n + 64 \times 64^n}{45} f_e(\frac{x}{2^n}) - \frac{80 \times 4^n - 1360 \times 16^n + 1280 \times 64^n}{45} f_e(\frac{x}{2^{n+1}}) \\
+\frac{1024 \times 4^n - 5120 \times 16^n + 4096 \times 64^n}{45} f_e(\frac{x}{2^{n+2}}) \\
+\frac{2^n - 20 \times 8^n + 64 \times 32^n}{45} f_o(\frac{x}{2^n}) - \frac{40 \times 2^n - 680 \times 8^n + 640 \times 32^n}{45} f_o(\frac{x}{2^{n+1}}) \\
+\frac{256 \times 2^n - 1280 \times 8^n + 1024 \times 32^n}{45} f_o(\frac{x}{2^{n+2}})
\end{array} & \text{if } 6 < p, \\[2ex]
\begin{array}{l}
\frac{4^n - 5 \times 16^n}{180} f_e(\frac{x}{2^{n-1}}) - \frac{80 \times 4^n - 340 \times 16^n}{180} f_e(\frac{x}{2^n}) + \frac{1024 \times 4^n - 1280 \times 16^n}{180} f_e(\frac{x}{2^{n+1}}) \\
+\frac{4}{180 \times 64^n} \big(f_e(2^{n+1}x) - 20 f_e(2^n x) + 64 f_e(2^{n-1}x)\big) \\
+\frac{2^n - 20 \times 8^n + 64 \times 32^n}{45} f_o(\frac{x}{2^n}) - \frac{40 \times 2^n - 680 \times 8^n + 640 \times 32^n}{45} f_o(\frac{x}{2^{n+1}}) \\
+\frac{256 \times 2^n - 1280 \times 8^n + 1024 \times 32^n}{45} f_o(\frac{x}{2^{n+2}})
\end{array} & \text{if } 5 < p < 6, \\[2ex]
\begin{array}{l}
\frac{4^n - 5 \times 16^n}{180} f_e(\frac{x}{2^{n-1}}) - \frac{80 \times 4^n - 340 \times 16^n}{180} f_e(\frac{x}{2^n}) + \frac{1024 \times 4^n - 1280 \times 16^n}{180} f_e(\frac{x}{2^{n+1}}) \\
+\frac{4}{180 \times 64^n} \big(f_e(2^{n+1}x) - 20 f_e(2^n x) + 64 f_e(2^{n-1}x)\big) \\
+\frac{2^n - 5 \times 8^n}{90} f_o(\frac{x}{2^{n-1}}) - \frac{40 \times 2^n - 170 \times 8^n}{90} f_o(\frac{x}{2^n}) + \frac{256 \times 2^n - 320 \times 8^n}{90} f_o(\frac{x}{2^{n+1}}) \\
+\frac{4}{90 \times 32^n} \big(f_o(2^{n+1}x) - 10 f_o(2^n x) + 16 f_o(2^{n-1}x)\big)
\end{array} & \text{if } 4 < p < 5, \\[2ex]
\begin{array}{l}
\frac{4 f_e(2^{n+1}x)}{180 \times 64^n} - \frac{80 f_e(2^n x)}{180 \times 64^n} + \frac{256 f_e(2^{n-1}x)}{180 \times 64^n} - \frac{5 f_e(2^{n+1}x)}{180 \times 16^n} + \frac{340 f_e(2^n x)}{180 \times 16^n} - \frac{1280 f_e(2^{n-1}x)}{180 \times 16^n} \\
+\frac{4^n}{180} f_e(\frac{x}{2^{n-1}}) - \frac{80 \times 4^n}{180} f_e(\frac{x}{2^n}) + \frac{1024 \times 4^n}{180} f_e(\frac{x}{2^{n+1}}) \\
+\frac{2^n - 5 \times 8^n}{90} f_o(\frac{x}{2^{n-1}}) - \frac{40 \times 2^n - 170 \times 8^n}{90} f_o(\frac{x}{2^n}) + \frac{256 \times 2^n - 320 \times 8^n}{90} f_o(\frac{x}{2^{n+1}}) \\
+\frac{4}{90 \times 32^n} \big(f_o(2^{n+1}x) - 10 f_o(2^n x) + 16 f_o(2^{n-1}x)\big)
\end{array} & \text{if } 3 < p < 4, \\[2ex]
\begin{array}{l}
\frac{4 f_e(2^{n+1}x)}{180 \times 64^n} - \frac{80 f_e(2^n x)}{180 \times 64^n} + \frac{256 f_e(2^{n-1}x)}{180 \times 64^n} - \frac{5 f_e(2^{n+1}x)}{180 \times 16^n} + \frac{340 f_e(2^n x)}{180 \times 16^n} - \frac{1280 f_e(2^{n-1}x)}{180 \times 16^n} \\
+\frac{4^n}{180} f_e(\frac{x}{2^{n-1}}) - \frac{80 \times 4^n}{180} f_e(\frac{x}{2^n}) + \frac{1024 \times 4^n}{180} f_e(\frac{x}{2^{n+1}}) \\
+\frac{4 f_o(2^{n+1}x)}{90 \times 32^n} - \frac{40 f_o(2^n x)}{90 \times 32^n} + \frac{64 f_o(2^{n-1}x)}{90 \times 32^n} - \frac{5 f_o(2^{n+1}x)}{90 \times 8^n} + \frac{170 f_o(2^n x)}{90 \times 8^n} - \frac{320 f_o(2^{n-1}x)}{90 \times 8^n} \\
+\frac{2^n}{90} f_o(\frac{x}{2^{n-1}}) - \frac{40 \times 2^n}{90} f_o(\frac{x}{2^n}) + \frac{256 \times 2^n}{90} f_o(\frac{x}{2^{n+1}})
\end{array} & \text{if } 2 < p < 3, \\[2ex]
\begin{array}{l}
\frac{f_e(2^{n+2}x)}{2880 \times 64^n} - \frac{20 f_e(2^{n+1}x)}{2880 \times 64^n} + \frac{64 f_e(2^n x)}{2880 \times 64^n} - \frac{5 f_e(2^{n+2}x)}{2880 \times 16^n} + \frac{340 f_e(2^{n+1}x)}{2880 \times 16^n} - \frac{1280 f_e(2^n x)}{2880 \times 16^n} \\
+\frac{4 f_e(2^{n+2}x) - 320 f_e(2^{n+1}x) + 4096 f_e(2^n x)}{2880 \times 4^n} \\
+\frac{4 f_o(2^{n+1}x)}{90 \times 32^n} - \frac{40 f_o(2^n x)}{90 \times 32^n} + \frac{64 f_o(2^{n-1}x)}{90 \times 32^n} - \frac{5 f_o(2^{n+1}x)}{90 \times 8^n} + \frac{170 f_o(2^n x)}{90 \times 8^n} - \frac{320 f_o(2^{n-1}x)}{90 \times 8^n} \\
+\frac{2^n}{90} f_o(\frac{x}{2^{n-1}}) - \frac{40 \times 2^n}{90} f_o(\frac{x}{2^n}) + \frac{256 \times 2^n}{90} f_o(\frac{x}{2^{n+1}})
\end{array} & \text{if } 1 < p < 2, \\[2ex]
\begin{array}{l}
\frac{f_e(2^{n+2}x)}{2880 \times 64^n} - \frac{20 f_e(2^{n+1}x)}{2880 \times 64^n} + \frac{64 f_e(2^n x)}{2880 \times 64^n} - \frac{5 f_e(2^{n+2}x)}{2880 \times 16^n} + \frac{340 f_e(2^{n+1}x)}{2880 \times 16^n} - \frac{1280 f_e(2^n x)}{2880 \times 16^n} \\
+\frac{4 f_e(2^{n+2}x) - 320 f_e(2^{n+1}x) + 4096 f_e(2^n x)}{2880 \times 4^n} \\
+\frac{f_o(2^{n+2}x)}{720 \times 32^n} - \frac{10 f_o(2^{n+1}x)}{720 \times 32^n} + \frac{16 f_o(2^n x)}{720 \times 32^n} - \frac{5 f_o(2^{n+2}x)}{720 \times 8^n} + \frac{170 f_o(2^{n+1}x)}{720 \times 8^n} - \frac{320 f_o(2^n x)}{720 \times 8^n} \\
+\frac{f_o(2^{n+2}x) - 40 f_o(2^{n+1}x) + 256 f_o(2^n x)}{180 \times 2^n}
\end{array} & \text{if } p < 1
\end{cases}
$$

for all $x \in X$ and all nonnegative integers n. Then
$J_n f(x) - J_{n+1} f(x) =$

$$\begin{cases} (\frac{4^n}{45} - \frac{4\times 16^n}{9} + \frac{64\times 64^n}{45})\Delta f(\frac{x}{2^{n+3}}) + (\frac{2^n}{45} - \frac{4\times 8^n}{9} + \frac{64\times 32^n}{45})\Gamma f(\frac{x}{2^{n+3}}) & \text{if } 6 < p, \\ (\frac{4^n}{180} - \frac{16^n}{36})\Delta f(\frac{x}{2^{n+2}}) - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + (\frac{2^n}{45} - \frac{4\times 8^n}{9} + \frac{64\times 32^n}{45})\Gamma f(\frac{x}{2^{n+3}}) & \text{if } 5 < p < 6, \\ (\frac{4^n}{180} - \frac{16^n}{36})\Delta f(\frac{x}{2^{n+2}}) - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + (\frac{2^n}{90} - \frac{8^n}{18})\Gamma f(\frac{x}{2^{n+2}}) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 4 < p < 5, \\ \frac{4^n}{180}\Delta f(\frac{x}{2^{n+2}}) + \frac{\Delta f(2^{n-1}x)}{36\times 16^{n+1}} - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + (\frac{2^n}{90} - \frac{8^n}{18})\Gamma f(\frac{x}{2^{n+2}}) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 3 < p < 4, \\ \frac{4^n}{180}\Delta f(\frac{x}{2^{n+2}}) + \frac{\Delta f(2^{n-1}x)}{36\times 16^{n+1}} - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + \frac{2^n}{90}\Gamma f(\frac{x}{2^{n+2}}) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 2 < p < 3, \\ -\frac{\Delta f(2^n x)}{720\times 4^{n+1}} + \frac{\Delta f(2^n x)}{576\times 16^{n+1}} - \frac{\Delta f(2^n x)}{2880\times 64^{n+1}} + \frac{2^n}{90}\Gamma f(\frac{x}{2^{n+2}}) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 1 < p < 2, \\ -\frac{\Delta f(2^n x)}{720\times 4^{n+1}} + \frac{\Delta f(2^n x)}{576\times 16^{n+1}} - \frac{\Delta f(2^n x)}{2880\times 64^{n+1}} - \frac{\Gamma f(2^n x)}{180\times 2^{n+1}} + \frac{\Gamma f(2^n x)}{144\times 8^{n+1}} - \frac{\Gamma f(2^n x)}{720\times 32^{n+1}} & \text{if } p < 1 \end{cases} \quad (12)$$

for all $x \in X$ and all nonnegative integers n.

Lemma 4. *If $f : X \to Y$ is a mapping such that $Df(x,y) = 0$ for all $x, y \in X$, then $J_n f(x) = f(x)$ for all $x \in X$ and all positive integers n.*

Lemma 5. *If $f : X \to Y$ is a mapping such that $f(0) = 0$ and $Df(x,y) = 0$ for all $x, y \in X \backslash \{0\}$, then $Df(x,y) = 0$ for all $x, y \in X$.*

Proof. Since $Df(x,0) = 0$ and $Df(0,y) = Df(7y,-y) = 0$ for all $x, y \in X\backslash\{0\}$, the equality $Df(x,y) = 0$ holds for all $x, y \in X$. □

From Lemmas 4 and 5, we can prove the following Hyers–Ulam–Rassias stability of the sextic functional equation.

Theorem 3. *Let $p \neq 1,2,3,4,5,6$ be a fixed real number. Suppose that $f : X \to Y$ is a mapping such that:*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad (13)$$

for all $x, y \in X\backslash\{0\}$. Then, there exists a unique general sextic mapping F such that $\|f(x) - f(0) - F(x)\| \leq$:

$$
\begin{cases}
\Big(\frac{K}{(2^p-2)} + \frac{(128+44\times 2^p)K}{(2^p-32)(2^p-8)} + \frac{K'}{(2^p-4)} + \frac{(256+44\times 2^p)K'}{(2^p-64)(2^p-16)}\Big)\frac{\theta\|x\|^p}{45\times 2^{2p}} & \text{if } 6<p, \\
\Big(\frac{K}{2^p(2^p-2)} + \frac{(128+44\times 2^p)K}{(2^p-32)(2^p-8)2^p} + \frac{(2^p-1)K'}{(2^p-4)(2^p-16)} + \frac{K'}{(64-2^p)}\Big)\frac{\theta\|x\|^p}{45\times 2^{p}} & \text{if } 5<p<6, \\
\Big(\frac{(2\times 2^p-1)K}{(2^p-8)(2^p-2)} + \frac{2K}{(32-2^p)} + \frac{(2^p-1)K'}{(2^p-4)(2^p-16)} + \frac{K'}{(64-2^p)}\Big)\frac{\theta\|x\|^p}{45\times 2^{p}} & \text{if } 4<p<5, \\
\Big(\frac{(2\times 2^p-1)K}{(2^p-8)(2^p-2)} + \frac{2K}{(32-2^p)} + \frac{K'}{4(2^p-4)} + \frac{(256-2^p)K'}{4(64-2^p)(16-2^p)}\Big)\frac{\theta\|x\|^p}{45\times 2^{p}} & \text{if } 3<p<4, \\
\Big(\frac{K}{(2^p-2)} + \frac{(128-2^p)K}{(32-2^p)(8-2^p)} + \frac{K'}{2(2^p-4)} + \frac{(256-2^p)K'}{2(64-2^p)(16-2^p)}\Big)\frac{\theta\|x\|^p}{90\times 2^{p}} & \text{if } 2<p<3, \\
\Big(\frac{K}{2^p(2^p-2)} + \frac{(128-2^p)K}{(32-2^p)(8-2^p)2^p} + \frac{(44+2^p)K'}{32(16-2^p)(4-2^p)} + \frac{K'}{32(64-2^p)}\Big)\frac{\theta\|x\|^p}{90} & \text{if } 1<p<2, \\
\Big(\frac{(22+2^p)K}{(8-2^p)(2-2^p)} + \frac{K}{(32-2^p)} + \frac{(44+2^p)K'}{4(16-2^p)(4-2^p)} + \frac{K'}{4(64-2^p)}\Big)\frac{\theta\|x\|^p}{720} & \text{if } p<1
\end{cases}
\tag{14}
$$

for all $x \in X\backslash\{0\}$ and $F(0) = 0$, where $K := 166 + 43\times 2^p + 112\times 3^p + 6^p$ and $K' := 184 + 57\times 2^p + 112\times 3^p + 6^p$.

Proof. If $\tilde{f}$ is the mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then $D\tilde{f}(x,y) = Df(x,y)$ and $\tilde{f}(0) = 0$. By (13) and the definitions of $\Gamma\tilde{f}$ and $\Delta\tilde{f}$, we have:

$$
\begin{aligned}
\|\Gamma\tilde{f}(x)\| =& \big\|Df_o(-6x,2x) + 6Df_o(-x,x) + 42Df_o(-2x,x) + 112Df_o(-3x,x)\big\| \\
\leq& \theta(166 + 43\times 2^p + 112\times 3^p + 6^p)\|x\|^p, \\
\|\Delta\tilde{f}(x)\| =& \big\|Df_e(-6x,2x) + 8Df_e(-x,x) + 56Df_e(-2x,x) + 112Df_e(-3x,x)\big\| \\
\leq& \theta(184 + 57\times 2^p + 112\times 3^p + 6^p)\|x\|^p
\end{aligned}
$$

for all $x \in X\backslash\{0\}$. It follows from (12) and (13) that $\|J_n\tilde{f}(x) - J_{n+1}\tilde{f}(x)\| \leq$:

$$
\begin{cases}
\Big(\frac{2^nK}{45\times 2^{(n+3)p}} - \frac{4\times 8^nK}{9\times 2^{(n+3)p}} + \frac{64\times 32^nK}{45\times 2^{(n+3)p}} + \frac{4^nK'}{45\times 2^{(n+3)p}} - \frac{4\times 16^nK'}{9\times 2^{(n+3)p}} + \frac{64\times 64^nK'}{45\times 2^{(n+3)p}}\Big)\theta\|x\|^p & \text{if } 6<p, \\
\Big(\frac{2^nK}{45\times 2^{(n+3)p}} - \frac{4\times 8^nK}{9\times 2^{(n+3)p}} + \frac{64\times 32^nK}{45\times 2^{(n+3)p}} + \frac{(5\times 16^n-4^n)K'}{180\times 2^{(n+2)p}} + \frac{2\times 2^{(n-1)p}K')}{90\times 64^{n+1}}\Big)\theta\|x\|^p & \text{if } 5<p<6, \\
\Big(\frac{(5\times 8^n-2^n)K}{90\times 2^{(n+2)p}} + \frac{2\times 2^{(n-1)p}K)}{45\times 32^{n+1}} + \frac{(5\times 16^n-4^n)K'}{180\times 2^{(n+2)p}} + \frac{2\times 2^{(n-1)p}K')}{90\times 64^{n+1}}\Big)\theta\|x\|^p & \text{if } 4<p<5, \\
\Big(\frac{(5\times 8^n-2^n)K}{90\times 2^{(n+2)p}} + \frac{2\times 2^{(n-1)p}K)}{45\times 32^{n+1}} + \frac{4^nK'}{180\times 2^{(n+2)p}} + \frac{(5\times 4^{n+1}-4)\times 2^{(n-1)p}K'}{180\times 64^{n+1}}\Big)\theta\|x\|^p & \text{if } 3<p<4, \\
\Big(\frac{2^nK}{90\times 2^{(n+2)p}} + \frac{(5\times 4^{n+1}-4)\times 2^{(n-1)p}K}{90\times 32^{n+1}} + \frac{4^nK'}{180\times 2^{(n+2)p}} + \frac{(5\times 4^{n+1}-4)\times 2^{(n-1)p}K'}{180\times 64^{n+1}}\Big)\theta\|x\|^p & \text{if } 2<p<3, \\
\Big(\frac{2^nK}{90\times 2^{(n+2)p}} + \frac{(5\times 4^{n+1}-4)\times 2^{(n-1)p}K}{90\times 32^{n+1}} + \frac{2^{np}K'}{720\times 4^{n+1}} - \frac{2^{np}K'}{576\times 16^{n+1}} + \frac{2^{np}K'}{2880\times 64^{n+1}}\Big)\theta\|x\|^p & \text{if } 1<p<2, \\
\Big(\frac{2^{np}K}{180\times 2^{n+1}} - \frac{2^{np}K}{144\times 8^{n+1}} + \frac{2^{np}K}{720\times 32^{n+1}} + \frac{2^{np}K'}{720\times 4^{n+1}} - \frac{2^{np}K'}{576\times 16^{n+1}} + \frac{2^{np}K'}{2880\times 64^{n+1}}\Big)\theta\|x\|^p & \text{if } p<1
\end{cases}
$$

for all $x \in X\backslash\{0\}$. Together with the equality $J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x) = \sum_{i=n}^{n+m-1}(J_i\tilde{f}(x) - J_{i+1}\tilde{f}(x))$ for all $x \in X$, we obtain that
$\|J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x)\| \leq$:

$$\begin{cases} \sum_{i=n}^{n+m-1}(\frac{2^iK - 20\times 8^iK + 64\times 32^iK + 4^iK' - 20\times 16^iK' + 64\times 64^iK'}{45\times 2^{(i+3)p}})\theta\|x\|^p & \text{if } 6 < p, \\ \sum_{i=n}^{n+m-1}(\frac{2^iK - 20\times 8^iK + 64\times 32^iK}{45\times 2^{(i+3)p}} + \frac{(5\times 16^i - 4^i)K'}{180\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K')}{90\times 64^{i+1}})\theta\|x\|^p & \text{if } 5 < p < 6, \\ \sum_{i=n}^{n+m-1}(\frac{(5\times 8^i - 2^i)K}{90\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K)}{45\times 32^{i+1}} + \frac{(5\times 16^i - 4^i)K'}{180\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K')}{90\times 64^{i+1}})\theta\|x\|^p & \text{if } 4 < p < 5, \\ \sum_{i=n}^{n+m-1}(\frac{(5\times 8^i - 2^i)K}{90\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K)}{45\times 32^{i+1}} + \frac{4^iK'}{180\times 2^{(i+2)p}} + \frac{(5\times 4^{i+1} - 4)\times 2^{(i-1)p}K'}{180\times 64^{i+1}})\theta\|x\|^p & \text{if } 3 < p < 4, \\ \sum_{i=n}^{n+m-1}(\frac{32\times 64^iK + (5\times 4^{i+1} - 4)\times 2^{(2i+1)p}K}{90\times 32^{i+1}\times 2^{(i+2)p}} + \frac{4^{4i+3} + (5\times 4^{i+1} - 4)\times 2^{(2i+1)p}K'}{180\times 64^{i+1}2^{(i+2)p}})\theta\|x\|^p & \text{if } 2 < p < 3, \\ \sum_{i=n}^{n+m-1}(\frac{32\times 64^iK + (5\times 4^{i+1} - 4)\times 2^{(2i+1)p}K}{90\times 32^{i+1}\times 2^{(i+2)p}} + \frac{(4^{i+2} - 5)2^{ip}K'}{2880\times 16^{i+1}} + \frac{2^{ip}K'}{2880\times 64^{i+1}})\theta\|x\|^p & \text{if } 1 < p < 2, \\ \sum_{i=n}^{n+m-1}(\frac{2^{ip}K}{180\times 2^{i+1}} - \frac{2^{ip}K}{144\times 8^{i+1}} + \frac{2^{ip}K}{720\times 32^{i+1}} + \frac{(4^{i+2} - 5)2^{ip}K'}{2880\times 16^{i+1}} + \frac{2^{ip}K'}{2880\times 64^{i+1}})\theta\|x\|^p & \text{if } p < 1 \end{cases} \tag{15}$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from (15) that the sequence $\{J_n\tilde{f}(x)\}$ is a Cauchy sequence for all $x \in X\backslash\{0\}$. Since Y is complete and $\tilde{f}(0) = 0$, the sequence $\{J_n\tilde{f}(x)\}$ converges for all $x \in X$. Hence, we can define a mapping $F : X \to Y$ by:

$$F(x) := \lim_{n\to\infty} J_n\tilde{f}(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (15), we get the inequality (14). For the case $2 < p < 3$, from the definition of F, we easily get:

$$\begin{aligned} \|DF(x,y)\| = \lim_{n\to\infty} \Bigg\| & \frac{Df_e(2^{n+1}x, 2^{n+1}y)}{45\times 64^n} - \frac{20Df_e(2^nx, 2^ny)}{45\times 64^n} + \frac{64Df_o(2^{n-1}x, 2^{n-1}y)}{45\times 64^n} \\ & - \frac{Df_e(2^{n+1}x, 2^{n+1}y) - 68Df_e(2^nx, 2^ny) + 256Df_e(2^{n-1}x, 2^{n-1}y)}{36\times 16^n} \\ & + \frac{4^n}{180}\left[Df_e\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) - 80Df_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 1024Df_e\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right)\right] \\ & + \frac{4Df_o(2^{n+1}x, 2^{n+1}y)}{90\times 32^n} - \frac{40Df_o(2^nx, 2^ny)}{90\times 32^n} + \frac{64Df_o(2^{n-1}x, 2^{n-1}y)}{90\times 32^n} \\ & - \frac{5[Df_o(2^{n+1}x, 2^{n+1}y) - 34Df_o(2^nx, 2^ny) + 64Df_o(2^{n-1}x, 2^{n-1}y)]}{90\times 8^n} \\ & + \frac{2^n}{90}\left[Df_o\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) - 40Df_o\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 256Df_o\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right)\right]\Bigg\| \\ \leq \lim_{n\to\infty} \Bigg(& \frac{(4^p + 20\times 2^p + 64)2^{(n-1)p}}{45\times 64^n} + \frac{(4^p + 68\times 2^p + 256)2^{(n-1)p}}{36\times 16^n} \\ & + \frac{4^n(4^p + 80\times 2^p + 1024)}{180\times 2^{(n+1)p}} + \frac{4(4^p + 10\times 2^p + 16)2^{(n-1)p}}{90\times 32^n} \\ & + \frac{(4^p + 34\times 2^p + 64)2^{(n-1)p}}{18\times 8^n} + \frac{2^n(4^p + 40\times 2^p + 256)}{90\times 2^{(n+1)p}}\Bigg) \times \theta(\|x\|^p + \|y\|^p) \\ =0 & \end{aligned}$$

for all $x, y \in X\backslash\{0\}$. Since $DF(x,y) = 0$ for all $x, y \in X\backslash\{0\}$, $F : X \to Y$ satisfies the equality $DF(x,y) = 0$ for all $x, y \in X$ by Lemma 5. Furthermore, we easily show that $DF(x,y) = 0$ by a similar method for the other cases, either $p < 1$, or $1 < p < 2$, or $3 < p < 4$, or $4 < p < 5$, or $5 < p < 6$, or $6 < p$. To prove the uniqueness of F, let $F' : X \to Y$ be another sextic mapping satisfying (14) and

$F'(0) = 0$. By Lemma 4, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $2 < p < 3$, we have:

$$\begin{aligned}
&\|J_n\tilde{f}(x) - F'(x)\| \\
&\quad =\|J_n\tilde{f}(x) - J_nF'(x)\| \\
&\quad \le \frac{\|(\tilde{f}_e - F_e)(2^{n+1}x)\|}{45 \times 64^n} + \frac{20\|(\tilde{f}_e - F_e)(2^n x)\|}{45 \times 64^n} + \frac{64\|(\tilde{f}_e - F_e)(2^{n-1}x)\|}{45 \times 64^n} \\
&\qquad + \frac{\|(\tilde{f}_e - F_e)(2^{n+1}x)\|}{36 \times 16^n} + \frac{68\|(\tilde{f}_e - F_e)(2^n x)\|}{36 \times 16^n} + \frac{256\|(\tilde{f}_e - F_e)(2^{n-1}x)\|}{36 \times 16^n} \\
&\qquad + \frac{4^n}{180}\left[\left\|(\tilde{f}_e - F_e)\left(\frac{2x}{2^n}\right)\right\| + 80\left\|(\tilde{f}_e - F_e)\left(\frac{x}{2^n}\right)\right\| + 1024\left\|(\tilde{f}_e - F_e)\left(\frac{x}{2^{n+1}}\right)\right\|\right] \\
&\qquad + \frac{4\|(\tilde{f}_o - F_o)(2^{n+1}x)\|}{90 \times 32^n} + \frac{40\|(\tilde{f}_o - F_o)(2^n x)\|}{90 \times 32^n} + \frac{64\|(\tilde{f}_o - F_o)(2^{n-1}x)\|}{90 \times 32^n} \\
&\qquad + \frac{5\|(\tilde{f}_o - F_o)(2^{n+1}x)\|}{90 \times 8^n} + \frac{170\|(\tilde{f}_o - F_o)(2^n x)\|}{90 \times 8^n} + \frac{320\|(\tilde{f}_o - F_o)(2^{n-1}x)\|}{90 \times 8^n} \\
&\qquad + \frac{2^n}{90}\left[\left\|(\tilde{f}_o - F_o)\left(\frac{2x}{2^n}\right)\right\| + 40\left\|(\tilde{f}_o - F_o)\left(\frac{x}{2^n}\right)\right\| + 256\left\|(\tilde{f}_o - F_o)\left(\frac{x}{2^{n+1}}\right)\right\|\right] \\
&\quad \le \Bigg(\frac{(4^p + 20 \times 2^p + 64)2^{(n-1)p}}{45 \times 64^n} + \frac{(4^p + 68 \times 2^p + 256)2^{(n-1)p}}{36 \times 16^n} \\
&\qquad + \frac{(4^p + 80 \times 2^p + 1024)2^n}{180 \times 2^{(n+1)p}} + \frac{2(4^p + 10 \times 2^p + 16)2^{(n-1)p}}{45 \times 32^n} \\
&\qquad + \frac{(4^p + 34 \times 2^p + 64)2^{(n-1)p}}{18 \times 8^n} + \frac{(4^p + 40 \times 2^p + 256)2^n}{90 \times 2^{(n+1)p}}\Bigg) \\
&\qquad \times \left(\frac{K}{2^p - 2} + \frac{(128 - 2^p)K}{(32 - 2^p)(8 - 2^p)} + \frac{K'}{2(2^p - 4)} + \frac{(256 - 2^p)K'}{2(64 - 2^p)(16 - 2^p)}\right)\frac{\theta\|x\|^p}{90 \times 2^p}
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and all positive integers n. Taking the limit in the above inequality as $n \to \infty$, we obtain the equality $F'(x) = \lim_{n\to\infty} J_n\tilde{f}(x)$ for all $x \in X\backslash\{0\}$, which means that $F(x) = F'(x)$ for all $x \in X\backslash\{0\}$. Furthermore, we easily show that $F(x) = F'(x)$ by a similar method for the other cases, either $p < 1$, or $1 < p < 2$, or $3 < p < 4$, or $4 < p < 5$, or $5 < p < 6$, or $6 < p$. $\square$

From Theorem 3, we also prove the hyperstability of the sextic functional equation when $p < 0$.

Theorem 4. *Let $p < 0$ be a real number. If a mapping $f : X \to Y$ satisfies the inequality (13) for all $x, y \in X\backslash\{0\}$, then $f : X \to Y$ is a sextic mapping itself.*

Proof. According to Theorem 3, there is a unique sextic mapping F of the functional equation $DF(x,y) = 0$ such that:

$$\|\tilde{f}(x) - F(x)\| \le \left(\frac{(22 + 2^p)K}{(8 - 2^p)(2 - 2^p)} + \frac{K}{(32 - 2^p)} + \frac{(44 + 2^p)K'}{4(16 - 2^p)(4 - 2^p)} + \frac{K'}{4(64 - 2^p)}\right)\frac{\theta\|x\|^p}{720}$$

for all $x \in X\backslash\{0\}$ and $F(0) = 0$. From the equality:

$$\begin{aligned}
D\tilde{f}(nx, -(n-1)x) =& D\tilde{f}(nx, -(n-1)x) - DF(nx, -(n-1)x) \\
=& \sum_{i=0}^{7} {}_7C_i(-1)^{7-i}(\tilde{f} - F)(nx - i(n-1)x)
\end{aligned}$$

for all $x \in X \backslash \{0\}$ and $n \in \mathbb{N}$, we have the inequality:

$$
\begin{aligned}
\|{}_7C_1(\tilde{f}-F)(x)\| =& \Big\| Df(nx, -(n-1)x) + (\tilde{f}-F)(nx) \\
& - \sum_{i=2}^{7} {}_7C_i(-1)^{7-i}(\tilde{f}-F)(nx - i(n-1)x) \Big\| \\
\leq& \theta \|x\|^p \Bigg[n^p + (n-1)^p + \Big(n^p + \sum_{i=2}^{7} {}_7C_i(i(n-1)-n)^p \Big) \\
& \times \Big(\frac{(22+2^p)K}{(8-2^p)(2-2^p)} + \frac{K}{(32-2^p)} + \frac{(44+2^p)K'}{4(16-2^p)(4-2^p)} + \frac{K'}{4(64-2^p)} \Big) \Bigg]
\end{aligned}
$$

for all $x \in X \backslash \{0\}$ and $n \in \mathbb{N} \backslash \{1,2\}$. Since $\tilde{f}(0) = F(0)$ and n^p, $\sum_{i=2}^{7} {}_7C_i(i(n-1)-n)^p$, and $(n-1)^p$ tends to zero as $n \to \infty$, we get $\tilde{f}(x) = F(x)$ for all $x \in X$. Therefore, $Df(x,y) = D\tilde{f}(x,y) = DF(x,y) = 0$ for all $x, y \in X$. □

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

Reference

1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224.
3. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300.
4. Gajda, Z. On stability of additive mappings. *Int. J. Math. Math. Sci.* **1991**, *14*, 431–434.
5. Găvruta, P. A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436.
6. Isac, G.; Rassias, T.M. On the Hyers-Ulam stability of ψ-additive mappings. *J. Approx. Theory* **1993**, *72*, 131–137.
7. Jung, S.-M. On the Hyers–Ulam–Rassias stability of approximately additive mappings. *J . Math. Anal. Appl.* **1996**, *204*, 221–226.
8. Lee, Y.-H.; Jun, K.W. On the stability of approximately additive mappings. *Proc. Am. Math. Soc.* **2000**, *128*, 1361–1369.
9. Gordji, M.E.; Alizadeh, B.; de la Sen, M.; Ghaemi, M.B. Fixed points and approximately C*-ternary quadratic higher derivations. *Int. J. Geom. Methods Mod. Phys.* **2012**, *2012*, 1320017.
10. Yang, X.; Shen, G.; Liu, G.; Chang, L. The Hyers–Ulam–Rassias stability of the quartic functional equation in fuzzy β-normed spaces. *J. Inequal. Appl.* **2015**, *2015*, 342.
11. Baker, J. A general functional equation and its stability. *Proc. Natl. Acad. Sci. USA* **2005**, *133*, 1657–1664.
12. Jun, K.-W.; Kim, H.-M. On the Hyers–Ulam–Rassias stability of a general cubic functional equation. *Math. Inequal. Appl.* **2003**, *6*, 289–302.
13. Lee, Y.-H. On the generalized Hyers-Ulam stability of the generalized polynomial function of degree 3. *Tamsui Oxf. J. Math. Sci.* **2008**, *24*, 429–444.
14. Lee, Y.-H. On the Hyers–Ulam–Rassias stability of the generalized polynomial function of degree 2. *J. Chungcheong Math. Soc.* **2009**, *22*, 201–209.
15. Lee, Y.-H. On the Hyers–Ulam–Rassias stability of a general quartic functional equation. *East Asian Math. J.* **2019**, *35*, 351–356.
16. Lee, Y.-H. Stability of a monomial functional equation on a restricted domain. *Mathematics* **2017**, *5*, 53.