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Non-Parametric Threshold Estimation for the Wiener–Poisson Risk Model

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Abstract: In this paper, we consider the Wiener–Poisson risk model, which consists of a Wiener process and a compound Poisson process. Given the discrete record of observations, we use a threshold method and a regularized Laplace inversion technique to estimate the survival probability. In addition, we also construct an estimator for the distribution function of jump size and study its consistency and asymptotic normality. Finally, we give some simulations to verify our results.

Keywords: Wiener–Poisson risk model; survival probability; Nonparametric threshold estimation

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1. Introduction

Let $S = \{S_t\}_{t\geq 0}$ with $S_0 = 0$ be a compound Poisson process defined as

$$S_t = \sum_{i=1}^{N_t} \gamma_i, \quad t \geq 0,$$

where $\{N_t\}_{t\geq 0}$ is a Poisson process with unknown intensity $\lambda > 0$, and $\gamma_1, \gamma_2, \gamma_3, \ldots$ are independent and identically distributed positive sequence of random variables with unknown distribution function F supported on $(0, \infty)$.

The Wiener–Poisson risk process is defined by

$$X_t = x + ct + \sigma W_t - \sum_{i=1}^{N_t} \gamma_i, \quad t \geq 0, \tag{1}$$

where x is a given positive constant, $\sigma > 0$ is an unknown constant, the corresponding process $\{N_t, t \geq 0\}$ is called the claim number process, $\{\gamma_i\}_{i=1,2,\ldots}$ is a sequence of claims, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. Suppose that $\{N_t\}_{t\geq 0}$, $\{W_t\}_{t\geq 0}$ and $\{\gamma_i\}_{i=1,2,\ldots}$ are independent of each other, and the mean and variance of claim are finite, i.e., $\mu_\gamma = \int_0^\infty xF(dx) < \infty$, $\sigma_\gamma^2 = \int_0^\infty x^2F(dx) - \mu_\gamma^2 < \infty$. Further, we assume that the risk model in Equation (1) has a relative safety loading $\omega = \frac{c}{\lambda\mu_\gamma} - 1 > 0$. Let $\tau(x) = \inf\{t > 0; X_t \leq 0, X_0 = x\}$. The survival probability of the risk model in Equation (1) is defined by:

$$\Phi(x) = \mathbf{P}\left(\tau(x) = \infty\right), \tag{2}$$

and $\Psi(x) = 1 - \Phi(x)$, the probability of ruin.

In the last few decades, many works have been contributed to the survival probability for the risk model in Equation (1) and its extended risk model. In [1], the author first introduced the risk model in Equation (1) and established an asymptotic estimate for $\Psi(x)$. In [2], the authors showed that $\Phi(x)$ satisfies a defective renewal equation. By renewal theory, they obtained the Pollaczeck–Khinchin formula of $\Phi(x)$. Accurate calculation and approximation for $\Psi(x)$ has always been an inspiration and an important source of technological development for actuarial mathematics (see, e.g., [3–9]). Although various approximations to the probability of ruin (e.g., importance sampling or saddle-point approximations) are now available, developing alternative approximations of different nature is still an interesting and practical problem.

In recent years, many authors studied the ruin probability by using statistical methods (see, e.g., [10–18]). In [17], the author assumed that $\{X_{t_i^n} | t_i^n = ih_n; i = 0, 1, 2, ..., n\}$ and $\{\gamma_1, \gamma_2, ..., \gamma_{N_{t_n^n}}\}$ are observed, where $h_n = t_i^n - t_{i-1}^n$ is the sampling interval and the time of claims are known. The author constructed an estimator for Gerber–Shiu function and obtained its asymptotic property. Please refer to Equation (1.2) in [17] for the details of Gerber–Shiu function.

In our work, we suppose that a sample $\{X_{t_1^n}, X_{t_2^n}, X_{t_3^n}, ..., X_{t_n^n}\}$ can be observed, where $h_n = t_i^n - t_{i-1}^n$ is the sampling interval. However, we cannot observe the exact time and size of claims. To estimate $\Phi(x)$, we have to estimate F, λ, $\lambda\mu_\gamma$ and σ^2. Given the discrete record of observations, we need to judge whether a claim occurs in the interval $(t_{i-1}^n, t_i^n]$. The threshold method from [19–22] is to determine that a single jump has occurred within $(t_{i-1}^n, t_i^n]$ if and only if the increment $|\Delta_i X| = |X_{t_i^n} - X_{t_{i-1}^n}|$ is larger than a suitable threshold function. By the threshold method and the work in [21,22], we can estimate F, λ, μ_γ and σ^2.

In [14,17], the authors estimated the ruin probability and Gerber–Shiu function by a regularized Laplace inversion technique. Using the threshold method and the work in [23], it is easy to obtain an estimator for the Laplace transform of $\Phi(x)$. To estimate $\Phi(x)$, the regularized Laplace inversion technique is used. Finally, we also obtain a rate of convergence for the estimator of $\Phi(x)$ in a sense of the integrated squared error (ISE).

This paper is organized as follows. In Section 2, we give some estimators for σ^2, $\lambda\mu_\gamma$, λ, F and its Laplace–Stieltjes transform. In Section 3, we study the asymptotic properties for the estimators. Finally, we give some conclusions in Section 5. All the technical proofs are presented in Appendix A.

2. Estimation of Survival Probability

To give the estimators for σ^2, $\lambda\mu_\gamma$, λ, F and the Laplace transform of F, we introduce the following filter:

$$C_i^n(\vartheta(h_n)) = \{\omega \in \Omega; |\Delta_i X(\omega)| > \vartheta(h_n)\}, \tag{3}$$

where $\vartheta(h_n)$ is a threshold function and $\mathcal{D}_i^n(\vartheta(h_n))$ is a complement of $C_i^n(\vartheta(h_n))$. In [19,20], the threshold function $\vartheta(h_n)$ satisfies $\lim_{h_n \to 0} \vartheta(h_n) = 0$ and $\lim_{h_n \to 0} \frac{\sqrt{h_n \log(\frac{1}{h_n})}}{\vartheta(h_n)} = 0$. In [21], the author gave an expression of threshold function $\vartheta(h_n) = L h_n^b$, where $L > 0$ is a constant and $b \in (0, \frac{1}{2})$. Obviously, the expression of $\vartheta(h_n)$ from [21] satisfies the two conditions. In our work, the expression of $\vartheta(h_n)$ is similar to that in [21].

We first estimate F. Using $\{|\Delta_i X|; 0 \le i \le n, \mathbf{I}_{C_i^n(\vartheta(h_n))} = 1\}$ and empirical distribution function, we can try to construct an estimator of F as follows:

$$\hat{F}_n(u) = \frac{1}{\sum_{i=1}^n \mathbf{I}_{C_i^n(\vartheta(h_n))}} \sum_{i=1}^n \mathbf{I}_{\{|\Delta_i X| \le u\}} \mathbf{I}_{C_i^n(\vartheta(h_n))}, \quad u \ge 0. \tag{4}$$

By Equations (3.4) and (3.6) in [21], the estimators of σ^2 and λ are

$$\widetilde{\sigma^2}_n = \frac{\sum_{i=1}^n |\Delta_i X - c h_n|^2 \mathbf{I}_{\mathcal{D}_i^n(\vartheta(h_n))}}{T_n}, \quad \widetilde{\lambda}_n = \frac{\sum_{i=1}^n \mathbf{I}_{C_i^n(\vartheta(h_n))}}{T_n}.$$

By Equation (3.10) in [21], an estimator of $\lambda\mu_\gamma$ is given by

$$\widetilde{\lambda\mu_{\gamma}}_n = \frac{\sum_{i=1}^{n} |\Delta_i X| \mathbf{I}_{C_i^n(\vartheta(h_n))}}{T_n}.$$

Let $\rho = \frac{\lambda\mu_\gamma}{c}$. Obviously, the estimator of ρ is given by

$$\widetilde{\rho}_n = \frac{1}{c}\frac{\sum_{i=1}^{n} |\Delta_i X| \mathbf{I}_{C_i^n(\vartheta(h_n))}}{T_n}.$$

The Laplace transform of F is defined by $l_F = E[e^{-s\gamma_1}] = \int_0^\infty e^{-su}F(du)$. An estimator of l_F is given by

$$\widetilde{l_F}_n(s) = \frac{\sum_{i=1}^{n} e^{-s|\Delta_i X|}\mathbf{I}_{C_i^n(\vartheta(h_n))}}{\sum_{i=1}^{n} \mathbf{I}_{C_i^n(\vartheta(h_n))}},$$

where $s \in \mathbb{E}$ and $\mathbb{E}$ is a compact subset of $(0, \infty)$.

By the work in [23], the Laplace transform of $\Phi(x)$ can be obtained as follows:

$$\begin{aligned} L_\Phi(s) &= \int_0^\infty e^{-sx}\Phi(x)dx \\ &= \frac{1-\rho}{D(s)}, \quad s > 0 \end{aligned} \tag{5}$$

where $\rho = \frac{\lambda\mu}{c}$ and $D(s) = s + \frac{\sigma^2}{2c}s^2 - \frac{\lambda}{c}(1 - l_F(s))$.

Let us define an estimator of $L_\Phi(s)$ as follows:

$$\widetilde{L_\Phi}(s) = \frac{1-\widetilde{\rho}_n}{\widetilde{D}(s)}, \quad \widetilde{D}(s) = s + \frac{\widetilde{\sigma^2}_n}{2c}s^2 - \frac{\widetilde{\lambda}_n}{c}(1 - \widetilde{l_F}_n(s)), \quad s > 0. \tag{6}$$

To estimate $\Phi(x)$, we use the L^2-inversion method proposed from [24]. Now, we give the L^2-inversion method by Definition 1. We say that $f \in L^2(0, \infty)$ if $(\int_0^\infty |f(t)|^2 dt)^{\frac{1}{2}} < \infty$.

Definition 1. *Let $m > 0$ be a constant. The regularized Laplace inversion $L_m^{-1} : L^2(0, \infty) \to L^2(0, \infty)$ is given by*

$$L_m^{-1}g(t) = \frac{1}{\pi^2}\int_0^\infty \int_0^\infty \Psi_m(y)y^{-\frac{1}{2}}e^{-tvy}g(v)dvdy \tag{7}$$

for a function $g \in L^2(0, \infty)$ and $t \in (0, \infty)$, where

$$\Psi_m(y) = \int_0^{a_m} \cosh(\pi x)\cos(x \log y)dx$$

and $a_m = \pi^{-1}\cosh^{-1}(\pi m) > 0$.

For further information, and details of L_m^{-1}, please refer to [24].

To use Definition 1, it requires to verify $\widetilde{L_\Phi}(s) \in L^2(0, \infty)$. As n is sufficiently large, for $\mathbf{P}$-almost all $\omega \in \Omega$ and $s > 0$, we have

$$\mathbf{P}(\{\omega \in \Omega; (1-\widetilde{\rho}_n)s \le \widetilde{D}(s) \le s + \frac{\widetilde{\sigma^2}_n}{2c}s^2\}) = 1. \tag{8}$$

From Equations (6) and (8), it is obvious that $\widetilde{L_\Phi}(s) \notin L^2(0, \infty)$. The L^2-inversion method in Definition 1 cannot be applied at once.

Therefore, to use Definition 1, we have to amend $\widetilde{L_\Phi}(s)$.
Let

$$\Phi_\theta(x) = e^{-\theta x}\Phi(x), \quad x > 0$$

for arbitrary fixed $\theta > 0$. It is obvious that

$$L_{\Phi_\theta}(s) = L_\Phi(s+\theta), \quad s > 0.$$

An estimator of L_{Φ_θ} is given by

$$\widetilde{L_{\Phi_\theta}}(s) = \widetilde{L_\Phi}(s+\theta), \quad s > 0.$$

Obviously, $\widetilde{L_{\Phi_\theta}} \in L^2(0,\infty)$.
Finally, an estimator of $\Phi(x)$ is given by

$$\widetilde{\Phi}_{m(n)}(x) = e^{\theta x}\widetilde{\Phi_{\theta,m(n)}}(x), \quad x > 0, \tag{9}$$

where $\widetilde{\Phi_{\theta,m(n)}}(x) = L^{-1}_{m(n)}\widetilde{L_{\Phi_\theta}}(s)$ and $m(n) > 0$.

3. Asymptotic Properties

According to Theorem 3.1 in [19], the author assumed that $\sigma < Q$ and $\gamma_i \geq \Gamma$ with $Q > 0, \Gamma > 0$. In our work, Assumption 1 is used to prove the asymptotic properties of estimators.

Assumption 1. *There exist two positive constants Q and Γ such that $\sigma < Q$ and $\mathbf{P}(\{\omega \in \Omega; \gamma_i \geq \Gamma\}) = 1$ for $i = 1, 2, \ldots$.*

Let $\bar{F} = 1 - F$. With Equation (4), an estimator of $\bar{F}$ is given by

$$\hat{\bar{F}}_n(u) \quad = \quad \frac{1}{\sum_{i=1}^n \mathbf{I}_{\mathcal{C}_i^n(\vartheta(h_n))}} \sum_{i=1}^n \mathbf{I}_{\{|\Delta_i X| > u\}}\mathbf{I}_{\mathcal{C}_i^n(\vartheta(h_n))}. \tag{10}$$

Let $\mathcal{N}(m,n)$ be a normal distribution with expectation m and variance n. Theorem 1 gives the asymptotic properties of $\hat{F}_n(u)$.

Theorem 1. *Suppose that $T_n = nh_n \to \infty$, $nh_n^2 \to 0$, $h_n \to 0$ as $n \to \infty$ and Assumption 1 is satisfied, then*

$$\sqrt{T_n}\left(\hat{\bar{F}}_n(u) - \bar{F}(u)\right) \xrightarrow{\mathbf{D}} \mathcal{N}(0, \frac{\overline{F}(u)(1-\overline{F}(u))}{\lambda}). \tag{11}$$

Obviously,

$$\sqrt{T_n}\left(\hat{F}_n(u) - F(u)\right) \xrightarrow{\mathbf{D}} \mathcal{N}(0, \frac{F(u)(1-F(u))}{\lambda}).$$

Remark 1. *By Dvoretzky–Kiefer–Wolfowitz inequality, we have*

$$\mathbf{P}(\sup_{u\in[0,\infty)} |\hat{F}_n(u) - F(u)| > x) \leq Ce^{-2\lambda T_n x^2}, \quad x > 0,$$

where C is a positive constant, not depending on F. Note that this inequality may be expression in the form :

$$\mathbf{P}(\sqrt{T_n}\sup_{u\in[0,\infty)} |\hat{F}_n(u) - F(u)| > x) \leq Ce^{-2\lambda x^2}, \quad x > 0,$$

which clearly demonstrate that

$$\sqrt{T_n} \sup_{u \in [0,\infty)} |\hat{F}_n(u) - F(u)| = O_{\mathbf{P}}(1).$$

The asymptotic properties of $\widetilde{\sigma^2}_n$ are given by the following Lemma 1.

Lemma 1. *Suppose that $T_n = nh_n \to \infty$, $nh_n^2 \to 0$ and $h_n \to 0$ as $n \to \infty$, then*

$$\widetilde{\sigma^2}_n \xrightarrow{\mathbf{P}} \sigma^2, \quad n \to \infty. \tag{12}$$

$$\sqrt{n}(\widetilde{\sigma^2}_n - \sigma^2) \xrightarrow{\mathbf{D}} \mathcal{N}(0, 2\sigma^4), \quad n \to \infty. \tag{13}$$

Lemma 2. *Suppose that $T_n = nh_n \to \infty$, $nh_n^{\beta} \to 0$ for some $\beta \in (1,2]$, $h_n \to 0$ as $n \to \infty$ and Assumption 1 is satisfied. Then,*

$$\widetilde{\lambda}_n \xrightarrow{\mathbf{P}} \lambda, \quad \sup_{\{s|s\in\mathbb{E}\}} |\widetilde{\lambda}_n \widetilde{l}_{Fn}(s) - \lambda l_F(s)| \xrightarrow{\mathbf{P}} 0, \tag{14}$$

$$\sqrt{T_n}(\widetilde{\lambda}_n - \lambda) \xrightarrow{\mathbf{D}} \mathcal{N}(0, \lambda), \tag{15}$$

$$\widetilde{\rho}_n \xrightarrow{\mathbf{P}} \rho \tag{16}$$

and

$$\sqrt{n}(\widetilde{\rho}_n - \rho) \xrightarrow{\mathbf{D}} \mathcal{N}(0, \frac{\lambda\sigma^2}{c^2}), \tag{17}$$

as $n \to \infty$.

Let $\|f\|_B^2 = \int_0^B |f(t)|^2 dt$ for any function f and $B > 0$. Theorem 2 gives a rate of convergence for $\widetilde{\Phi}_{m(n)}(x)$ in a sense of ISE.

Theorem 2. *Suppose that there exists a constant $K > 0$ such that $0 \le \Phi'(x) = g(x) \le K < \infty$ and the conditions in Lemma 2 are satisfied. Then, for $m(n) = \sqrt{\frac{T_n}{\log T_n}}$ and for any constant $B > 0$, we have*

$$\|\widetilde{\Phi}_{m(n)} - \Phi\|_B^2 = O_P((\log T_n)^{-1}), \quad n \to \infty.$$

Remark 2. *The explicit expression for $\widetilde{\Phi}_{m(n)}(x)$ is*

$$\widetilde{\Phi}_{m(n)}(x) = \frac{e^{x\theta}}{\pi^2} \int_0^{\infty} \int_0^{\infty} e^{-xsy} \widetilde{L_{\Phi_\theta}}(s) \Psi_{m(n)}(y) y^{-\frac{1}{2}} ds dy$$

where $\Psi_{m(n)}(y) = \int_0^{a_{m(n)}} \cosh(\pi x) \cos(x \log(y)) dx$ and $a_{m(n)} = \pi^{-1} \cosh^{-1}(\pi m(n)) > 0$ and $m(n) = \sqrt{\frac{T_n}{\log T_n}}$.

When c, λ, σ, F, θ, $\vartheta(h_n)$ and sample size n are known, $\widetilde{\Phi}_{m(n)}(x)$ can be evaluated with the command integral2$(f; 0; \infty; 0; \infty)$ of Matlab.

4. Simulation

If $F(x) = 1 - e^{-\frac{1}{\mu_\gamma}x}$, the survival probability is given by

$$\Phi(x) = 1 - \frac{r_1 + \frac{1}{\mu_\gamma} + \frac{2\lambda\mu_\gamma}{\sigma^2}}{r_1 - r_2} e^{r_1 x} - \frac{r_2 + \frac{1}{\mu_\gamma} + \frac{2\lambda\mu_\gamma}{\sigma^2}}{r_2 - r_1} e^{r_2 x}, \quad x \ge 0, \tag{18}$$

where $r_2 < r_1 < 0$ are negative roots of the following equation

$$\frac{1}{2}\sigma^2 s + c - \frac{\lambda}{s + \frac{1}{\mu_\gamma}} = 0.$$

By the work in [25], Equation (18) is obtained easily.

Let $c = \lambda = 10$, $\mu_\gamma = \frac{1}{2}$, $\sigma = 5$, $\theta = 0.075$, $\vartheta(h_n) = h_n^b$, $b = \frac{1}{4}$ and $h_n = n^{-\frac{4}{5}}$.

Firstly, we computed $\widetilde{L_{\Phi_\theta}}(s)$. In Figure 1, we plot the mean points with sample sizes n =5000, 10,000, 30,000, 50,000, 80,000, which were computed based on 5000 simulation experiments.

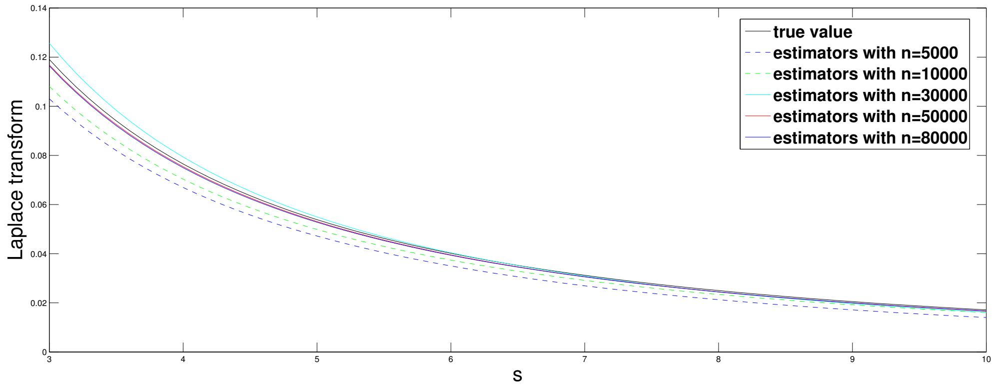

Figure 1. The estimator of $\mathcal{L}_{\Phi_\theta}$ with sample sizes n =5000, 10,000, 30,000, 50,000, 80,000.

In Remark 2, $\widetilde{\Phi}_{m(n)}(u)$ is a double complex integrals. Using Matlab to compute $\widetilde{\Phi}_{m(n)}(u)$ would take a long time. As shown in Figure 1, $\hat{\mathcal{L}}_{\Phi_\theta}$ is very close to $\mathcal{L}_{\Phi_\theta}$ as $n \geq$ 30,000. To improve computational efficiency, let

$$\Phi_p(x) = \frac{e^{x\theta}}{\pi^2}\int_0^\infty \int_0^\infty e^{-xsy}[\hat{\mathcal{L}}_{\Phi_\theta}(s)]_{n=30000}\Psi_p(y)y^{-\frac{1}{2}}dsdy,$$

where $[\hat{\mathcal{L}}_{\Phi_\theta}(s)]_{n=30000} = \frac{1-\frac{1}{c30000(30000)^{-\frac{4}{5}}}\sum_{k=1}^{30000}(c(30000)^{-\frac{4}{5}}-Z_k)\mathbf{I}_{\mathcal{D}_k^{30000}}}{\frac{1}{c(30000)^{-\frac{4}{5}}}(\frac{1}{30000}\sum_{k=1}^{30000}e^{sZ_k}-1)}$, $\Psi_p(y) = \int_0^{a_p}\cosh(\pi x)\cos(x\log(y))dx$ and $a_p = \pi^{-1}\cosh^{-1}(\pi p) > 0$.

In Figure 2, we plot the mean points with sample sizes n = 30,000 and $p = 100, 500, 800, 1000, 3000$, which were computed based on 5000 simulation experiments.

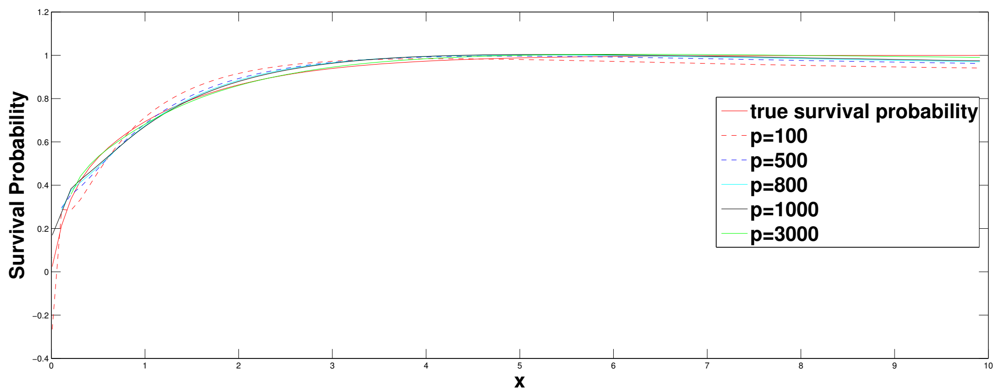

Figure 2. $\Phi_p(x)$ with sample size n = 30,000 and $p = 100, 500, 800, 1000, 3000$

5. Conclusions

In this paper, we use the threshold estimation technique and regularized Laplace inversion technique to constructed an estimator of survival probability for the Wiener–Poisson risk model. The rate of convergence for the estimator is a logarithmic rate. We adopt a method proposed by Cai et al. [26] to improve the speed in simulated calculation. The further work is to improve the speed of convergence for the estimator. We will combine the threshold estimation technique with Fourier transform (inversion) technique to construct an estimator of survival probability. We hope some further studies will be done when the risk model is the compound Poisson model with the barrier dividend strategy and investment. The Gerber–Shiu function and dividend function will be estimated by some statistical methods.

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Appendix A. Proofs of Theorems

Proof of Lemma 1. The proof of Lemma 1 is easily obtained by Theorem 3.1 in [21]. □

Proof of Lemma 2. The proof of Equation (14) is given as Theorem 3.2 in [21]. By Theorem 3.1 in [19], we can get Equation (15). It is easy to get Equations (16) and (17) by Proposition 3.4 in [19] and Theorem 3 in [20]. □

To prove Theorem 1, we need the following Proposition, which can be easily obtained in Section 3.2 of [19].

Proposition A1. *Following from the condition of Theorem 1, for any $\epsilon > 0$,*

$$\lim_{n\to\infty} P\left(\left||\Delta_i X|\mathbf{I}_{C_i^n(\vartheta(h_n))} - \gamma_{\tau^{(i)}}\mathbf{I}_{\{\Delta_i N\geq 1\}}\right| > \epsilon\right) = 0, \tag{A1}$$

where $\gamma_{\tau^{(i)}}$ is the size of the eventual jump in time interval $(t_{i-1}^n, t_i^n]$.

Proof of Theorem 1. By Equation (10),

$$\sqrt{T_n}\left(\hat{\bar{F}}_n(u) - \bar{F}(u)\right) = \frac{J}{\tilde{\lambda}_n}, \tag{A2}$$

where

$$J = \frac{\sum_{i=1}^n \left(\mathbf{I}_{\{|\Delta_i X|>u\}} - \bar{F}(u)\right)\mathbf{I}_{C_i^n(\vartheta(h_n))}}{\sqrt{T_n}}.$$

As $n \to \infty$, the expectation of J is

$$\begin{aligned}\lim_{n\to\infty}\mathbf{E}[J] &= \lim_{n\to\infty}\frac{n}{\sqrt{T_n}}\mathbf{E}\left[(\mathbf{I}_{\{|\Delta_i X|>u\}} - \bar{F}(u))\mathbf{I}_{C_i^n(\vartheta(h_n))}\right] \\ &= \lim_{n\to\infty}\frac{n}{\sqrt{T_n}}\left[\mathbf{P}(|\Delta_i X|\mathbf{I}_{C_i^n(\vartheta(h_n))} > u) - \bar{F}(u)\mathbf{P}(|\Delta_i X| > \vartheta(h_n))\right].\end{aligned}$$

By Proposition A1, we have

$$\lim_{n\to\infty} \mathbf{P}(|\Delta_i X|\mathbf{I}_{\mathcal{C}_i^n(\vartheta(h_n))} > u) = \lim_{n\to\infty} \mathbf{P}(\gamma_{\tau^{(i)}}\mathbf{I}_{\{\Delta_i N\geq 1\}} > u) = \bar{F}(u)(\lambda h_n + o(h_n)). \quad \text{(A3)}$$

By the work in [19], it is obvious that

$$\lim_{n\to\infty} \mathbf{P}(|\Delta_i X| > \vartheta(h_n)) \quad = \quad \lambda h_n + o(h_n). \quad \text{(A4)}$$

Therefore,

$$\lim_{n\to\infty} \mathbf{E}[J] = 0.$$

As $n \to \infty$, the variance of J is

$$\begin{aligned} \lim_{n\to\infty} \mathbf{Var}[J] &= \lim_{n\to\infty} \frac{n}{T_n} \mathbf{Var}\left[(\mathbf{I}_{\{|\Delta_i X|>u\}} - \bar{F}(u))\mathbf{I}_{\mathcal{C}_i^n(\vartheta(h_n))}\right] \\ &= \lim_{n\to\infty} \frac{n}{T_n} \mathbf{E}[\mathbf{I}_{\{|\Delta_i X|\mathbf{I}_{\mathcal{C}_i^n(\vartheta(h_n))}>u\}}] + \lim_{n\to\infty} \frac{n}{T_n} \mathbf{E}[(\bar{F}(u))^2\mathbf{I}_{\mathcal{C}_i^n(\vartheta(h_n))}]. \\ &\quad - \lim_{n\to\infty} \frac{n}{T_n} \mathbf{E}[2\bar{F}(u)\mathbf{I}_{\{|\Delta_i X|\mathbf{I}_{\mathcal{C}_i^n(\vartheta(h_n))}>u\}}]. \end{aligned}$$

By Equations (A3) and (A4),

$$\lim_{n\to\infty} \mathbf{Var}[J] \quad = \quad \lambda\bar{F}(u)(1-\bar{F}(u)).$$

With the central limit theorem, Slutsky's theorem and Lemma 2, we have

$$\sqrt{T_n}\left(\bar{\hat{F}}_n(u) - \bar{F}(u)\right) \xrightarrow{\mathbf{D}} \mathcal{N}(0, \frac{\bar{F}(u)(1-\bar{F}(u))}{\lambda}), \quad n \to \infty.$$

□

To prove Theorem 2, we need the following Lemma A1.

Lemma A1. *Suppose that* $\int_0^\infty [t(t^{\frac{1}{2}}f(t))']^2 t^{-1} dt < \infty$ *for a function* $f \in L^2(0,\infty)$ *with the derivative* f'. *Then,*

$$\|L_n^{-1}L_f - f\| = O\left((\log n)^{-\frac{1}{2}}\right), \quad n \to \infty.$$

By Theorem 3.2 in [24], the proof of Lemma A1 can be found.

Proof of Theorem 2. By Equation (9),

$$\begin{aligned} \|\widetilde{\Phi}_{m(n)} - \Phi\|_B^2 &\leq e^{2\theta B}\|\widetilde{\Phi}_{\theta,m(n)} - \Phi_\theta\|_B^2 \\ &\leq 2e^{2\theta B}\{\|L_{m(n)}^{-1}\widetilde{L_{\Phi_\theta}} - L_{m(n)}^{-1}L_{\Phi_\theta}\|^2 + \|\Phi_{\theta,m(n)} - \Phi_\theta\|^2\}. \end{aligned} \quad \text{(A5)}$$

Let $\Phi_\theta' = g_\theta$. Now, we show that $\Phi_{\theta,m(n)}$ satisfies the condition of Lemma A1.

$$\begin{aligned}
\int_0^\infty [x(\sqrt{x}\Phi_\theta(x))']^2 \frac{1}{x} dx &= \int_0^\infty [x(\frac{1}{2\sqrt{x}}\Phi_\theta(x) + x\sqrt{x} g_\theta(x))]^2 \frac{1}{x} dx \\
&\leq \int_0^\infty 2\frac{1}{x}[x\frac{1}{2\sqrt{x}}\Phi_\theta(x)]^2 + \int_0^\infty 2\frac{1}{x}[x\sqrt{x} g_\theta(x)]^2 dx \\
&= \int_0^\infty \frac{1}{2}\Phi_\theta^2(x)dx + 2\int_0^\infty x^2 g_\theta^2(x)dx \\
&\leq \int_0^\infty \frac{1}{2} e^{-2\theta x} dx + 2\int_0^\infty x^2[g(x)e^{-\theta x} - \theta\Phi(x)e^{-\theta x}]^2 dx \\
&\leq \frac{1}{4\theta} + 4\int_0^\infty x^2 g^2(x) e^{-2\theta x} dx + 4\theta^2 \int_0^\infty \Phi^2(x) x^2 e^{-2\theta x} dx \\
&\leq \frac{1}{4\theta} + 4(K^2+\theta^2)\int_0^\infty x^2 e^{-2\theta x} dx < \infty.
\end{aligned}$$

Therefore, by Lemma A1, we have

$$\|\Phi_{\theta,m(n)} - \Phi_\theta\|^2 = O(\frac{1}{\log m(n)}), \quad n \to \infty. \tag{A6}$$

By Equations (5) and (6),

$$\|\widetilde{L_{\Phi_\theta}} - L_\theta\|^2 = \int_0^\infty \left(\frac{(1-\rho)(\widetilde{D}(s+\theta) - D(s+\theta))}{\widetilde{D}(s+\theta)D(s+\theta)} + \frac{(\widetilde{\rho}_n - \rho)}{\widetilde{D}(s+\theta)} \right)^2 ds. \tag{A7}$$

Exploiting Equation (8) and $\mathbf{P}(\{\omega \in \Omega; \widetilde{\rho}_n = 1\}) = 0$, the right-hand side of Equation (A7) is bounded by

$$2\int_0^\infty \frac{(\widetilde{D}(s+\theta) - D(s+\theta))^2}{(s+\theta)^4(1-\widetilde{\rho}_n)^2} ds + 2\int_0^\infty (\frac{\widetilde{\rho}_n - \rho}{1-\widetilde{\rho}_n})^2 \frac{1}{(s+\theta)^2} ds. \tag{A8}$$

By Lemmas 1 and 2, the term

$$\begin{aligned}
\widetilde{D}_n(s+\theta) - D(s+\theta) &= \frac{(s+\theta)^2}{2c}(\widetilde{\sigma^2}_n - \sigma^2) + \frac{1}{c}\left((\lambda - \widetilde{\lambda}_n) + (\widetilde{\lambda}_n \widetilde{l}_{F_n}(s+\theta) - \lambda l_F(s+\theta))\right) \\
&= O_{\mathbf{P}}(T_n^{-\frac{1}{2}}) + \frac{1}{c}\frac{N_{T_n}}{T_n}\left(\int_0^\infty e^{-(s+\theta)x}(\hat{F}_n(dx) - F(dx))\right) \\
&= O_{\mathbf{P}}(T_n^{-\frac{1}{2}}) + \frac{1}{c}\frac{N_{T_n}}{T_n}\left(\int_0^\infty (s+\theta)e^{-(s+\theta)x}(\hat{F}_n(x) - F(x))dx\right) \\
&\leq O_{\mathbf{P}}(T_n^{-\frac{1}{2}}) + \frac{1}{c}\frac{N_{T_n}}{T_n} \sup_{x\in[0,\infty)} |\hat{F}_n(x) - F(x)| \\
&= O_{\mathbf{P}}(T_n^{-\frac{1}{2}})
\end{aligned} \tag{A9}$$

The last equality is obtained from Remark 1.
By Equation (A9) and Lemma 2, we have

$$2\int_0^\infty \frac{(\widetilde{D}(s+\theta) - D(s+\theta))^2}{(s+\theta)^4(1-\widetilde{\rho}_n)^2} ds = O_{\mathbf{P}}(T_n^{-1})$$

and

$$2\int_0^\infty (\frac{\widetilde{\rho}_n - \rho}{1-\widetilde{\rho}_n})^2 \frac{1}{(s+\theta)^2} ds = o_{\mathbf{P}}(T_n^{-1}).$$

Recall that $\|L^{-1}_{m(n)}\|^2 \leq \pi m^2(n)$ (see [24]), so

$$\|L^{-1}_{m(n)}\|^2 \|\widetilde{L_{\Phi_\theta}} - L_{\Phi_\theta}\|^2 \quad = \quad O_{\mathbf{P}}(\frac{m^2(n)}{T_n}). \tag{A10}$$

Combining Equations (A6) and (A10), we have

$$\|\widetilde{\Phi}_{m(n)} - \Phi\|^2_B = O_{\mathbf{P}}(\frac{m^2(n)}{T_n}) + O_{\mathbf{P}}(\frac{1}{\log m(n)}). \tag{A11}$$

with an optimal $m(n) = \sqrt{\frac{T_n}{\log T_n}}$ balancing the the two right-hand terms in Equation (A11), the order becomes $O_{\mathbf{P}}((\log T_n)^{-1})$. $\square$

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