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A Mizoguchi–Takahashi Type Fixed Point Theorem in Complete Extended *b*-Metric Spaces

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Abstract: In this paper, we prove a new fixed point theorem for a multi-valued mapping from a complete extended b-metric space $\mathbf{U}$ into the non empty closed and bounded subsets of $\mathbf{U}$, which generalizes Nadler's fixed point theorem. We also establish some fixed point results, which generalize our first result. Furthermore, we establish Mizoguchi–Takahashi's type fixed point theorem for a multi-valued mapping from a complete extended b-metric space $\mathbf{U}$ into the non empty closed and bounded subsets of $\mathbf{U}$ that improves many existing results in the literature.

Keywords: complete extended b-metric space; Hausdorff metric; fixed point theorems

1. Introduction

Throughout this paper, $(\mathbf{U}, d_\phi)$ is an extended b-metric space. We denote by $\mathcal{CL}(\mathbf{U})$ the set of all subsets of $\mathbf{U}$ that are non empty and closed, by $\mathcal{CLB}(\mathbf{U})$ the set of all subsets of $\mathbf{U}$ that are non empty closed and bounded and by $\mathcal{K}(\mathbf{U})$ the set of all subsets of $\mathbf{U}$ that are non empty compacts.

An element $u' \in \mathcal{U}$ is called a fixed point of a multi-valued map $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$ if $u' \in Fu'$. An orbit for a mapping $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$ at a point $u_0 \in \mathbf{U}$ denoted by $O(F)$ is a sequence $\{u_n\}_{n=0}^{\infty}$ in $\mathbf{U}$ such that $u_{n+1} \in Fu_n$. A mapping $f : \mathbf{U} \to \mathbb{R}$ is said to be F-orbitally lower semi-continuous if for any sequence $\{u_n\}_{n=0}^{\infty}$ in $O(F)$ and $u \in \mathbf{U}$, $u_n \to u$ implies $f(u) \leq \lim_{n\to\infty} \inf f(u_n)$.

Define a function $f : \mathbf{U} \to \mathbb{R}$ as $f(u) = d_\phi(u, Fu)$. For a constant $q \in (0,1)$, define the set $I_q^u \subset \mathbf{U}$ as

$$I_q^u = \{v \in Fu \mid q d_\phi(u, v) \leq d_\phi(u, Fu)\}.$$

The Pompeiu–Hausdorff distance measuring the distance between the subsets of a metric space was initiated by D. Pompeiu in [1]. The fixed point theory of set-valued contractions was initiated by Nadler [2], but later many authors extrapolated it multi directionally (see [3,4]).

Theorem 1 (Reich [5])**.** *Let $(\mathbf{U}, d)$ be a complete metric space and let $F : \mathbf{U} \to \mathcal{K}(\mathbf{U})$. Assume that there exists a map $\eta : [0,\infty) \to [0,1)$ such that*

$$\limsup_{s\to t^+} \eta(s) < 1, \textit{ for all } t \in (0,\infty),$$

and

$$\mathbf{H}(Fu, Fv) \le \eta(d(u,v))d(u,v), \quad \text{for all } u,v \in \mathbf{U}.$$

Then F has a fixed point.

In [5] Reich raised the question if the above theorem is also true for $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$. In [6], Mizoguchi and Takahashi gave supportive solution to the conjecture of [5] under the hypothesis $\limsup_{s \to t^+} \eta(s) < 1$, for all $t \in [0, \infty)$. In particular, they proved the following result:

Theorem 2 (Mizoguchi, Takahashi [6])**.** *Let $(\mathbf{U}, d)$ be a complete metric space and let $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$. Assume that there exists a map $\eta : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{s \to t^+} \eta(s) < 1, \text{ for all } t \in [0, \infty),$$

and

$$\mathbf{H}(Fu, Fv) \le \eta(d(u,v))d(u,v), \quad \text{for all } u,v \in \mathbf{U},\ u \neq v.$$

Then F has a fixed point.

In [7], Feng and Liu extended Nadler's fixed point theorem, other than the direction of Reich and Takahashi. They proved a theorem as follows:

Theorem 3 (Feng, Liu [7])**.** *Let $(\mathbf{U}, d)$ be a complete metric space and let $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$. Assume that:*

(i) *The map $f : \mathbf{U} \to \mathbb{R}$ defined by $f(u) = d(u, Fu)$, $u \in \mathbf{U}$, is lower semi-continuous;*
(ii) *There exist $p, q \in (0,1)$, $p < q$ such that for all $u \in \mathbf{U}$ there exists $v \in \{v \in Fu \mid qd(u,v) \le d(u, Fu)\}$ satisfying*

$$d(v, Fv) \le pd(u,v).$$

Then F has a fixed point.

Hicks and Rhodes [8] and Klim and Wardowski [9] proved the following results:

Theorem 4 ([8])**.** *Let $(\mathbf{U}, d)$ be a complete metric space and let $g : \mathbf{U} \to \mathbf{U}$, $0 \le h < 1$. Suppose there exists q such that*

$$d(gv, g^2v) \le hd(v, gv), \quad for\ every\ y \in \{x, gx, g^2x, \dots\}.$$

Then

(i) $\lim_n g^n x = q$ *exists;*
(ii) $d(g^n x, q) \le \frac{h^n}{1-h} d(x, gx)$*;*
(iii) *q is a fixed point of g iff $G(x) = d(x, gx)$ is g-orbitally lower semi-continuous at q.*

Theorem 5 ([9])**.** *Let $(\mathbf{U}, d)$ be a complete metric space and let $F : \mathbf{U} \to \mathcal{K}(\mathbf{U})$. Assume that the following conditions hold:*

(i) *The map $f : \mathbf{U} \to \mathbb{R}$ defined by $f(u) = d(u, Fu)$, $u \in \mathbf{U}$, is lower semi-continuous;*
(ii) *There exists a map $\eta : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{s \to t^+} \eta(s) < 1, \quad \text{for all } t \in (0, \infty),$$

and for all $u \in \mathbf{U}$ there exists $v \in \{v \in Fu : d(u,v) \le d(u, Fu)\}$ satisfying

$$d(v, Fv) \le \eta(d(u,v))d(u,v).$$

Then F has a fixed point.

In 2007, Kamran [10] logically presented Mizoguchi–Takahashi's type fixed point theorem, that simply generalizes Theorems 4 and 5.

The idea of generalizing metric spaces into b-metric spaces was initiated from the works of Bakhtin [11], Bourbaki [12], and Czerwik [13,14]. In [15], the notion of b-metric space was generalized further by introducing the concept of extended b-metric spaces (see also [16–18]) as follows:

Definition 1 ([15]). *Let $\mathbf{U}$ be a non empty set and $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$. A function $d_\phi : \mathbf{U} \times \mathbf{U} \to [0, \infty)$ is called an extended b-metric, if for all $u_1, u_2, u_3 \in \mathbf{U}$ it satisfies:*

(i) $d_\phi(u_1, u_2) = 0$ *if and only if* $u_1 = u_2$,

(ii) $d_\phi(u_1, u_2) = d_\phi(u_2, u_1)$,

(iii) $d_\phi(u_1, u_3) \leq \phi(u_1, u_3)[d_\phi(u_1, u_2) + d_\phi(u_2, u_3)]$.

The pair $(\mathbf{X}, d_\phi)$ is called an extended b-metric space.

Remark 1 ([15]). *Every b-metric space is an extended b-metric space with a constant function $\phi(x_1, x_2) = s$, for $s \geq 1$, but its converse is not true in general.*

Example 1. *Let $\mathbf{U} = \{u \in \mathbb{R} : u \geq 1\}$. Define $d_\phi : \mathbf{U} \times \mathbf{U} \to [0, \infty)$ and $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$ as follows:*

$$d_\phi(u_1, u_2) = (u_1 - u_2)^2, \;\; \phi(u_1, u_2) = 1 + u_1 + u_2,$$

for all $u_1,\ u_2 \in \mathbf{U}$. Then $(\mathbf{U}, d_\phi)$ is an extended b-metric space.

For more examples and recent results see [19]. Also, in [20] Muhammad Usman Ali et al. established fixed point results for new F-contractions of Hardy–Rogers type in the setting of b-metric space and proved the existence theorem for Volterra-type integral inclusion. Their results generalized many existence results in the literature. Finally in [21], authors introduced the notion of a generalized Pompeiu–Hausdorff metric induced by the extended b-metric as follows:

Definition 2. *([21]) Let $(\mathbf{U}, d_\phi)$ be an extended b-metric space, where $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$ is bounded. Then for all $\mathbf{A}, \mathbf{B} \in \mathcal{CLB}(\mathbf{U})$, where $\mathcal{CLB}(\mathbf{U})$ denotes the family of all non empty closed and bounded subsets of $\mathbf{U}$, the Hausdorff–Pompieu metric on $\mathcal{CLB}(\mathbf{U})$ induced by d_ϕ is defined by*

$$\mathbf{H}_\Phi(\mathbf{A}, \mathbf{B}) = \max\{\sup_{a \in \mathbf{A}} d_\phi(a, \mathbf{B}), \sup_{b \in \mathbf{B}} d_\phi(b, \mathbf{A})\},$$

where for every $a \in \mathbf{A}$, $d_\phi(a, \mathbf{B}) = \inf\{d_\phi(a, b) : b \in \mathbf{B}\}$ and $\Phi : \mathcal{CLB}(\mathbf{U}) \times \mathcal{CLB}(\mathbf{U}) \to [1, \infty)$ is such that

$$\Phi(\mathbf{A}, \mathbf{B}) = \sup\{\phi(a, b) : a \in \mathbf{A}, b \in \mathbf{B}\}.$$

Theorem 6. *([21]) Let $(\mathbf{U}, d_\phi)$ be an extended b-metric space. Then $(\mathcal{CLB}(\mathbf{U}), \mathbf{H}_\Phi)$ is an extended Hausdorff–Pompieu b-metric space.*

In this paper, we extend Nadler's fixed point theorem for the extended b-metric space. Moreover, we improve Mizoguchi–Takahashi's type fixed point theorem (Theorem 1.2, [10]) for the extended b-metric space when F is a multi-valued mapping from $\mathbf{U}$ to $\mathcal{CLB}(\mathbf{U})$. Our results generalize Theorems 4 and 5 in the setting of extended b-metric spaces which in turn generalize many existing results including Theorems 1–3.

2. Main Results

We start with the following lemma.

Lemma 1. *Let* $\mathbf{X}, \mathbf{Y} \in \mathcal{CLB}(\mathbf{U})$*, then for every* $\eta > 0$ *and* $y \in \mathbf{Y}$ *there exists* $x \in \mathbf{X}$ *such that*

$$d_\phi(x, y) \le \mathbf{H}_\Phi(\mathbf{X}, \mathbf{Y}) + \eta.$$

Proof. By definition of the Hausdorff metric, for $\mathbf{X}, \mathbf{Y} \in \mathcal{CLB}(\mathbf{U})$ and for any $y \in \mathbf{Y}$, we have

$$d_\phi(\mathbf{X}, y) \le \mathbf{H}_\Phi(\mathbf{X}, \mathbf{Y}).$$

By the definition of an infimum, we can let $\{x_n\}_{n=0}^{\infty}$ be a sequence in $\mathbf{X}$ such that

$$d_\phi(y, x_n) < d_\phi(y, \mathbf{X}) + \eta, \ \ \textit{where } \eta > 0. \tag{1}$$

We know that $\mathbf{X}$ is closed and bounded, so there exists $x \in \mathbf{X}$ such that $x_n \to x$. Therefore by (1), we have

$$d_\phi(x, y) < d_\phi(\mathbf{X}, y) + \eta \le \mathbf{H}_\Phi(\mathbf{X}, \mathbf{Y}) + \eta.$$

□

Theorem 7. *Let* $(\mathbf{U}, d_\phi)$ *be a complete extended b-metric space. If* $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$ *satisfies the inequality*

$$\mathbf{H}_\Phi(Fu, Fv) \le \eta d_\phi(u, v), \ \ \textit{for all } u,\ v \in \mathbf{U}, \tag{2}$$

where $\eta \in [0, 1)$ *is a real constant such that* $\lim_{n,m \to \infty} \eta\phi(u_n, u_m) < 1$*, then* F *has a fixed point.*

Proof. Let us consider $\eta > 0$. Let $u_0 \in \mathbf{U}$ and choose $u_1 \in Fu_0$. Since $Fu_0, Fu_1 \in \mathcal{CLB}\mathbf{U})$ and $u_1 \in Fu_0$, then by Lemma 1, there exists $u_2 \in Fu_1$ such that

$$d_\phi(u_1, u_2) \le \mathbf{H}_\Phi(Fu_0, Fu_1) + \eta.$$

Now since $Fu_1, Fu_2 \in \mathcal{CLB}\mathbf{U})$ and $u_2 \in Fu_1$, there is a point $u_3 \in Fu_2$ such that

$$d_\phi(u_2, u_3) \le \mathbf{H}_\Phi(Fu_1, Fu_2) + \eta^2.$$

Continuing in this fashion, we obtain a sequence $\{u_n\}_{n=0}^{\infty}$ of elements of $\mathbf{U}$ such that $u_{n+1} \in Fu_n$ and

$$d_\phi(u_n, u_{n+1}) \le \mathbf{H}_\Phi(Fu_{n-1}, Fu_n) + \eta^n, \ \ \text{for all } \ n \ge 1.$$

By (2), we note that

$$\begin{aligned} d_\phi(u_n, u_{n+1}) &\le \eta d_\phi(u_{n-1}, u_n) + \eta^n \\ &\le \eta(\eta d_\phi(u_{n-2}, u_{n-1}) + \eta^{n-1}) + \eta^n \\ &\le \eta^2 d_\phi(u_{n-2}, u_{n-1}) + 2\eta^n. \end{aligned}$$

Continuing in this way, we have

$$d_\phi(u_n, u_{n+1}) \le \eta^n d_\phi(u_0, u_1) + n\eta^n, \ \ \text{for all } \ n \ge 1. \tag{3}$$

By the triangle inequality and (3) for $m > n$, we have

$$\begin{aligned} d_\phi(u_n, u_m) \quad \leq \quad & \phi(u_n, u_m)[\eta^n d_\phi(u_0, u_1) + n\eta^n] + \phi(u_n, u_m)\phi(u_{n+1}, u_m)[\eta^{n+1} d_\phi(u_0, u_1) \\ & +(n+1)\eta^{n+1}] + \ldots\ldots + \phi(u_n, u_m)\phi(u_{n+1}, u_m)\ldots\phi(u_{m-1}, u_m) \\ & [\eta^{m-1} d_\phi(u_0, u_1) + (m-1)\eta^{m-1}], \end{aligned}$$

$$\begin{aligned} d_\phi(u_n, u_m) \quad \leq \quad & d_\phi(u_0, u_1)[\phi(u_n, u_m)\eta^n + \phi(u_n, u_m)\phi(u_{n+1}, u_m)\eta^{n+1} + \ldots + \phi(u_n, u_m) \\ & \phi(u_{n+1}, u_m)\ldots\phi(u_{m-1}, u_m)\eta^{m-1}] + [\phi(u_n, u_m)n\eta^n \;+ \phi(u_n, u_m) \\ & \phi(u_{n+1}, u_m)(n+1)\eta^{n+1} + \ldots + \phi(u_n, u_m)\phi(u_{n+1}, u_m)\ldots \\ & \phi(u_{m-1}, u_m)(m-1)\eta^{m-1}], \end{aligned}$$

$$\begin{aligned} d_\phi(u_n, u_m) \quad \leq \quad & d_\phi(u_0, u_1)[\phi(u_1, u_m)\phi(u_2, u_m)\ldots\phi(u_n, u_m)\eta^n + \phi(u_1, u_m)\phi(u_2, u_m)\ldots \\ & \phi(u_{n+1}, u_m)\eta^{n+1} + \ldots + \phi(u_1, u_m)\phi(u_2, u_m)\ldots\phi(u_n, u_m)\phi(u_{n+1}, u_m)\ldots \\ & \phi(u_{m-1}, u_m)\eta^{m-1}] + [\phi(u_1, u_m)\phi(u_2, u_m)\ldots\phi(u_n, u_m)n\eta^n + \phi(u_1, u_m) \\ & \phi(u_2, u_m)\ldots\phi(u_{n+1}, u_m)(n+1)\eta^{n+1} + \ldots + \phi(u_1, u_m)\phi(u_2, u_m)\ldots \\ & \phi(u_n, u_m)\phi(u_{n+1}, u_m)\ldots\phi(u_{m-1}, u_m)(m-1)\eta^{m-1}]. \end{aligned}$$

Since $\lim_{n,m\to\infty} \phi(u_{n+1}, u_m)\eta < 1$, the series

$$\sum_{n=1}^{\infty} \eta^n \prod_{i=1}^{n} \phi(u_i, u_m) \;\text{ and }\; \sum_{n=1}^{\infty} n\eta^n \prod_{i=1}^{n} \phi(u_i, u_m)$$

converges by the ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} \eta^n \prod_{i=1}^{n} \phi(u_i, u_m), \;\; S_n \;=\; \sum_{j=1}^{n} \eta^j \prod_{i=1}^{j} \phi(u_i, u_m),$$

and

$$S' = \sum_{n=1}^{\infty} n\eta^n \prod_{i=1}^{n} \phi(u_i, u_m), \;\; S'_n = \sum_{j=1}^{n} j\eta^j \prod_{i=1}^{j} \phi(u_i, u_m).$$

Thus for $m > n$, the above inequality implies

$$d_\phi(u_n, u_m) \leq d_\phi(u_0, u_1)[S_{m-1} - S_n] + [S'_{m-1} - S'_n].$$

By letting $n \to \infty$, we conclude that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $\mathbf{U}$ is complete, there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$ (so $\lim_{n\to\infty} u_{n+1} = u$). Now by the triangle inequality

$$\begin{aligned} d_\phi(Fu, u) \quad & \leq \quad \phi(Fu, u)[d_\phi(Fu, u_n) + d_\phi(u_n, u)] \\ & \leq \quad \phi(Fu, u)[\eta d_\phi(u, u_{n-1}) + d_\phi(u_n, u)]. \end{aligned}$$

This implies that

$$d_\phi(Fu, u) \leq 0 \;\; as \;\; n \to \infty.$$

$$d_\phi(Fu, u) = 0.$$

Hence u is a fixed point of F. □

Theorem 8. *Let us consider a multi-valued mapping $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$, where $(\mathbf{U}, d_\phi)$ is a complete extended b-metric space. Furthermore, let us consider that the following two conditions hold:*

(i) *The map $f : \mathbf{U} \to \mathbb{R}$ defined by $f(u) = d_\phi(u, Fu)$, $u \in \mathbf{U}$, is lower semi-continuous;*
(ii) *There exist $p, q \in (0,1)$, $p < q$ such that for all $u \in \mathbf{U}$ there exists $v \in I_q^u$ satisfying*

$$d_\phi(v, Fv) \le p d_\phi(u, v).$$

Moreover $\lim_{n,m \to\infty} \alpha\phi(u_n, u_m) < 1$, for all $\alpha \in (0,1)$. Then F has a fixed point in $\mathbf{U}$.

Proof. As $Fu \in \mathcal{CLB}(\mathbf{U})$ for any $u \in \mathbf{U}$, I_q^u is non void for any constant $q \in (0,1)$. For some arbitrary point $u_0 \in \mathbf{U}$, there exists $u_1 \in I_q^{u_0}$ such that

$$d_\phi(u_1, Fu_1) \le p d_\phi(u_0, u_1).$$

And, for $u_1 \in \mathbf{U}$, there exists $u_2 \in I_q^{u_1}$ satisfying

$$d_\phi(u_2, Fu_2) \le p d_\phi(u_1, u_2).$$

Continuing in this fashion, we can get an iterative sequence $\{u_n\}_{u=0}^\infty$, where $u_{n+1} \in I_q^{u_n}$ and

$$d_\phi(u_{n+1}, Fu_{n+1}) \le p d_\phi(u_n, u_{n+1}),\ \ n = 0, 1, 2, \cdots.$$

Now we will prove that $\{u_n\}_{n=0}^\infty$ is a Cauchy sequence. On the one hand,

$$d_\phi(u_{n+1}, Fu_{n+1}) \le p d_\phi(u_n, u_{n+1}),\ \ n = 0, 1, 2, \cdots. \tag{4}$$

On the other hand, $u_{n+1} \in I_q^{u_n}$ implies

$$q d_\phi(u_n, u_{n+1}) \le d_\phi(u_n, Fu_n),\ \ n = 0, 1, 2, \cdots.$$

By the above two equations, we have

$$d_\phi(u_{n+1}, u_{n+2}) \le \frac{p}{q} d_\phi(u_n, u_{n+1}),\ \ n = 0, 1, 2, \cdots, \tag{5}$$

$$d_\phi(u_{n+1}, Fu_{n+1}) \le \frac{p}{q} d_\phi(u_n, Fu_n),\ \ n = 0, 1, 2, \cdots.$$

By inequality (5), it is easy to prove that

$$d_\phi(u_n, u_{n+1}) \le \frac{p^n}{q^n} d_\phi(u_0, u_1),\ \ n = 0, 1, 2, \cdots,$$

$$d_\phi(u_n, Fu_n) \le \frac{p^n}{q^n} d_\phi(u_0, Fu_0),\ \ n = 0, 1, 2, \cdots. \tag{6}$$

Let $\alpha = \frac{p}{q}$. Since $p < q$ we have $\alpha = \frac{p}{q} < 1$. By taking $n \to \infty$ in (6), we obtain

$$\lim_{n\to\infty} d_\phi(u_n, Fu_n) = 0. \tag{7}$$

By the triangle inequality and (6), for $m, n \in \mathbb{N}$, $m > n$

$$d_\phi(u_n, u_m) \le \phi(u_n, u_m)[d_\phi(u_n, u_{n+1}) + d_\phi(u_{n+1}, u_m)],$$

$$d_\phi(u_n, u_m) \le \phi(u_n, u_m) d_\phi(u_n, u_{n+1}) + \phi(u_n, u_m)\phi(u_{n+1}, u_m)[d_\phi(u_{n+1}, u_{n+2}) + d_\phi(u_{n+2}, u_m)],$$

$$\begin{aligned} d_\phi(u_n, u_m) &\leq \phi(u_n, u_m) d_\phi(u_n, u_{n+1}) + \phi(u_n, u_m)\phi(u_{n+1}, u_m) d_\phi(u_{n+1}) + \cdots\cdots \\ &+ \phi(u_n, u_m)\phi(u_{n+1}, u_m) \ldots \phi(u_{m-1}, u_m) d_\phi(u_{m-1}, u_m), \end{aligned}$$

$$\begin{aligned} d_\phi(u_n, u_m) &\leq \phi(u_n, u_m)\alpha^n d_\phi(u_0, u_1) + \phi(u_n, u_m)\phi(u_{n+1}, u_m)\alpha^{n+1} d_\phi(u_0, u_1) + \cdots\cdots \\ &+ \phi(u_n, u_m)\phi(u_{n+1}, u_m) \ldots \phi(u_{m-1}, u_m)\alpha^{m-1} d_\phi(u_0, u_1), \end{aligned}$$

$$\begin{aligned} d_\phi(u_n, u_m) &\leq d_\phi(u_0, u_1)[\phi(u_1, u_m)\phi(u_2, u_m) \ldots \phi(u_n, u_m)\alpha^n + \phi(u_1, u_m)\phi(u_2, u_m) \ldots \\ &\phi(u_{n+1}, u_m)\alpha^{n+1} + \ldots + \phi(u_1, u_m)\phi(u_2, u_m) \ldots \phi(u_n, u_m)\phi(u_{n+1}, u_m) \\ &\ldots \phi(u_{m-1}, u_m)\alpha^{m-1}]. \end{aligned}$$

Since $\alpha < 1$ so $\lim_{n,m \to \infty} \alpha\phi(u_n, u_m) < 1$. Therefore the series $\sum_{n=1}^{\infty} \alpha^n \prod_{i=1}^{n} \phi(u_i, u_m)$ converges by ratio test for all $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} \alpha^n \prod_{i=1}^{n} \phi(u_i, u_m), \quad \text{and} \quad S_n = \sum_{j=1}^{n} \alpha^j \prod_{i=1}^{j} \phi(u_i, u_m).$$

Thus for $m > n$ the above inequality implies

$$d_\phi(u_n, u_m) \leq d_\phi(u_0, u_1)[S_{m-1} - S_n].$$

By taking $n \to \infty$, we conclude that $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As $\mathbf{U}$ is complete, there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$.

On the other hand as f is lower semi-continuous, so from (7) we have

$$0 \leq f(u) \leq \lim_{n\to\infty} \inf f(u_n) = 0.$$

Hence $f(u) = d_\phi(u, Fu) = 0$. Finally, by the closeness of Fu, we have $u \in Fu$. □

Theorem 9. *Let us consider a multi-valued mapping $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$, where $(\mathbf{U}, d_\phi)$ is a complete extended b-metric space. Furthermore, let us consider that the following two conditions hold:*

(*i*) *The map $f : \mathbf{U} \to \mathbb{R}$ defined by $f(u) = d_\phi(u, Fu)$, $u \in \mathbf{U}$, is lower semi-continuous;*
(*ii*) *There exist $q \in (0, 1)$ and $\eta : [0, \infty) \to [0, q)$ such that*

$$\limsup_{s\to t^+} \eta(s) < q, \text{ for all } t \in [0, \infty) \tag{8}$$

and for all $u \in \mathbf{U}$, there exists $v \in I_q^u$ satisfying

$$d_\phi(v, Fv) \leq \eta(d_\phi(u, v)) d_\phi(u, v), \text{ for all } u \in \mathbf{U} \text{ and } v \in Fu. \tag{9}$$

Moreover $\lim_{n,m \to\infty} \alpha\phi(u_n, u_m) < 1$, for all $\alpha \in (0, 1)$. Then F has a fixed point in $\mathbf{U}$.

Proof. Let us assume that F has no fixed point, so $d_\phi(u, Fu) > 0$ for each $u \in \mathbf{U}$. Since $Fu \in \mathcal{CLB}(\mathbf{U})$, for any $u \in \mathbf{U}$, I_q^u is non void for any constant $q \in (0, 1)$. If $v = u$ then $u \in Fu$, which is a contradiction. Hence for all $q \in (0, 1)$ and $u \in \mathbf{U}$, there exist $v \in Tu$ with $u \neq v$ such that

$$q d_\phi(u, v) \leq d_\phi(u, Fu). \tag{10}$$

Let us take an arbitrary point $u_0 \in \mathbf{U}$. By (10) and (*ii*), there exists $u_1 \in Fu_0$ with $u_1 \neq u_0$, satisfying

$$qd_{\phi}(u_0, u_1) \leq d_{\phi}(u_0, Fu_0), \tag{11}$$

and

$$d_{\phi}(u_1, Fu_1) \leq \eta(d_{\phi}(u_0, u_1))d_{\phi}(u_0, u_1), \quad \eta(d_{\phi}(u_0, u_1) < q. \tag{12}$$

From (11) and (12), we have

$$\begin{aligned} d_{\phi}(u_0, Fu_0) - d_{\phi}(u_1, Fu_1) &\geq qd_{\phi}(u_0, u_1) - \eta(d_{\phi}(u_0, u_1))d_{\phi}(u_0, u_1) \\ &\geq [q - \eta(d_{\phi}(u_0, u_1))]d_{\phi}(u_0, u_1) > 0. \end{aligned}$$

Further, for u_1, there exists $u_2 \in Fu_1$, $u_2 \neq u_1$, such that

$$qd_{\phi}(u_1, u_2) \leq d_{\phi}(u_1, Fu_1), \tag{13}$$

and

$$d_{\phi}(u_2, Fu_2) \leq \eta(d_{\phi}(u_1, u_2))d_{\phi}(u_1, u_2), \quad \eta(d_{\phi}(u_1, u_2) < q. \tag{14}$$

By (13) and (14), we have

$$\begin{aligned} d_{\phi}(u_1, Fu_1) - d_{\phi}(u_2, Fu_2) &\geq qd_{\phi}(u_1, u_2) - \eta(d_{\phi}(u_1, u_2))d_{\phi}(u_1, u_2) \\ &\geq [q - \eta(d_{\phi}(u_1, u_2))]d_{\phi}(u_1, u_2) > 0. \end{aligned}$$

Furthermore from (12) and (13)

$$d_{\phi}(u_1, u_2) \leq \frac{1}{q}d_{\phi}(u_1, Fu_1) \leq \frac{1}{q}\eta(d_{\phi}(u_0, u_1))d_{\phi}(u_0, u_1) < d_{\phi}(u_0, u_1).$$

Continuing in this fashion, for u_n, $n > 1$, there exists $u_{n+1} \in Fu_n$, $u_{n+1} \neq u_n$ satisfying

$$qd_{\phi}(u_n, u_{n+1}) \leq d_{\phi}(u_n, Fu_n), \tag{15}$$

and

$$d_{\phi}(u_{n+1}, Fu_{n+1}) \leq \eta(d_{\phi}(u_n, u_{n+1}))d_{\phi}(u_n, u_{n+1}), \quad \eta(d_{\phi}(u_n, u_{n+1}) < q. \tag{16}$$

From (15) and (16), we have

$$\begin{aligned} d_{\phi}(u_n, Fu_n) - d_{\phi}(u_{n+1}, Fu_{n+1}) &\geq qd_{\phi}(u_n, u_{n+1}) - \eta(d_{\phi}(u_n, u_{n+1}))d_{\phi}(u_n, u_{n+1}) \\ &\geq [q - \eta(d_{\phi}(u_n, u_{n+1}))]d_{\phi}(u_n, u_{n+1}) > 0 \end{aligned}$$

and

$$d_{\phi}(u_n, u_{n+1}) < d_{\phi}(u_{n-1}, u_n). \tag{17}$$

From above both equations, it follows that the sequences $\{d_{\phi}(u_n, Fu_n)\}$ and $\{d_{\phi}(u_n, u_{n+1})\}$ are decreasing, and hence convergent. Now from (8), there exists $q^{'} \in [0, q)$ such that $\lim_{n\to\infty} \sup \eta(d_{\phi}(u_n, u_{n+1})) = q^{'}$. Therefore for any $q_0 \in (q^{'}, q)$, there exists $n_0 \in \mathbb{N}$ such that

$$\eta(d_{\phi}(u_n, u_{n+1})) < q_0, \quad \text{for all } n > n_0 \tag{18}$$

Consequently from (15) and (16), we have

$$d_{\phi}(u_n, u_{n+1}) < \alpha d_{\phi}(u_{n-1}, u_n), \tag{19}$$

where $\alpha = \frac{q_0}{q}$ and $n > n_0$. Furthermore, from (15)–(17), for $n > n_0$, we have

$$
\begin{aligned}
d_\phi(u_n, Fu_n) &\leq \eta d_\phi(u_{n-1}, u_n) \leq \frac{\eta(d_\phi(u_{n-1}, u_n))}{q} d_\phi(u_{n-1}, Fu_{n-1}) \\
&\leq \ldots \leq \frac{(\eta(d_\phi(u_{n-1}, u_n)) \ldots \eta(d_\phi(u_0, u_1))}{q^n} d_\phi(u_0, Fu_0) \\
&= \frac{\eta(d_\phi(u_{n-1}, u_n)) \ldots \eta(d_\phi(u_{n_0+1}, u_{n_0+2}))}{q^{n-n_0}} \\
&\quad \times \frac{\eta(d_\phi(u_{n_0}, u_{n_0+1})) \ldots \eta(d_\phi(u_0, u_1))}{q^{n_0}} d_\phi(u_0, Fu_0) \\
&< \left(\frac{q_0}{q}\right)^{n-n_0} \frac{\eta(d_\phi(u_{n_0}, u_{n_0+1})) \ldots \eta(d_\phi(u_0, u_1))}{q^{n_0}} d_\phi(u_0, Fu_0).
\end{aligned}
$$

Since $q_0 < q$, clearly $\lim_{n\to\infty}(\frac{q_0}{q})^{n-n_0} = 0$. This gives

$$\lim_{n\to\infty} d_\phi(u_n, Fu_n) = 0.$$

Let $m > n > n_0$, from the triangle inequality and (19), we have

$$
\begin{aligned}
d_\phi(u_n, u_m) \leq\; & \phi(u_n, u_m) d_\phi(u_n, u_{n+1}) + \phi(u_n, u_m)\phi(u_{n+1}, u_m) d_\phi(u_{n+1}) + \cdots\cdots \\
& + \phi(u_n, u_m)\phi(u_{n+1}, u_m) \ldots \phi(u_{m-1}, u_m) d_\phi(u_{m-1}, u_m),
\end{aligned}
$$

$$
\begin{aligned}
d_\phi(u_n, u_m) \leq\; & \phi(u_n, u_m)\alpha^n d_\phi(u_0, u_1) + \phi(u_n, u_m)\phi(u_{n+1}, u_m)\alpha^{n+1} d_\phi(u_0, u_1) + \cdots\cdots \\
& + \phi(u_n, u_m)\phi(u_{n+1}, u_m) \ldots \phi(u_{m-1}, u_m)\alpha^{m-1} d_\phi(u_0, u_1).
\end{aligned}
$$

By using the analogous procedure as in Theorem 8, there exists a Cauchy sequence $\{u_n\}_{n=0}^\infty$ such that $u_{n+1} \in Fu_n$, $u_{n+1} \neq u_n$. As $\mathbf{U}$ is complete, therefore there exists $u \in \mathbf{U}$ such that $u_n \to u$. By (i), we obtain

$$0 \leq d_\phi(u, Fu) \leq \lim_{n\to\infty} \inf d_\phi(u_n, Fu_n) = 0.$$

By the closedness of Fu, we have $u \in Fu$, which contradicts our assumption that F has no fixed point. □

Corollary 1. *Let $F : \mathbf{U} \to \mathcal{K}(\mathbf{U})$ be a multi-valued mapping, where $(\mathbf{U}, d_\phi)$ is a complete extended b-metric space. Furthermore, let us consider that the following conditions hold:*

(i) *The map $f : \mathbf{U} \to \mathbb{R}$ defined by $f(u) = d_\phi(u, Fu)$, $u \in \mathbf{U}$, is lower semi-continuous;*

(ii) *There exists $\eta : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{s\to t^+} \eta(s) < 1, \text{ for all } t \in [0, \infty),$$

and for all $u \in \mathbf{U}$, there exists $v \in I_1^u$ satisfying

$$d_\phi(v, Fv) \leq \eta(d_\phi(u, v)) d_\phi(u, v), \text{ for all } u \in \mathbf{U} \text{ and } v \in Fu.$$

Moreover $\lim_{n,m\to\infty} \alpha\phi(u_n, u_m) < 1$, for all $\alpha \in (0, 1)$. Then F has a fixed point in $\mathbf{U}$.

Proof. Let us assume that F has no fixed point, so $d_\phi(u, Fu) > 0$ for any $u \in \mathbf{U}$. Since $Fu \in \mathcal{K}(\mathbf{U})$ for any $u \in \mathbf{U}$, I_1^u is non empty. If $v = u$ then $u \in Fu$, which is a contradiction. Hence for all $u \in \mathbf{U}$, there exists $v \in Fu$ with $u \neq v$ such that

$$d_\phi(u, v) \leq d_\phi(u, Fu). \tag{20}$$

Let us consider an arbitrary point $u_0 \in \mathbf{U}$. From (20), by using the analogous procedure as in Theorem 9, we obtain the existence of a Cauchy sequence $\{u_n\}_{n=0}^{\infty}$ such that $u_{n+1} \in Fu_n$, $u_{n+1} \neq u_n$, satisfying

$$d_\phi(u_n, u_{n+1}) = d_\phi(u_n, Fu_n)$$

and

$$d_\phi(u_n, Fu_n) \leq \eta(d_\phi(u_{n-1}, u_n)) d_\phi(u_{n-1}, u_n), \qquad \eta(d_\phi(u_{n-1}, u_n)) < 1.$$

Since $\mathbf{U}$ is complete, there exists $u \in \mathbf{U}$ such that $u_n \to u$. By (i), we obtain

$$0 \leq d_\phi(u, Fu) \leq \lim_{n\to\infty} \inf d_\phi(u_n, Fu_n) = 0.$$

By the closedness of Fu, we have $u \in Fu$, which contradicts our assumption that F has no fixed point. □

Lemma 2. *Let $(\mathbf{U}, d_\phi)$ be an extended b-metric space. Then for any $u \in \mathbf{U}$ and $\alpha > 1$, there exists an element $x \in \mathbf{X}$, where $\mathbf{X} \in \mathcal{CLB}(\mathbf{U})$ such that*

$$d_\phi(u, x) \leq \alpha d_\phi(u, \mathbf{X}). \tag{21}$$

Proof. Let us suppose that $d_\phi(u, \mathbf{X}) = 0$ then $u \in \mathbf{X}$, since $\mathbf{X}$ is a closed subset of $\mathbf{U}$. Further, let us suppose that $x = u$, so (21) holds. Now, suppose that $d_\phi(u, \mathbf{X}) > 0$ and choose

$$\epsilon = (\alpha - 1) d_\phi(u, \mathbf{X}). \tag{22}$$

Then using the definition of $d_\phi(u, \mathbf{X})$, there exists $x \in \mathbf{X}$ such that

$$d_\phi(u, x) \leq d_\phi(u, \mathbf{X}) + \epsilon, \quad \text{where} \quad \epsilon > 0. \tag{23}$$

By putting (22) in (23), we get

$$d_\phi(u, x) \leq \alpha d_\phi(u, \mathbf{X}).$$

□

Theorem 10. *Let $(\mathbf{U}, d_\phi)$ be a complete extended b-metric space and $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$ be a multi-valued mapping satisfying*

$$d_\phi(v, Fv) \leq \eta(d_\phi(u, v)) d_\phi(u, v), \ \textit{for all } u \in \mathbf{U} \textit{ and } v \in Fu, \tag{24}$$

where $\eta : (0, \infty) \to [0, 1)$ such that

$$\limsup_{s\to t^+} \eta(s) < 1, \textit{ for all } t \in [0, \infty). \tag{25}$$

Moreover, let us suppose that $\lim_{n,m\to\infty} \alpha\phi(u_n, u_m) < 1$, for all $\alpha \in (0, 1)$. Then

(i) There exists an orbit $\{u_n\}_{n=0}^{\infty}$ of F for each $u_0 \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$ for $u \in \mathbf{U}$;

(ii) u is a fixed point of F, if and only if the function $f(u) = d_\phi(u, Fu)$ is F-orbitally lower semi-continuous at u.

Proof. Let us assume $u_0 \in \mathbf{U}$ and choose $u_1 \in Fu_0$, since $Fu_0 \neq 0$. If $u_0 = u_1$, then u_0 is a fixed point of F. Let $u_0 \neq u_1$, by taking $\alpha = \frac{1}{\sqrt{\eta(d_\phi(u_0,u_1))}}$, it follows from Lemma 2 that there exists $u_2 \in Fu_1$ such that

$$d_\phi(u_1, u_2) \leq \frac{1}{\sqrt{\eta(d_\phi(u_0, u_1))}} d_\phi(u_1, Fu_1).$$

Continuing in this fashion, we produce a sequence $\{u_n\}_{n=1}^{\infty}$ of points in $\mathbf{U}$ such that $u_{n+1} \in Fu_n$ and

$$d_\phi(u_n, u_{n+1}) \leq \frac{1}{\sqrt{\eta(d_\phi(u_{n-1}, u_n))}} d_\phi(u_n, Fu_n). \tag{26}$$

Now assume that $u_{n-1} \neq u_n$, for otherwise u_{n-1} is fixed point of F. Using (24), it follows from (26) that

$$\begin{aligned} d_\phi(u_n, u_{n+1}) &\leq \sqrt{\eta(d_\phi(u_{n-1}, u_n))} d_\phi(u_{n-1}, u_n) \\ &< d_\phi(u_{n-1}, u_n). \end{aligned} \tag{27}$$

Hence $\{d_\phi(u_n, u_{n+1})\}$ is a decreasing sequence, so it is converges to some non-negative real number. Let a be the limit of $\{d_\phi(u_n, u_{n+1})\}$. Clearly, $a = 0$, for otherwise by taking limits in (27), we obtain $a \leq \sqrt{ca}$, where $c = \limsup_{s \to a^+} \eta(s)$. From (27), we have

$$\begin{aligned} d_\phi(u_n, u_{n+1}) &\leq \sqrt{\eta(d_\phi(u_{n-1}, u_n))} \sqrt{\eta(d_\phi(u_{n-2}, u_{n-1}))} d_\phi(u_{n-2}, u_{n-1}) \ldots \\ &\ldots \leq \sqrt{\eta(d_\phi(u_{n-1}, u_n))} \ldots \sqrt{\eta(d_\phi(u_0, u_1))}] d_\phi(u_0, u_1). \end{aligned}$$

From (25), we can choose $\delta > 0$ and $\alpha \in (0, 1)$ such that

$$\eta(t) < \alpha^2, \text{ for } t \in (0, \delta).$$

Let N be such that $d_\phi(u_{n-1}, u_n) < \delta$ for $n \geq N$. From (27), we have

$$\begin{aligned} d_\phi(u_n, u_{n+1}) &\leq \alpha d_\phi(u_{n-1}, u_n) \leq \ldots \\ &\leq \alpha^{n-N+1} d_\phi(u_{N-1}, u_n). \end{aligned}$$

Hence from the inequality (27), we get

$$\begin{aligned} d_\phi(u_n, u_{n+1}) &\leq \alpha^{n-N+1} [\sqrt{\eta(d_\phi(u_{N-2}, u_{N-1}))} \ldots \sqrt{\eta(d_\phi(u_0, u_1))}] d_\phi(u_0, u_1) \\ &< \alpha^{n-N+1} d_\phi(u_0, u_1). \end{aligned} \tag{28}$$

Therefore from the triangle inequality and (28) for any $m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d_\phi(u_n, u_{n+m}) &\leq \phi(u_n, u_{n+m}) d_\phi(u_n, u_{n+1}) + \phi(u_n, u_{n+m}) \phi(u_{n+1}, u_{n+m}) d_\phi(u_{n+1}, u_{n+2}) + \\ &\quad \cdots\cdots + \phi(u_n, u_{n+m}) \phi(u_{n+1}, u_{n+m}) \ldots \phi(u_{n+m-1}, u_{n+m}) \\ &\quad d_\phi(u_{n+m-1}, u_{n+m}), \end{aligned}$$

$$\begin{aligned} d_\phi(u_n, u_{n+m}) &\leq \alpha^{n-N+1} [\phi(u_n, u_{n+m}) + \alpha^2 \phi(u_n, u_{n+m}) \phi(u_{n+1}, u_{n+m}) + \cdots\cdots + \alpha^{m-n-1} \\ &\quad \phi(u_n, u_{n+m}) \phi(u_{n+1}, u_{n+m}) \ldots \phi(u_{n+m-1}, u_{n+m})] d_\phi(u_0, u_1), \end{aligned}$$

$$\begin{aligned} d_\phi(u_n, u_{n+m}) &\leq \alpha^{n-N+1}[\phi(u_1, u_{n+m})\phi(u_2, u_{n+m}) \ldots \phi(u_n, u_{n+m}) + \phi(u_1, u_{n+m}) \\ &\phi(u_2, u_{n+m}) \ldots \phi(u_{n+m-1}, u_{n+m})]d_\phi(u_0, u_1). \end{aligned}$$

Since $\lim_{n,m \to \infty} \phi(u_n, u_m)\alpha < 1$, the series $\sum_{j=1}^{\infty} \alpha^j \prod_{i=1}^{j} \phi(u_j, u_{n+m})$ converges by the ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{j=1}^{\infty} \alpha^j \prod_{i=1}^{j} \phi(u_i, u_{n+m}), \quad S_n = \sum_{j=1}^{n} \alpha^j \prod_{i=1}^{j} \phi(u_i, u_{n+m}).$$

Thus for $m \in \mathbb{N}$ with $m > n$, the above inequality implies

$$d_\phi(u_n, u_{n+m}) \leq \alpha^{n-N+1}[S_{m-1} - S_n].$$

By letting $n \to \infty$, we conclude that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbf{U}$. As $\mathbf{U}$ is complete, there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$. Since $u_n \in Fu_{n-1}$, it follows from (24) that

$$\begin{aligned} d_\phi(u_n, Fu_n) &\leq \eta(d_\phi(u_{n-1}, u_n))d_\phi(u_{n-1}, u_n) \\ &< d_\phi(u_{n-1}, u_n). \end{aligned}$$

Letting $n \to \infty$, from the above inequality we have

$$\lim_{n\to\infty} d_\phi(u_n, Fu_n) = 0.$$

Suppose $f(u) = d_\phi(u, Fu)$ is F orbitally semi-continuous at u,

$$d_\phi(u, Fu) = f(u) \leq \lim_{n\to\infty} \inf f(u_n) = \lim_{n\to\infty} \inf d_\phi(u_n, Fu_n) = 0.$$

Hence $u \in Fu$, since Fu is closed. Conversely let us suppose that u is a fixed point of F ($u \in Fu$), then $f(u) = 0 \leq \lim_{n\to\infty} \inf f(u_n)$. Hence f is F orbitally semi-continuous at u. □

Remark 2. *Theorem 10 improves Theorem 1, since F may take values in $\mathcal{CLB}(\mathbf{U})$. Since $d_\phi(v, Fv) \leq H(Fu, Fv)$ for $v \in Fu$. We have the following corollary.*

Corollary 2. *Let $(\mathbf{U}, d_\phi)$ be a complete extended b-metric space and $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$ be such that*

$$\mathbf{H}_\Phi(Fu, Fv) \leq \eta(d_\phi(u, v))d_\phi(u, v), \quad \textit{for each} \quad u \in \mathbf{U} \textit{ and } v \in Fu,$$

where $\eta : (0, \infty) \to (0, 1]$ is such that

$$\limsup_{s\to t^+} \eta(s) < 1, \textit{ for all } t \in [0, \infty).$$

Then

- *(i) there exist an orbit $\{u_n\}_{n=0}^{\infty}$ of F for each $u_0 \in \mathbf{U}$ and $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$;*
- *(ii) u is a fixed point of F, if and only if the function $f(u) = d_\phi(u, Fu)$ is F-orbitally lower semi-continuous at u.*

Remark 3. *Theorem 7 extends Nadler's fixed point theorem when $\mathbf{U}$ is the extended b-metric space.*

Remark 4. *Theorem 8 is a generalization of 7. The following example shows that generalization.*

Example 2. *Let* $\mathbf{U} = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\} \cup \{0, 1\}$ *and* $d_\phi : \mathbf{U} \times \mathbf{U} \to [0, \infty)$ *be a mapping defined as* $d_\phi(u_1, u_2) = (u_1 - u_2)^2$, *for* $u_1,\ u_2 \in \mathbf{U}$, *where* $\phi : \mathbf{U} \times \mathbf{U} \to [1,\ \infty)$ *is a mapping defined by* $\phi(u_1,\ u_2) = u_1 + u_2 + 2$. *Then* $(\mathbf{U}, d_\phi)$ *is a complete extended b-metric space. Define* $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$ *as*

$$F(u) = \begin{cases} \{\frac{1}{2^{n+1}}, 1\}, & u = \frac{1}{2^n},\ n = 0, 1, 2, \ldots \\ \{0, \frac{1}{2}\}, & u = 0. \end{cases}$$

In a sense of Theorem 7, clearly F is not contractive, in fact

$$\mathbf{H}_\Phi\Big(F\Big(\frac{1}{2^n}\Big), F(0)\Big) = \frac{1}{2} \geq \frac{1}{2^{2n}} = d_\phi(u_1, u_2), \qquad for\ \ n = 1,\ 2,\ 3, \ldots.$$

On the other way,

$$f(u) = \begin{cases} (\frac{1}{2^{n+1}})^2, & u = \frac{1}{2^n},\ n = 1, 2, \ldots \\ u, & u = 0,\ 1 \end{cases}$$

Hence f *is continuous, so it is clearly lower semi-continuous. Furthermore there exists* $v \in I^u_{0.7}$ *for any* $u \in \mathbf{U}$ *such that*

$$d_\phi(v, F(v)) = \frac{1}{4} d_\phi(u, v).$$

Thus the existence of a fixed point follows from Theorem 8. Hence Theorem 8 is a generalization of Theorem 7.

Remark 5. *Theorem 9 is an extension of Theorem 8. In fact, let us consider a constant map* $\eta = c$, *where* $0 < c < q$. *Thus the hypotheses of Theorem 9 are fulfilled. On the other hand, there exists a map which fulfills the hypotheses of Theorem 9, but does not fulfill the hypotheses of Theorem 8. See the following example:*

Example 3. *Let* $\mathbf{U} = [0, 1]$ *and* $d_\phi : \mathbf{U} \times \mathbf{U} \to [0, \infty)$ *be a mapping defined as* $d_\phi(u_1, u_2) = (u_1 - u_2)^2$, *for* $u_1,\ u_2 \in \mathbf{U}$, *where* $\phi : \mathbf{U} \times \mathbf{U} \to [1,\ \infty)$ *is a mapping defined by* $\phi(u_1,\ u_2) = u_1 + u_2 + 2$. *Then* $(\mathbf{U}, d_\phi)$ *is a complete extended b-metric space. Let* $F : \mathbf{U} \to \mathcal{CLB}(\mathbf{U})$ *be such that*

$$F(u) = \begin{cases} \{\frac{1}{2u^2}\}, & u \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1], \\ \{\frac{17}{96}, \frac{1}{4}\}, & u = \frac{15}{32}. \end{cases}$$

Let $q = \frac{3}{4}$ *and let* $\eta : [0, \infty) \to [0, q)$ *be of the form*

$$\eta(t) = \begin{cases} \frac{3}{2}t, & for\ t \in [0, \frac{7}{24}) \cup (\frac{7}{24}, \frac{1}{2}), \\ \frac{425}{768}, & for\ t = \frac{7}{24}, \\ \frac{1}{2}, & for\ t = [\frac{1}{2}, \infty). \end{cases}$$

Since

$$f(u) = \begin{cases} (u - \frac{1}{2}u^2)^2, & for\ u \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1], \\ \frac{49}{1024}, & for\ u = \frac{15}{32}. \end{cases}$$

Obviously f *is a lower semi-continuous. Further, for any* $u \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1]$ *and* $v = \frac{1}{2}u^2$, *we have*

$$q d_\phi(u, v) \leq d_\phi(u, Fu),$$

and

$$d_\phi(v, Fv) \leq \eta(d_\phi(u, v)) d_\phi(u, v).$$

Of course these both inequalities hold for $u = \frac{15}{32}$ *and* $v = \frac{17}{96}$*. Hence all the hypotheses of Theorem 9 are satisfied and the fixed point of F is* $\{0\}$*. Next let us suppose that, if* $q \in (0, \frac{3}{4}]$ *and* $p \in (0, 1)$ *is such that* $p < q$*, then, for* $u = 1$*, we have* $v = 1/2$ *and consequently*

$$d_\phi\Big(\frac{1}{2}, F\Big(\frac{1}{2}\Big)\Big) > pd_\phi\Big(1, \frac{1}{2}\Big).$$

If $q \in (3/4, 1)$ *and* $p \in (0, 1)$ *is such that* $p < q$*, then for* $u = \frac{15}{32}$*, we have* $Fu = \{\frac{17}{96}, \frac{1}{4}\}$*. Thus, in the case* $v = \frac{17}{96}$*, we obtain*

$$qd_\phi\Big(\frac{15}{32}, \frac{17}{96}\Big) > d_\phi\Big(\frac{15}{32}, F\Big(\frac{15}{32}\Big)\Big),$$

and, in the case $v = \frac{1}{4}$*, we have*

$$d_\phi\Big(\frac{1}{4}, F\Big(\frac{1}{4}\Big)\Big) > pd_\phi\Big(\frac{15}{32}, \frac{1}{4}\Big).$$

Hence hypotheses of Theorem 8 are not fulfilled.

Remark 6. *Theorem 10 is an extension of (Theorem 2.1, [10]) for the case when F is a multi-valued mapping from* $\mathbf{U}$ *to* $\mathcal{CLB}(\mathbf{U})$ *and hence generalizes Theorems 4 and 5 and also the results of [2,5,7,22].*

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