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On Separate Fractional Sum-Difference Equations with n-Point Fractional Sum-Difference Boundary Conditions via Arbitrary Different Fractional Orders

Saowaluck Chasreechai [1] and Thanin Sitthiwirattham [2,*]

1 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; saowaluck.c@sci.kmutnb.ac.th

2 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10700, Thailand

* Correspondence: thanin_sit@dusit.ac.th

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Abstract: In this article, we study the existence and uniqueness results for a separate nonlinear Caputo fractional sum-difference equation with fractional difference boundary conditions by using the Banach contraction principle and the Schauder's fixed point theorem. Our problem contains two nonlinear functions involving fractional difference and fractional sum. Moreover, our problem contains different orders in $n+1$ fractional differences and $m+1$ fractional sums. Finally, we present an illustrative example.

Keywords: fractional sum-difference equations; boundary value problem; existence; uniqueness

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1. Introduction

Fractional calculus has recently been an attractive field to researchers because it is a powerful tool for explaining many engineering and scientific disciplines as the mathematical modeling of systems and processes which appear in nature, for example, ecology, biology, chemistry, physics, mechanics, networks, flow in porous media, electrical, control systems, viscoelasticity, mathematical biology, fitting of experimental data, and so forth. For example, Zhang et al. [1] proposed both analytical and numerical results from studying the propagation of optical beams in the fractional Schrödinger equation with a harmonic potential. In 2015, Zingales and Failla [2] solved the fractional-order heat conduction equation by using a pertinent finite element method. For Lazopoulos's [3] work, they defined the fractional curvature of plane curves, the fractional beam small deflection, the fractional curvature is approximate. In 2017, Sumelka and Voyiadjis [4] proposed a concept of short memory connected with the definition of damage parameter evolution in terms of fractional calculus for hyperelastic materials.

Basic definitions and properties of fractional difference calculus, appear in the book [5]. In particular, fractional calculus is a powerful tool for the processes which appear in nature, e.g., ecology, biology and other areas, one may see the papers [6–8] and the references therein. The interesting papers related to discrete fractional boundary value problems can be found in [9–29] and references cited therein. For previous works, Goodrich [10] considered the discrete fractional boundary value problem

$$\begin{cases} -\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) = f(t+\mu_1+\mu_2+\mu_3-1, y(t+\mu_1+\mu_2+\mu_3-1)), \\ y(0) = 0 = y(b+2), \end{cases} \tag{1}$$

where $t \in \mathbb{N}_{2-\mu_1-\mu_2-\mu_3, b+2-\mu_1-\mu_2-\mu_3}$, $0 < \mu_1, \mu_2, \mu_3 < 1$, $1 < \mu_2 + \mu_3 < 2$, $1 < \mu_1 + \mu_2 + \mu_3 < 2$, $f : \mathbb{N}_0 \times \mathbb{R} \to [0, +\infty)$ is a continuous function, and Δ^μ is the Riemann-Liouville fractional difference operator of order μ. Existence of positive solutions are obtained by the use of the Krasnosel'skii fixed point theorem.

Weidong [12] examined the sequential fractional boundary value problem with a p-Laplacian

$$\begin{cases} \Delta_C^\beta[\phi_p(\Delta_C^\alpha x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \quad t \in \mathbb{N}_{0,b}, \\ \Delta_C^\beta x(\beta-1) + \Delta_C^\beta x(\beta+b) = 0, \\ x(\alpha+\beta-2) + x(\alpha+\beta+b) = 0, \end{cases} \tag{2}$$

where $0 < \alpha, \beta \le 1$, $1 < \alpha+\beta \le 2$, $f : \mathbb{N}_{\alpha+\beta-1, \alpha+\beta+T-1} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, ϕ_p is the p-Laplacian operator, and Δ_C^β is the Caputo fractional difference operator of order β. Existence and uniqueness of solutions are obtained by using the Schaefer's fixed point theorem.

Recently, Sitthiwirattham [19,20] investigated three-point fractional sum boundary value problems for sequential fractional difference equations of the forms

$$\begin{cases} \Delta_C^\alpha[\phi_p(\Delta_C^\beta x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \\ \Delta_C^\beta x(\alpha-1) = 0, \quad x(\alpha+\beta+T) = \rho\Delta^{-\gamma}x(\eta+\gamma), \end{cases} \tag{3}$$

and

$$\begin{cases} \Delta_\alpha^\alpha(\Delta_{\alpha+\beta-1}^\beta + \lambda E_\beta)x(t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \\ x(\alpha+\beta-2) = 0, \quad x(\alpha+\beta+T) = \rho\Delta_{\alpha+\beta-1}^{-\gamma}x(\eta+\gamma), \end{cases} \tag{4}$$

where $t \in \mathbb{N}_{0,T}$, $0 < \alpha, \beta \le 1$, $1 < \alpha+\beta \le 2$, $0 < \gamma \le 1$, $\eta \in \mathbb{N}_{\alpha+\beta-1, \alpha+\beta+T-1}$, ρ is a constant, $f : \mathbb{N}_{\alpha+\beta-2, \alpha+\beta+T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $E_\beta x(t) = x(t+\beta-1)$ and ϕ_p is the p-Laplacian operator. Existence and uniqueness of solutions are obtained by using the Banach fixed point theorem and the Schaefer's fixed point theorem.

The results mentioned above are the motivation for this research. In this paper, we consider a separate nonlinear Caputo fractional sum-difference equation of the form

$$\begin{aligned} \left[\Delta_C^\alpha + (e+1)\,\Delta_C^{\alpha-1}\right] u(t) \;&=\; \lambda F\Big(t+\alpha-1, u(t+\alpha-1), (\Upsilon^\vartheta u)(t+\alpha-\vartheta)\Big) \\ &\quad + \mu H\Big(t+\alpha-1, u(t+\alpha-1), (\Psi^\gamma u)(t+\alpha+\gamma-1)\Big), \end{aligned} \tag{5}$$

with the fractional sum-difference boundary value conditions

$$\begin{aligned} u(\alpha-n) \;&=\; \Delta_C^{\beta_1} u\,(\alpha-n-\beta_1+2) \;=\; \Delta_C^{\beta_1+\beta_2} u\,(\alpha-n-\beta_1-\beta_2+4) \;=\; \ldots \\ &=\; \Delta_C^{\sum_{i=1}^{n-2}\beta_i} u\left(\alpha+n-4-\sum_{i=1}^{n-2}\beta_i\right) \;=\; 0, \\ u(T+\alpha) \;&=\; \tau\,\Delta^{-\sum_{i=1}^m \theta_i} g\left(\eta+\sum_{i=1}^m \theta_i\right) u\left(\eta+\sum_{i=1}^m \theta_i\right), \end{aligned} \tag{6}$$

where $t \in \mathbb{N}_{0,T} := \{0, 1, \ldots, T\}$, $\tau < \dfrac{\Gamma\left(\sum_{i=1}^m \theta_i\right) \sum_{s=\alpha-n}^{T+\alpha-n+1} e^{-s}}{\sum_{r=\alpha-n}^{\eta} \sum_{s=\alpha-n}^{r-n+1} \left(\eta+\sum_{i=1}^m \theta_i - \sigma(r)\right)^{\underline{\sum_{i=1}^m \theta_i - 1}} e^{-s} g\left(\eta+\sum_{i=1}^m \theta_i\right)}$, $\alpha \in (n-1, n]$, $\beta_i, \theta_i, \gamma \in (0, 1]$, $m, n \in \mathbb{N}_4$, $m < n$, $T > n-3$, $\sum_{i=1}^{n-2} \beta_i \in (n-3, n-2]$, $\sum_{i=1}^m \theta_i \in (m-1, m]$ and $\lambda, \mu \in \mathbb{R}$ are given constants; $F \in C\big(\mathbb{N}_{\alpha-n, T+\alpha} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$, $H \in C\big(\mathbb{N}_{\alpha-n, T+\alpha} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$, $g \in C\big(\mathbb{N}_{\alpha-n, T+\alpha}, \mathbb{R}^+\big)$, and for $\varphi, \phi \in C\big(\mathbb{N}_{\alpha-n, T+\alpha} \times \mathbb{N}_{\alpha-n, T+\alpha}, [0, \infty)\big)$, we defined the operators

$$
\begin{aligned}
(\Upsilon^{\vartheta}u)(t-\vartheta+1) &:= [\Delta_C^{\vartheta}\phi\, u](t-\vartheta+1)\\
&= \frac{1}{\Gamma(1-\vartheta)}\sum_{s=\alpha-n+\vartheta-1}^{t+\vartheta-1}(t-\sigma(s))^{\underline{-\vartheta}}\phi(t,s-\vartheta+1)\Delta u(s-\vartheta+1),
\end{aligned}
$$

$$
\text{and}\ \ (\Psi^{\gamma}u)(t+\gamma) := [\Delta^{-\gamma}\varphi\, u](t+\gamma) = \frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-n-\gamma}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}\varphi(t,s+\gamma)u(s+\gamma).
$$

The plan of this paper is as follows. In Section 2 we recall some definitions and basic lemmas. We derive a representation for the solution of (5) by converting the problem to an equivalent summation equation. In Section 3, we prove existence results of the problem (5) by using the Banach contraction principle and the Schauder's theorem. Finally, an illustrative example is presented in Section 4.

2. Preliminaries

The notations, definitions, and lemmas which are used in the main results are as follows.

Definition 1. *We define the generalized falling function by* $t^{\underline{\alpha}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, *for any* t *and* α *for which the right-hand side is defined. If* $t+1-\alpha$ *is a pole of the Gamma function and* $t+1$ *is not a pole, then* $t^{\underline{\alpha}} = 0$.

Lemma 1 ([16]). *Assume the factorial functions are well defined. If* $t \le r$, *then* $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ *for any* $\alpha > 0$.

Definition 2. *For* $\alpha > 0$ *and* f *defined on* $\mathbb{N}_a := \{a, a+1, \ldots\}$, *the* α*-order fractional sum of* f *is defined by*

$$
\Delta^{-\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}f(s),
$$

where $t \in \mathbb{N}_{a+\alpha}$ *and* $\sigma(s) = s+1$.

Definition 3. *For* $\alpha > 0$ *and* f *defined on* $\mathbb{N}_a$, *the* α*-order Caputo fractional difference of* f *is defined by*

$$
\Delta_C^{\alpha}f(t) := \Delta^{-(N-\alpha)}\Delta^N f(t) = \frac{1}{\Gamma(N-\alpha)}\sum_{s=a}^{t-(N-\alpha)}(t-\sigma(s))^{\underline{N-\alpha-1}}\Delta^N f(s),
$$

where $t \in \mathbb{N}_{a+N-\alpha}$ *and* $N \in \mathbb{N}$ *is chosen so that* $0 \le N-1 < \alpha < N$.

Lemma 2 ([14]). *Assume that* $\alpha > 0$ *and* $0 \le N-1 < \alpha \le N$. *Then*

$$
\Delta^{-\alpha}\Delta_C^{\alpha}y(t) = y(t) + C_0 + C_1 t^{\underline{1}} + C_2 t^{\underline{2}} + \ldots + C_{N-1}t^{\underline{N-1}},
$$

for some $C_i \in \mathbb{R}$, $0 \le i \le N-1$.

To investigate the solution of the boundary value problem (5) we need the following lemma involving a linear variant of the boundary value problem (5).

Lemma 3. *Let* $\tau < \frac{\Gamma\left(\sum_{i=1}^{m}\theta_i\right)\sum_{s=\alpha-n}^{T+\alpha-n+1}e^{-s}}{\sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1}\left(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r)\right)^{\underline{\sum_{i=1}^{m}\theta_i-1}}e^{-s}g\left(\eta+\sum_{i=1}^{m}\theta_i\right)}$, $\alpha \in (n-1,n]$, $\beta_i, \theta_i, \gamma \in (0,1]$, $m,n \in \mathbb{N}_4$, $m < n$, $T > n-3$, $\sum_{i=1}^{n-2}\beta_i \in (n-3,n-2]$, $\sum_{i=1}^{m}\theta_i \in (m-1,m]$ *and* $h \in C\left(\mathbb{N}_{\alpha-n,T+\alpha}\times\mathbb{R},\mathbb{R}\right)$, $g \in C\left(\mathbb{N}_{\alpha-n,T+\alpha},\mathbb{R}^+\right)$ *be given. Then the problem*

$$\left[\Delta_C^{\alpha} + (e-1)\,\Delta_C^{\alpha-1}\right] u(t) = h(t+\alpha-1), \quad t \in \mathbb{N}_{0,T} \tag{7}$$

$$\begin{aligned} u(\alpha-n) &= \Delta_C^{\beta_1} u\,(\alpha-n-\beta_1+2) \;=\; \Delta_C^{\beta_1+\beta_2} u\,(\alpha-n-\beta_1-\beta_2+4) \\ &= \;...\; = \Delta_C^{\sum_{i=1}^{n-2}\beta_i} u\left(\alpha+n-4-\sum_{i=1}^{n-2}\beta_i\right) \;=\; 0, \end{aligned} \tag{8}$$

$$u(T+\alpha) = \tau\,\Delta^{-\sum_{i=1}^{m}\theta_i} g\left(\eta+\sum_{i=1}^{m}\theta_i\right) u\left(\eta+\sum_{i=1}^{m}\theta_i\right), \tag{9}$$

has the unique solution

$$\begin{aligned} u(t) \;&=\; \frac{\mathcal{O}[h]}{\Lambda}\sum_{s=\alpha-n}^{t-n+1} e^{-s} \\ &+\frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{t-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n} e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}} h(\xi+\alpha-1), \end{aligned} \tag{10}$$

where the functional $\mathcal{O}[h]$ and the constant Λ are defined by

$$\begin{aligned} \mathcal{O}[h] \;&=\; \sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n} \frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma\,(\sum_{i=1}^{m}\theta_i)\,\Gamma(\alpha-n)}\times \\ &\quad e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}} g\left(\eta+\sum_{i=1}^{m}\theta_i\right) h(\xi+\alpha-1) \\ &\quad -\frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{T+\alpha-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n} e^{v-s}(v-\sigma(x))^{\underline{\alpha-n-1}} h(x+\alpha-1), \end{aligned} \tag{11}$$

$$\begin{aligned} \Lambda \;&=\; \sum_{s=\alpha-n}^{T+\alpha-n+1} e^{-s} - \sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1} \frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma\,(\sum_{i=1}^{m}\theta_i)}\times \\ &\quad e^{-s} g\left(\eta+\sum_{i=1}^{m}\theta_i\right), \textit{ respectively.} \end{aligned} \tag{12}$$

Proof. Using the fractional sum of order $\alpha : \Delta^{-\alpha}$ for (7), we obtain

$$\begin{aligned} u(t)+(e-1)\,\Delta^{-1}u(t) \;&=\; C_1 + C_2 t^{\underline{1}} + C_3 t^{\underline{2}} + ... + C_n t^{\underline{n-1}} \\ &\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1), \quad t \in \mathbb{N}_{\alpha-n,T+\alpha}. \end{aligned} \tag{13}$$

For the forward difference of order $n : \Delta^n$ for (13), we have

$$\Delta^n u(t) + (e-1)\,\Delta^{n-1}u(t) \;=\; \frac{1}{\Gamma(\alpha-n)}\sum_{s=0}^{t-\alpha+n}(t-\sigma(s))^{\underline{\alpha-n-1}} h(s+\alpha-1).$$

Therefore,

$$\Delta\left[e^t\,\Delta^{n-1}u(t)\right] \;=\; \frac{e^t}{\Gamma(\alpha-n)}\sum_{s=0}^{t-\alpha+n}(t-\sigma(s))^{\underline{\alpha-n-1}} h(s+\alpha-1). \tag{14}$$

Taking the sum: Δ^{-1} to (14), we get

$$e^t\,\Delta^{n-1}u(t) \;=\; C_n + \frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{t-1}\sum_{v=0}^{s-\alpha+n} e^s(s-\sigma(v))^{\underline{\alpha-n-1}}h(v+\alpha-1). \tag{15}$$

Next, taking the sum of order $n-1:\Delta^{-(n-1)}$ to (15), we obtain

$$\begin{aligned} u(t) \;=\; & C_1 + C_2t^{\underline{1}} + C_3t^{\underline{2}} + \ldots + C_{n-1}t^{\underline{n-2}} + C_n\sum_{s=\alpha-n}^{t-n+1} e^{-s} \\ & + \frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{t-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{x=0}^{v-\alpha+n} e^{v-s}(v-\sigma(x))^{\underline{\alpha-n-1}}h(x+\alpha-1),\;\; t\in\mathbb{N}_{\alpha-n,T+\alpha}. \end{aligned} \tag{16}$$

Using the Caputo fractional differences of order β_i for (16) where $i=1$ to $i=n-2$, we obtain

$$\begin{aligned} & \Delta_C^{\beta_1}u(t) \\ = \; & C_2\Delta_C^{\beta_1}t^{\underline{1}} + C_3\Delta_C^{\beta_1}t^{\underline{2}} + \ldots + C_{n-1}\Delta_C^{\beta_1}t^{\underline{n-2}} \\ & + C_n\sum_{r=\alpha-n}^{t+\beta_1-1}\frac{(t-\sigma(r))^{\underline{-\beta_1}}}{\Gamma(1-\beta_1)}\Delta_r\left\{\sum_{s=\alpha-n}^{r-n+1}e^{-s}\right\} \\ & + \sum_{r=\alpha-n}^{t+\beta_1-1}\frac{(t-\sigma(r))^{\underline{-\beta_1}}}{\Gamma(1-\beta_1)}\Delta_r\left\{\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{x=0}^{v-\alpha+n}e^{v-s}\frac{(v-\sigma(x))^{\underline{\alpha-n-1}}}{\Gamma(\alpha-n)}h(x+\alpha-1)\right\}, \end{aligned} \tag{17}$$

for $t\in\mathbb{N}_{\alpha-n+1-\beta_1,T+\alpha+1-\beta_1}$.

$$\begin{aligned} & \Delta_C^{\beta_1+\beta_2}u(t) \\ = \; & C_3\Delta_C^{\beta_1+\beta_2}t^{\underline{2}} + \ldots + C_{n-1}\Delta_C^{\beta_1+\beta_2}t^{\underline{n-2}} \\ & + C_n\sum_{r=\alpha-n}^{t+\beta_1+\beta_2-2}\frac{(t-\sigma(r))^{\underline{1-\beta_1-\beta_2}}}{\Gamma(2-\beta_1-\beta_2)}\Delta_r^2\left\{\sum_{s=\alpha-n}^{r-n+1}e^{-s}\right\} \\ & + \sum_{r=\alpha-n}^{t+\beta_1+\beta_2-2}\frac{(t-\sigma(r))^{\underline{1-\beta_1-\beta_2}}}{\Gamma(2-\beta_1-\beta_2)}\Delta_r^2\left\{\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{x=0}^{v-\alpha+n}e^{v-s}\frac{(v-\sigma(x))^{\underline{\alpha-n-1}}}{\Gamma(\alpha-n)}h(x+\alpha-1)\right\}, \end{aligned} \tag{18}$$

for $t\in\mathbb{N}_{\alpha-n+2-\beta_1-\beta_2,T+\alpha+2-\beta_1-\beta_2}$.

$$\vdots$$

$$\begin{aligned} & \Delta_C^{\sum_{i=1}^{n-2}\beta_i}u(t) \\ = \; & C_{n-1}\Delta_C^{\sum_{i=1}^{n-2}\beta_i}t^{\underline{n-2}} + C_n\sum_{r=\alpha-n}^{t-n+2+\sum_{i=1}^{n-2}\beta_i}\frac{(t-\sigma(r))^{\underline{n-3-\sum_{i=1}^{n-2}\beta_i}}}{\Gamma\left(n-2-\sum_{i=1}^{n-2}\beta_i\right)}\Delta_r^{n-2}\left\{\sum_{s=\alpha-n}^{r-n+1}e^{-s}\right\} \\ & + \sum_{r=\alpha-n}^{t-n+2+\sum_{i=1}^{n-2}\beta_i}\frac{(t-\sigma(r))^{\underline{n-3-\sum_{i=1}^{n-2}\beta_i}}}{\Gamma\left(n-2-\sum_{i=1}^{n-2}\beta_i\right)}\times \\ & \Delta_r^{n-2}\left\{\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{x=0}^{v-\alpha+n}e^{v-s}\frac{(v-\sigma(x))^{\underline{\alpha-n-1}}}{\Gamma(\alpha-n)}h(x+\alpha-1)\right\}, \end{aligned} \tag{19}$$

for $t \in \mathbb{N}_{\alpha-2-\sum_{i=1}^{n-2}\beta_i, T+\alpha+n-2-\sum_{i=1}^{n-2}\beta_i}$.

Employing the conditions of (8), we have the system of $n-1$ equations

$$\begin{aligned}
&(E_1) \quad C_1 + C_2(\alpha-n) + C_3(\alpha-n)^{\underline{2}} + \ldots + C_{n-1}(\alpha-n)^{\underline{n-2}} \;=\; 0,\\
&(E_2) \quad C_2\Delta_C^{\beta_1}(\alpha-n+2-\beta_1) + \ldots + C_{n-1}\Delta_C^{\beta_1}(\alpha-n+2-\beta_1)^{\underline{n-2}} \;=\; 0,\\
&(E_3) \quad C_3\Delta_C^{\beta_1+\beta_2}(\alpha-n+4-\beta_1-\beta_2)^{\underline{2}} + \ldots + C_{n-1}\Delta_C^{\beta_1+\beta_2}(\alpha-n+4-\beta_1-\beta_2)^{\underline{n-2}} \;=\; 0,\\
&\ldots\\
&(E_{n-1}) \quad C_{n-1}\Delta_C^{\sum_{i=1}^{n-2}\beta_i}\left(\alpha+n-4-\sum_{i=1}^{n-2}\beta_i\right)^{\underline{n-2}} \;=\; 0.
\end{aligned} \tag{20}$$

Using the fractional sum of order $\sum_{i=1}^{m}\theta_i$ for (16), we have

$$\begin{aligned}
&\Delta_C^{-\sum_{i=1}^{m}\theta_i}u(t)\\
=&\sum_{r=\alpha-n}^{t-\sum_{i=1}^{m}\theta_i}\frac{(t-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}\left[C_1 + C_2\Delta_C^{\beta_1}s^{\underline{1}} + C_3\Delta_C^{\beta_1}s^{\underline{2}} + \ldots + C_{n-1}\Delta_C^{\beta_1}s^{\underline{n-2}}\right]\\
&+C_n\sum_{r=\alpha-n}^{t-\sum_{i=1}^{m}\theta_i}\sum_{s=\alpha-n}^{r-n+1}\frac{(t-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}e^{-s}\\
&+\sum_{r=\alpha-n}^{t-\sum_{i=1}^{m}\theta_i}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{x=0}^{v-\alpha+n}\frac{(t-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}\frac{(v-\sigma(x))^{\underline{\alpha-n-1}}}{\Gamma(\alpha-n)}e^{v-s}h(x+\alpha-1),
\end{aligned} \tag{21}$$

for $t \in \mathbb{N}_{\alpha-n+\sum_{i=1}^{m}\theta_i, T+\alpha+\sum_{i=1}^{m}\theta_i}$.

By substituting $t = \eta + \sum_{i=1}^{m}\theta_i$ into (21) and using the second condition of (9), we finally get

$$\begin{aligned}
(E_n) \quad & C_1\left\{1 - \sum_{r=\alpha-n}^{\eta}\frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}g\left(\eta+\sum_{i=1}^{m}\theta_i\right)\right\}\\
&+C_2\left\{(T+\alpha)^{\underline{1}} - \sum_{r=\alpha-n}^{\eta}\frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}s}{\Gamma(\sum_{i=1}^{m}\theta_i)}g\left(\eta+\sum_{i=1}^{m}\theta_i\right)\right\}\\
&+C_3\left\{(T+\alpha)^{\underline{2}} - \sum_{r=\alpha-n}^{\eta}\frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}s^{\underline{2}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}g\left(\eta+\sum_{i=1}^{m}\theta_i\right)\right\}\\
&+\cdots\\
&+C_{n-1}\left\{(T+\alpha)^{\underline{n-2}} - \sum_{r=\alpha-n}^{\eta}\frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}s^{\underline{n-2}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}g\left(\eta+\sum_{i=1}^{m}\theta_i\right)\right\}\\
&+C_n\left\{\sum_{s=\alpha-n}^{T+\alpha-n+1}e^{-s} - \sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1}\frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}e^{-s}g\left(\eta+\sum_{i=1}^{m}\theta_i\right)\right\}\\
=&\sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{x=0}^{v-\alpha+n}\frac{\tau(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma(\sum_{i=1}^{m}\theta_i)}e^{v-s}\times\\
&\frac{(v-\sigma(x))^{\underline{\alpha-n-1}}}{\Gamma(\alpha-n)}g\left(\eta+\sum_{i=1}^{m}\theta_i\right)h(x+\alpha-1)\\
&-\sum_{s=\alpha-n}^{T+\alpha-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{x=0}^{v-\alpha+n}e^{v-s}\frac{(v-\sigma(x))^{\underline{\alpha-n-1}}}{\Gamma(\alpha-n)}h(x+\alpha-1).
\end{aligned} \tag{22}$$

Solving the system of Equations $(E_1)-(E_n)$, we obtain

$$C_1 = C_2 = \dots = C_{n-1} = 0 \quad \text{and} \quad C_n = \frac{\mathcal{O}[h]}{\Lambda},$$

where $\mathcal{O}[h], \Lambda$ are defined by (11), (12), respectively. Substituting the constants $C_1 - C_n$ into (17), we obtain (10). This completes the proof. □

3. Main Results

The goal of this section is to show the existence results for the problem (5). To accomplish this, we denote $\mathcal{C} = C(\mathbb{N}_{\alpha-n,T+\alpha}, \mathbb{R})$, the Banach space of all functions u with the norm is defined by

$$\|u\|_{\mathcal{C}} = \|u\| + \|\Delta_C^{\vartheta} u\| + \|\Delta^{-\gamma} u\|,$$

where $\|u\| = \max_{t\in\mathbb{N}_{\alpha-n,T+\alpha}} |u(t)|$, $\|\Delta_C^{\vartheta} u\| = \max_{t\in\mathbb{N}_{t\in\mathbb{N}_{\alpha-n,T+\alpha}}} |\Delta_C^{\vartheta} x(t-\vartheta+1)|$ and $\|\Delta^{-\gamma} u\| = \max_{t\in\mathbb{N}_{t\in\mathbb{N}_{\alpha-n,T+\alpha}}} |\Delta^{-\gamma} x(t+\gamma)|$. In addition, we define the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}u)(t) = {} & \frac{\mathcal{O}[F(u)+H(u)]}{\Lambda} \sum_{s=\alpha-n}^{t-n+1} e^{-s} + \frac{1}{\Gamma(\alpha-n)} \sum_{s=\alpha-n}^{t-n+1} \sum_{v=\alpha-n}^{s-1} \sum_{\xi=0}^{v-\alpha+n} e^{v-s} \times \\ & (v-\sigma(\xi))^{\underline{\alpha-n-1}} \Big[\lambda F\Big(\xi+\alpha-1, u(\xi+\alpha-1), (\Upsilon^{\vartheta} u)(\xi+\alpha-\vartheta)\Big) \\ & + \mu H\Big(\xi+\alpha-1, u(\xi+\alpha-1), (\Psi^{\gamma} u)(\xi+\alpha+\gamma-1)\Big)\Big], \end{aligned} \tag{23}$$

where Λ is defined by (12) and the functional $\mathcal{O}[F(u)+H(u)]$ is defined by

$$\begin{aligned} & \mathcal{O}[F(u)+H(u)] \\ = {} & \sum_{r=\alpha-n}^{\eta} \sum_{s=\alpha-n}^{r-n+1} \sum_{v=\alpha-n}^{s-1} \sum_{\xi=0}^{v-\alpha+n} \frac{\tau\left(\eta+\sum_{i=1}^m \theta_i - \sigma(r)\right)^{\underline{\sum_{i=1}^m \theta_i - 1}}}{\Gamma\left(\sum_{i=1}^m \theta_i\right)\Gamma(\alpha-n)} e^{v-s} g\left(\eta+\sum_{i=1}^m \theta_i\right) \times \\ & (v-\sigma(\xi))^{\underline{\alpha-n-1}} \Big[\lambda F\Big(\xi+\alpha-1, u(\xi+\alpha-1), (\Upsilon^{\vartheta} u)(\xi+\alpha-\vartheta)\Big) \\ & + \mu H\Big(\xi+\alpha-1, u(\xi+\alpha-1), (\Psi^{\gamma} u)(\xi+\alpha+\gamma-1)\Big)\Big] \\ & - \frac{1}{\Gamma(\alpha-n)} \sum_{s=\alpha-n}^{T+\alpha-n+1} \sum_{v=\alpha-n}^{s-1} \sum_{\xi=0}^{v-\alpha+n} e^{v-s} (v-\sigma(x))^{\underline{\alpha-n-1}} \times \\ & \Big[\lambda F\Big(\xi+\alpha-1, u(\xi+\alpha-1), (\Upsilon^{\vartheta} u)(\xi+\alpha-\vartheta)\Big) \\ & + \mu H\Big(\xi+\alpha-1, u(\xi+\alpha-1), (\Psi^{\gamma} u)(\xi+\alpha+\gamma-1)\Big)\Big]. \end{aligned} \tag{24}$$

Clearly, the problem (5) has solutions if and only if the operator $\mathcal{F}$ has fixed points. The first show the existence and uniqueness of a solution to the problem (5) by using the Banach contraction principle.

Theorem 1. *Assume that* $F, H : \mathbb{N}_{\alpha-n,T+\alpha} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous,* $\varphi, \phi : \mathbb{N}_{\alpha-n,T+\alpha} \times \mathbb{N}_{\alpha-n,T+\alpha} \to [0,\infty)$ *are continuous with* $\varphi_0 = \max\{\varphi(t-1,s) : (t,s) \in \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha}\}$ *and* $\phi_0 = \max\{\phi(t-1,s) : (t,s) \in \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha}\}$*. In addition, suppose that:*

(H_1) *there exist constants* $L_1, L_2 > 0$ *such that for each* $t \in \mathbb{N}_{\alpha-n,T+\alpha}$ *and* $u, v \in \mathcal{C}$

$$|F(t,u(t),(\Upsilon^{\vartheta}u)(t-\vartheta+1)) - F(t,v(t),(\Upsilon^{\vartheta}v)(t-\vartheta+1))| \leq L_1|u-v| + L_2|(\Upsilon^{\vartheta}u)-(\Upsilon^{\vartheta}v)|,$$

(H_2) *there exist constants* $\ell_1, \ell_2 > 0$ *such that for each* $t \in \mathbb{N}_{\alpha-n,T+\alpha}$ *and* $u, v \in \mathcal{C}$

$$|H(t,u(t),(\Psi^{\gamma}u)(t+\gamma)) - f(t,v(t),(\Psi^{\gamma}v)(t+\gamma))| \leq \ell_1|u-v| + \ell_2|(\Psi^{\gamma}u)-(\Psi^{\gamma}v)|,$$

(H_3) $0 < g(t) < K \neq \frac{(e^{T-n+3}-1)e^{\eta-2n+2}\Gamma(m+1)}{(\eta-\alpha+n+m)^{\underline{m}}\tau\, e^{T+\alpha-2n+2}(\eta-\alpha+n+m)^{\underline{m}}}$ *for each* $t \in \mathbb{N}_{\alpha-n,T+\alpha}$.

$$\begin{aligned} \textit{If} \;\; \chi := & \Big[\lambda\Big(L_1 + L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big) + \mu\Big(\ell_1 + \ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\Big] \times \\ & \big(\Omega_1+\Omega_2+\Omega_3\big) \; < \; 1, \end{aligned} \tag{25}$$

then the problem (5) has a unique solution on $\mathbb{N}_{\alpha-n,T+\alpha}$, *where*

$$\Omega_1 = \frac{\Theta e^{n-\alpha}(T+2)}{|\Lambda|} + \frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}, \tag{26}$$

$$\Omega_2 = \left[\frac{\Theta e^{2n-\alpha-2}}{|\Lambda|} + \frac{e^{T-1}(T+\alpha-n+3)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}\right]\frac{(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}, \tag{27}$$

$$\Omega_3 = \left[\frac{\Theta e^{n-\alpha}(T+2)}{|\Lambda|} + \frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}\right]\frac{(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}, \tag{28}$$

$$\Theta = \frac{\tau K e^{\eta-\alpha-1}(\eta-\alpha+2)^{\underline{\alpha-n+2}}(\eta-\alpha+n+m)^{\underline{m}}}{\Gamma(m+1)\Gamma(\alpha-n+3)} - \frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}. \tag{29}$$

Proof. We shall show that $\mathcal{F}$ is a contraction. For any $u, v \in \mathcal{C}$ and for each $t \in \mathbb{N}_{\alpha-n,T+\alpha}$, we have

$$\begin{aligned} & \Big|\mathcal{O}[F(u)+H(u)] - \mathcal{O}[F(v)+H(v)]\Big| \\ \leq & \sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}\frac{\tau\,(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r))^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma\left(\sum_{i=1}^{m}\theta_i\right)\Gamma(\alpha-n)}\,e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}}\times \\ & \Big[\lambda\Big(L_1|u-v| + L_2|(\Upsilon^{\vartheta}u)-(\Upsilon^{\vartheta}v)|\Big) + \mu\Big(\ell_1|u-v| + \ell_2|(\Psi^{\gamma}u)-(\Psi^{\gamma}v)|\Big)\Big]\times \\ & g\left(\eta+\sum_{i=1}^{m}\theta_i\right) - \frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{T+\alpha-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}e^{v-s}(v-\sigma(x))^{\underline{\alpha-n-1}}\times \\ & \Big[\lambda\Big(L_1|u-v| + L_2|(\Upsilon^{\vartheta}u)-(\Upsilon^{\vartheta}v)|\Big) + \mu\Big(\ell_1|u-v| + \ell_2|(\Psi^{\gamma}u)-(\Psi^{\gamma}v)|\Big)\Big] \\ \leq & \Big[\lambda\Big(L_1 + L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big) + \mu\Big(\ell_1 + \ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\Big]\Bigg|\,\|u-v\|_{\mathcal{C}}\times \\ & \Bigg|\frac{\tau K e^{\eta-\alpha-1}(\eta-\alpha+2)^{\underline{\alpha-n+2}}(\eta-\alpha+n+m)^{\underline{m}}}{\Gamma(m+1)\Gamma(\alpha-n+3)} - \frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)} \\ = & \Big[\lambda\Big(L_1 + L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big) + \mu\Big(\ell_1 + \ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\Big]\|u-v\|_{\mathcal{C}}\Theta, \end{aligned} \tag{30}$$

and

$$\begin{aligned}
&\big|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\big| \\
\leq\ & \frac{1}{\Lambda}\Big|\mathcal{O}[F(u)+H(u)]-\mathcal{O}[F(v)+H(v)]\Big|\sum_{s=\alpha-n}^{t-n+1}e^{-s} \\
&+\frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{t-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}}\times \\
&\Big[\lambda\Big(L_1|u-v|+L_2|(\Upsilon^{\vartheta}u)-(\Upsilon^{\vartheta}v)|\Big)+\mu\Big(\ell_1|u-v|+\ell_2|(\Psi^{\gamma}u)-(\Psi^{\gamma}v)|\Big)\Big] \\
\leq\ & \Big[\lambda\Big(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big)+\mu\Big(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\Big]\|u-v\|_{\mathcal{C}}\times \\
&\left\{\frac{\Theta}{\Lambda}\sum_{s=\alpha-n}^{t-n+1}e^{-s}+\frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{t-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}}\right\} \\
\leq\ & \Big[\lambda\Big(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big)+\mu\Big(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\Big]\|u-v\|_{\mathcal{C}}\times \\
&\left(\frac{\Theta e^{n-\alpha}(T+2)}{|\Lambda|}+\frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}\right) \\
\leq\ & \Big[\lambda\Big(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big)+\mu\Big(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\Big]\|u-v\|_{\mathcal{C}}\Omega_1.
\end{aligned} \tag{31}$$

Next, we consider the following $(\Delta_C^{\vartheta}\mathcal{F}u)$ and $(\Delta^{\gamma}\mathcal{F}u)$ as

$$\begin{aligned}
&(\Delta_C^{\vartheta}\mathcal{F}u)(t-\vartheta+1) \\
&= \frac{\mathcal{O}[F(u)+H(u)]}{\Lambda\Gamma(1-\vartheta)}\sum_{s=\alpha-n}^{t}(t-\vartheta+1-\sigma(s))^{\underline{-\vartheta}}\Delta_s\sum_{v=\alpha-n}^{s-n+1}e^{-v} \\
&\quad+\frac{1}{\Gamma(\alpha-n)\Gamma(1-\vartheta)}\sum_{r=\alpha-n}^{t}(t-\vartheta+1-\sigma(r))^{\underline{-\vartheta}}\Delta_r\Bigg\{\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}e^{v-s}\times \\
&\quad(v-\sigma(\xi))^{\underline{\alpha-n-1}}\Big[\lambda F\Big(\xi+\alpha-1,u(\xi+\alpha-1),(\Upsilon^{\vartheta}u)(\xi+\alpha-\vartheta)\Big) \\
&\quad+\mu H\Big(\xi+\alpha-1,u(\xi+\alpha-1),(\Psi^{\gamma}u)(\xi+\alpha+\gamma-1)\Big)\Big]\Bigg\} \\
&= \frac{\mathcal{O}[F(u)+H(u)]}{\Lambda\Gamma(1-\vartheta)}\sum_{s=\alpha-n}^{t}(t-\vartheta+1-\sigma(s))^{\underline{-\vartheta}}e^{n-s-2} \\
&\quad+\frac{1}{\Gamma(\alpha-n)\Gamma(1-\vartheta)}\sum_{r=\alpha-n}^{t}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}(t-\vartheta+1-\sigma(r))^{\underline{-\vartheta}}e^{v-s}\times \\
&\quad(v-\sigma(\xi))^{\underline{\alpha-n-1}}\Big[\lambda F\Big(\xi+\alpha-1,u(\xi+\alpha-1),(\Upsilon^{\vartheta}u)(\xi+\alpha-\vartheta)\Big) \\
&\quad+\mu H\Big(\xi+\alpha-1,u(\xi+\alpha-1),(\Psi^{\gamma}u)(\xi+\alpha+\gamma-1)\Big)\Big],
\end{aligned} \tag{32}$$

$$
\begin{aligned}
(\Delta^{-\gamma}\mathcal{F}u)(t+\gamma) = {} & \frac{\mathcal{O}[F(u)+H(u)]}{\Lambda\Gamma(\gamma)} \sum_{s=\alpha-n}^{t}\sum_{v=\alpha-n}^{s-n+1}(t+\gamma-\sigma(s))^{\underline{\gamma-1}}e^{-v} \\
& + \frac{1}{\Gamma(\alpha-n)\Gamma(\gamma)}\sum_{r=\alpha-n}^{t}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}(t+\gamma-\sigma(r))^{\underline{\gamma-1}}e^{v-s}\times \\
& (v-\sigma(\xi))^{\underline{\alpha-n-1}}\Big[\lambda F\Big(\xi+\alpha-1,u(\xi+\alpha-1),(\Upsilon^{\vartheta}u)(\xi+\alpha-\vartheta)\Big) \\
& + \mu H\Big(\xi+\alpha-1,u(\xi+\alpha-1),(\Psi^{\gamma}u)(\xi+\alpha+\gamma-1)\Big)\Big]. \qquad (33)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \left|(\Delta_C{}^{\vartheta}\mathcal{F}u)(t-\vartheta+1)-(\Delta_C^{\vartheta}\mathcal{F}v)(t-\vartheta+1)\right| \\
\leq {} & \left[\lambda\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)+\mu\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\right]\|u-v\|_{\mathcal{C}}\times \\
& \left[\frac{\Theta e^{2n-\alpha-2}}{|\Lambda|}+\frac{e^{T-1}(T+\alpha-n+3)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}\right]\frac{(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)} \qquad (34) \\
\leq {} & \left[\lambda\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)+\mu\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\right]\|u-v\|_{\mathcal{C}}\Omega_2,
\end{aligned}
$$

and

$$
\begin{aligned}
& \left|(\Delta^{-\gamma}\mathcal{F}u)(t+\gamma)-(\Delta^{-\gamma}\mathcal{F}v)(t+\gamma)\right| \\
\leq {} & \left[\lambda\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)+\mu\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\right]\|u-v\|_{\mathcal{C}}\times \\
& \left[\frac{\Theta e^{n-\alpha}(T+2)}{|\Lambda|}+\frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}\right]\frac{(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)} \qquad (35) \\
\leq {} & \left[\lambda\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)+\mu\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\right]\|u-v\|_{\mathcal{C}}\Omega_3.
\end{aligned}
$$

Hence (31), (34) and (35) imply that

$$
\begin{aligned}
\left\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\right\|_{\mathcal{C}} \leq {} & \left[\lambda\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)\right. \\
& \left.+\mu\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\right](\Omega_1+\Omega_2+\Omega_3)\|u-v\|_{\mathcal{C}} \\
= {} & \chi\|u-v\|_{\mathcal{C}}. \qquad (36)
\end{aligned}
$$

By (H_4), we have $\left\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\right\|_{\mathcal{C}} < \|u-v\|_{\mathcal{C}}$.

Consequently, $\mathcal{F}$ is a contraction. Therefore, by the Banach fixed point theorem, we get that $\mathcal{F}$ has a fixed point which is a unique solution of the problem (5) on $t \in \mathbb{N}_{\alpha-n,T+\alpha}$. □

In the second result, we deduce the existence of at least one solution of (5) by the following, the Schauder's fixed point theorem.

Lemma 4 ([30])**.** *(Arzelá-Ascoli theorem) A set of function in* $C[a,b]$ *with the sup norm is relatively compact if and only it is uniformly bounded and equicontinuous on* $[a,b]$*.*

Lemma 5 ([30])**.** *If a set is closed and relatively compact then it is compact.*

Lemma 6 ([31]). *(Schauder fixed point theorem) Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Theorem 2. *Assuming that $(H_1) - (H_3)$ hold, problem (5) has at least one solution on $\mathbb{N}_{\alpha-n,T+\alpha}$.*

Proof. We divide the proof into three steps as follows.
Step I. Verify $\mathcal{F}$ map bounded sets into bounded sets in $B_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \le R\}$. We consider $B_R = \{u \in C(\mathbb{N}_{\alpha-n,T+\alpha}) : \|u\|_{\mathcal{C}} \le R\}$.

Let $\max\limits_{t\in\mathbb{N}_{\alpha-n,T+\alpha}} |F(t,0,0)| = M$, $\max\limits_{t\in\mathbb{N}_{\alpha-n,T+\alpha}} |H(t,0,0)| = N$ and choose a constant

$$R \ge \frac{(M+N)\big(\Omega_1+\Omega_2+\Omega_3\big)}{1-\big(\Omega_1+\Omega_2+\Omega_3\big)\Big\{\lambda\Big(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big)+\mu\Big(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\Big\}}. \tag{37}$$

Noting that

$$\begin{aligned}
|\mathcal{S}(t,u,0)| &= \Big|F\big(t+\alpha-1,u(t+\alpha-1),\Delta_C^{\vartheta}u(t+\alpha-\vartheta)\big)-F\big(t+\alpha-1,0,0\big)\big| \\
&\quad+\big|F\big(t+\alpha-1,0,0\big)\Big|, \\
|\mathcal{T}(t,u,0)| &= \Big|H\big(t+\alpha-1,u(t+\alpha-1),\Delta^{-\gamma}u(t+\alpha+\gamma-1)\big)-H\big(t+\alpha-1,0,0\big)\big| \\
&\quad+\big|H\big(t+\alpha-1,0,0\big)\Big|,
\end{aligned}$$

for each $u \in B_R$, we obtain

$$\begin{aligned}
&\Big|\mathcal{O}[F(u)+H(u)]\Big| \\
&\le \sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}\frac{\tau\left(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r)\right)^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma\left(\sum_{i=1}^{m}\theta_i\right)\Gamma(\alpha-n)}e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}}\times \\
&\quad\Big[\big|\mathcal{S}(\xi,u,0)\big|+\mu\big|\mathcal{T}(\xi,u,0)\big|\Big] \\
&\le \sum_{r=\alpha-n}^{\eta}\sum_{s=\alpha-n}^{r-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}\frac{\tau\left(\eta+\sum_{i=1}^{m}\theta_i-\sigma(r)\right)^{\underline{\sum_{i=1}^{m}\theta_i-1}}}{\Gamma\left(\sum_{i=1}^{m}\theta_i\right)\Gamma(\alpha-n)}e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}}\times \\
&\quad\Big[\lambda\Big(L_1\|u\|+L_2\|\Upsilon^{\vartheta}u\|+M\Big)+\mu\Big(\ell_1\|u\|+\ell_2\|\Psi^{\gamma}u\|+N\Big)\Big] \\
&\quad-\frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{T+\alpha-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n}e^{v-s}(v-\sigma(x))^{\underline{\alpha-n-1}}\times \\
&\quad\Big[\lambda\Big(L_1|u-v|+L_2|(\Upsilon^{\vartheta}u)-(\Upsilon^{\vartheta}v)|+M\Big)+\mu\Big(\ell_1|u-v|+\ell_2|(\Psi^{\gamma}u)-(\Psi^{\gamma}v)|+N\Big)\Big] \\
&\le \Big\{\lambda\Big[\Big(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big)\|u\|_{\mathcal{C}}+M\Big] \\
&\quad+\mu\Big[\Big(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\|u\|_{\mathcal{C}}+N\Big]\Big\}\times \\
&\quad\left|\frac{\tau K e^{\eta-\alpha-1}(\eta-\alpha+2)^{\underline{\alpha-n+2}}(\eta-\alpha+n+m)^{\underline{m}}}{\Gamma(m+1)\Gamma(\alpha-n+3)}-\frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}\right| \\
&\le \Bigg\{\lambda\Big[\Big(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\Big)\|u\|_{\mathcal{C}}+M\Big] \\
&\quad+\mu\Big[\Big(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\Big)\|u\|_{\mathcal{C}}+N\Big]\Bigg\}\Theta,
\end{aligned} \tag{38}$$

and

$$
\begin{aligned}
&\left|(\mathcal{F}u)(t)\right| \\
&\leq \frac{1}{\Lambda}\left|\mathcal{O}[F(u)+H(u)]\right|\sum_{s=\alpha-n}^{t-n+1} e^{-s} \\
&+\frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{t-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n} e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}}\Big[\lambda\left|\mathcal{S}(\xi,u,0)\right|+\mu\left|\mathcal{T}(\xi,u,0)\right|\Big] \\
&\leq \Big[\lambda\Big(L_1\|u\|+L_2\|\Upsilon^{\vartheta}u\|+M\Big)+\mu\Big(\ell_1\|u\|+\ell_2\|\Psi^{\gamma}u\|+N\Big)\Big]\times \\
&\quad\left\{\frac{\Theta}{|\Lambda|}\sum_{s=\alpha-n}^{t-n+1} e^{-s}+\frac{1}{\Gamma(\alpha-n)}\sum_{s=\alpha-n}^{t-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n} e^{v-s}(v-\sigma(\xi))^{\underline{\alpha-n-1}}\right\} \\
&\leq \left\{\lambda\left[\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)\|u\|_{\mathcal{C}}+M\right]\right. \\
&\quad\left.+\mu\left[\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_{\mathcal{C}}+N\right]\right\}\left(\frac{\Theta e^{n-\alpha}(T+2)}{|\Lambda|}+\frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)}\right) \\
&\leq \left\{\lambda\left[\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)\|u\|_{\mathcal{C}}+M\right]\right. \\
&\quad\left.+\mu\left[\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_{\mathcal{C}}+N\right]\right\}\Omega_1. \qquad (39)
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\left|(\Delta_{C}^{\vartheta}\mathcal{F}u)(t-\vartheta+1)\right| \leq &\left\{\lambda\left[\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)\|u\|_{\mathcal{C}}+M\right]\right. \\
&\left.+\mu\left[\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_{\mathcal{C}}+N\right]\right\}\Omega_2, \qquad (40)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|(\Delta^{-\gamma}\mathcal{F}u)(t+\gamma)\right| \leq &\left\{\lambda\left[\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)\|u\|_{\mathcal{C}}+M\right]\right. \\
&\left.+\mu\left[\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_{\mathcal{C}}+N\right]\right\}\Omega_3. \qquad (41)
\end{aligned}
$$

Hence (39)–(41) imply that

$$
\begin{aligned}
\left\|(\mathcal{F}u)(t)\right\|_{\mathcal{C}} \leq &\left\{\lambda\left[\left(L_1+L_2\frac{\phi_0(T+n-\vartheta+1)^{\underline{1-\vartheta}}}{\Gamma(2-\vartheta)}\right)\|u\|_{\mathcal{C}}+M\right]\right. \\
&\left.+\mu\left[\left(\ell_1+\ell_2\frac{\varphi_0(T+n+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_{\mathcal{C}}+N\right]\right\}(\Omega_1+\Omega_2+\Omega_3) \\
\leq\ & R. \qquad (42)
\end{aligned}
$$

So, $\|\mathcal{F}u\|_{\mathcal{C}} \leq R$. This implies that $\mathcal{F}$ is uniformly bounded.

Step II. Since F and H are continuous, the operator $\mathcal{F}$ is continuous on B_R.

Step III. Examine $\mathcal{F}$ is equicontinuous on B_R. For any $\epsilon > 0$, there exists a positive constant $\rho^* = \max\{\delta_1, \delta_2, \delta_3, \delta_4\}$ such that for $t_1, t_2 \in \mathbb{N}_{\alpha-n,T+\alpha}$

$$\begin{aligned}
&\Big|t_2 - t_1\Big| < \tfrac{\epsilon\Lambda}{6e^{T-1}\Theta\left[\lambda\|F\|+\mu\|H\|\right]}, \qquad \text{whenever } |t_2 - t_1| < \delta_1,\\
&\Big|(t_2 - n + 1)^{\underline{\alpha-n+2}} - (t_1 - n + 1)^{\underline{\alpha-n+2}}\Big| < \tfrac{\epsilon\Gamma(\alpha-n+3)}{6e^{n-\alpha}\Theta\left[\lambda\|F\|+\mu\|H\|\right]}, \qquad \text{whenever } |t_2 - t_1| < \delta_2,\\
&\Big|(t_2 - \alpha + n - \vartheta + 1)^{\underline{1-\vartheta}} - (t_1 - \alpha + n - \vartheta + 1)^{\underline{1-\vartheta}}\Big|\\
&\quad < \frac{\epsilon}{3\left[\lambda\|F\| + \mu\|H\|\right]\left(\frac{\Theta e^{2n-\alpha-2}}{\Lambda\Gamma(2-\vartheta)} + \frac{e^{T-1}(T+\alpha-n+3)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)\Gamma(2-\vartheta)}\right)}\\
&\quad < \frac{\epsilon\,\Gamma(1-\nu)\Gamma(\beta)\Gamma(\alpha)}{4\widetilde{\Theta}_R}, \qquad \text{whenever } |t_2 - t_1| < \delta_3,\\
&\Big|(t_2 - \alpha + n + \gamma)^{\underline{\gamma}} - (t_1 - \alpha + n + \gamma)^{\underline{\gamma}}\Big| < \frac{\epsilon}{3\left[\lambda\|F\| + \mu\|H\|\right]\left(\frac{\Theta e^{n-\alpha}(T+2)}{\Lambda\Gamma(\gamma+1)} + \frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)\Gamma(\gamma+1)}\right)},\\
&\qquad \text{whenever } |t_2 - t_1| < \delta_4.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\big|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\big|\\
&\leq \frac{1}{|\Lambda|}\mathcal{O}[F(u) + H(u)]\left|\sum_{s=\alpha-n}^{t_2-n+1} e^{-s} - \sum_{s=\alpha-n}^{t_1-n+1} e^{-s}\right| + \frac{1}{\Gamma(\alpha-n)}\left|\sum_{s=\alpha-n}^{t_2-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n} e^{v-s} \times\right.\\
&\quad (v - \sigma(\xi))^{\underline{\alpha-n-1}}\Bigg[\lambda F\Big(\xi + \alpha - 1, u(\xi + \alpha - 1), (\Upsilon^{\vartheta}u)(\xi + \alpha - \vartheta)\Big)\\
&\quad + \mu H\Big(\xi + \alpha - 1, u(\xi + \alpha - 1), (\Psi^{\gamma}u)(\xi + \alpha + \gamma - 1)\Big)\Bigg] - \sum_{s=\alpha-n}^{t_1-n+1}\sum_{v=\alpha-n}^{s-1}\sum_{\xi=0}^{v-\alpha+n} e^{v-s} \times\\
&\quad (v - \sigma(\xi))^{\underline{\alpha-n-1}}\Bigg[\lambda F\Big(\xi + \alpha - 1, u(\xi + \alpha - 1), (\Upsilon^{\vartheta}u)(\xi + \alpha - \vartheta)\Big)\\
&\quad \left. + \mu H\Big(\xi + \alpha - 1, u(\xi + \alpha - 1), (\Psi^{\gamma}u)(\xi + \alpha + \gamma - 1)\Big)\Bigg]\right|\\
&\leq \left[\lambda\|F\| + \mu\|H\|\right]\left\{\frac{\Theta e^{n-\alpha}}{|\Lambda|}\Big|t_2 - t_1\Big| + \frac{e^{T-1}}{\Gamma(\alpha-n+3)}\Big|(t_2 - n + 1)^{\underline{\alpha-n+2}} - (t_1 - n + 1)^{\underline{\alpha-n+2}}\Big|\right\}\\
&< \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.
\end{aligned} \tag{43}$$

Furthermore, we have

$$\begin{aligned}
&\big|(\Delta_C^{\vartheta}\mathcal{F}u)(t_2 - \vartheta + 1) - (\Delta_C^{\vartheta}\mathcal{F}u)(t_1 - \vartheta + 1)\big|\\
&\leq \left[\lambda\|F\| + \mu\|H\|\right]\left\{\left[\frac{\Theta e^{2n-\alpha-2}}{|\Lambda|\Gamma(2-\vartheta)} + \frac{e^{T-1}(T + \alpha - n + 3)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)\Gamma(2-\vartheta)}\right]\times\right.\\
&\quad \left.\Big|(t_2 - \alpha + n - \vartheta + 1)^{\underline{1-\vartheta}} - (t_1 - \alpha + n - \vartheta + 1)^{\underline{1-\vartheta}}\Big|\right\} \leq \frac{\epsilon}{3},
\end{aligned} \tag{44}$$

and

$$\begin{aligned}&\left|(\Delta^{-\gamma}\mathcal{F}u)(t_2+\gamma)-(\Delta^{-\gamma}\mathcal{F}u)(t_1+\gamma)\right|\\ \leq\ &[\lambda\|F\|+\mu\|H\|]\left\{\left[\frac{\Theta e^{n-\alpha}(T+2)}{|\Lambda|\Gamma(\gamma+1)}+\frac{e^{T-1}(T+\alpha-n+2)^{\underline{\alpha-n+2}}}{\Gamma(\alpha-n+3)\Gamma(\gamma+1)}\right]\times\right.\\ &\left.\left|(t_2-\alpha+n+\gamma)^{\underline{\gamma}}-(t_1-\alpha+n+\gamma)^{\underline{\gamma}}\right|\right\}\ \leq\ \frac{\epsilon}{3}. \end{aligned} \tag{45}$$

Hence

$$\left\|(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)\right\|_{\mathcal{C}}\ <\ \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\ =\ \epsilon. \tag{46}$$

This implies that the set $\mathcal{F}(B_R)$ is an equicontinuous set. As a consequence of Steps I to III together with the Arzelá-Ascoli theorem, we find that $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ is completely continuous. By Schauder fixed point theorem, we can conclude that problem (5) has at least one solution. The proof is completed. □

4. An Example

In order to study the existence of a solution to our problem, we obtain the conditions provided in Section 3. Since our designated problem is a theoretical problem, it is rare to find the application related to our results. However, for thorough explanation, we provide the following example to illustrate our results. Consider the following fractional difference boundary value problem

$$\begin{aligned}\left[\Delta_C^{\frac{9}{2}}+(e+1)\Delta_C^{\frac{7}{2}}\right]u(t)=&\frac{e^{-\cos^2(2\pi(t+\frac{7}{2})+10)}}{100+e^{\sin^2(2\pi(t+\frac{7}{2}))}}\cdot\frac{|u\left(t+\frac{7}{2}\right)|+\left|\Upsilon^{\frac{1}{3}}u\left(t+\frac{26}{6}\right)\right|}{[1+|u\left(t+\frac{7}{2}\right)|]}\\ &+\frac{\left(t+\frac{227}{2}\right)^{-2}|u\left(t+\frac{7}{2}\right)|+\left|\Psi^{\frac{4}{5}}u\left(t+\left(t+\frac{43}{10}\right)\right)\right|}{\left(t+\frac{27}{2}\right)^3[1+|u\left(t+\frac{7}{2}\right)|]},\\ u\left(-\frac{1}{2}\right)=&D_C^{\frac{1}{4}}u\left(\frac{5}{4}\right)=D_C^{\frac{1}{2}}u\left(\frac{11}{4}\right)=D_C^{\frac{3}{4}}u\left(4\right)=0\\ u\left(\frac{27}{2}\right)=&\frac{1}{e^5}\Delta^{-\frac{77}{60}}e^{-sin\left(\frac{587\pi}{60}\right)}u\left(\frac{587}{60}\right),\end{aligned} \tag{47}$$

where $$(\Psi^{\frac{1}{3}}u)\left(t+\frac{26}{6}\right)=\sum_{s=-\frac{7}{6}}^{t-\frac{2}{3}}\frac{(t-\sigma(s))^{\underline{-\frac{1}{3}}}}{\Gamma\left(\frac{2}{3}\right)}\frac{e^{-(s+\frac{2}{3})}}{(t+50)^3}\Delta u\left(s+\frac{2}{3}\right),$$

$$(\Psi^{\frac{4}{5}}u)\left(t+\frac{4}{5}\right)=\sum_{s=-\frac{13}{10}}^{t-\frac{4}{5}}\frac{(t-\sigma(s))^{\underline{-\frac{1}{5}}}}{\Gamma\left(\frac{4}{5}\right)}\frac{e^{-(s+\frac{4}{5})}}{(t+100)^2}u\left(s+\frac{4}{5}\right).$$

Set $\alpha=\frac{9}{2}$, $n=5$, $\vartheta=\frac{1}{3}$, $\gamma=\frac{4}{5}$, $\beta_1=\frac{1}{4}$, $\beta_2=\frac{1}{2}$, $\beta_3=\frac{3}{4}$, $\lambda=e^{-10}$, $\mu=1$, $T=6$, $\eta=\frac{17}{2}$, $T=6$, $\tau=e^{-5}$, $m=4$, $\theta_1=\frac{1}{2}$, $\theta_2=\frac{1}{3}$, $\theta_3=\frac{1}{4}$, $\theta_4=\frac{1}{5}$, $g(t)=e^{\sin(\pi t)}$, $\phi(t,s-\vartheta+1)=\frac{e^{-(s+\frac{2}{3})}}{(t+50)^3}$ and $\varphi(t,s+\gamma)=\frac{e^{-s+\frac{4}{5}}}{(t+100)^2}$.

We can show that

$$\begin{gathered}\Theta=545.5721,\ \ |\Lambda|=202.553,\ \ \Omega_1=2189.264,\ \ \Omega_2=15715.32\\ \Omega_3=17049.09\ \ \text{and}\ \ \phi_0=\left(\tfrac{2}{99}\right)^3e^{-\frac{1}{6}},\ \varphi_0\leq\left(\tfrac{2}{199}\right)^2e^{-\frac{3}{10}}.\end{gathered}$$

Nothing that $(H_1)-(H_3)$ hold, for each $t\in\frac{1}{2}\mathbb{N}_{-\frac{1}{2},\frac{27}{2}}$, we obtain

$$\left|F[t,u,\Upsilon^{\frac{1}{3}}u]-F[t,v,\Upsilon^{\frac{1}{3}}v]\right|\leq\tfrac{1}{101}|u-v|+\tfrac{1}{101}|\Upsilon^{\frac{1}{3}}u-\Upsilon^{\frac{1}{3}}v|,$$

$$\left|F[t,u,\Psi^{\frac{4}{5}}u] - F[t,v,\Psi^{\frac{4}{5}}v]\right| \leq \left(\tfrac{2}{199}\right)^2 \left(\tfrac{2}{19}\right)^3 |u-v| + \left(\tfrac{2}{19}\right)^3 |\Psi^{\frac{4}{5}}u - \Psi^{\frac{4}{5}}v|,$$
$$\text{and} \quad \tfrac{1}{e} < g(t) < e,$$

so, $L_1 = L_2 = 0.0099$, $\ell_1 = 1.178 \times 10^{-7}, \ell_2 = 0.0012$, $K = e$.

Finally, we find that

$$\chi \;=\; 0.0443 \;<\; 1.$$

Hence, by Theorem 1, the problem (47) has a unique solution on $\frac{1}{2}\mathbb{N}_{-\frac{1}{2},\frac{27}{2}}$.

5. Conclusions

We study the existence and unique results of the solution for a separate nonlinear Caputo fractional sum-difference equation with fractional sum-difference boundary conditions. Some conditions are obtained when Banach contraction principle is used as a tool. In addition, the conditions for the case of at least one solution are obtained by using the Schauder fixed point theorem.

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