

Construction of Fair Dice Pairs

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Abstract: An interesting and challenging problem in mathematics is how to construct fair dice pairs. In this paper, by means of decomposing polynomials in a residue class ring and applying the Discrete Fourier Transformation, we present all the 2000 fair dice pairs and their 8 equivalence classes in a four-person game, identifying what we call the mandarin duck property of fair dice pairs.

Keywords: fair dice pairs; discrete Fourier transformation; discrete convolution

1. Introduction

In this paper, we refer to dice exclusively as small throwable regular cubes made of uniform materials. The six faces of a **standard die** are marked using numbers (or dots, also called pips) from 1 to 6. As for its uniformity and symmetry, the number shown on the upper face is equally probable when we throw (or roll) a standard die. In practice, people often roll standard dice and use the resulting number to ensure fairness in activities such as education [1], games [2], gambling and other entertainment.

More specifically, we denote the number appearing when a standard die is thrown as X , thus X is a random variable with **uniform distribution** on $\{1, 2, 3, 4, 5, 6\}$, that is, $P(X = i) = \frac{1}{6}$ for $1 \leq i \leq 6$. Given this situation, a standard die serves as a **True Random Number Generator**. For example, if the number of participants is exactly six, using a standard die to determine the fairness of opportunity would be the most appropriate tool. In many other cases, two (or more) dice are thrown simultaneously and the total value of the dice is used to choose the first player (or the starting position) of a game. In this case, can one guarantee that the result is fair by using the sum of a number of standard dice? As we will see below, this is impossible in general.

In Reference [3], P. R. Halmos et al. proved in 1951 that, for any weighted (a.k.a. loaded) pair of dice, it is impossible to ensure that the sum of the two numbers forms a uniform distribution on $\{2, 3, \dots, 12\}$. Thus, it is clearly impossible to use the sum of a pair of dice as a fair match for 11 outcomes no matter how we design the dice. Even if we modify the number on some faces, it is still impossible. In 1978, M. Gardner [4] identified a pair of dice called Sicherman dice with numbers $(1, 2, 2, 3, 3, 4)$ and $(1, 3, 4, 5, 6, 8)$ that generate same distribution as a pair of standard dice. Since then, the question of modifying (or renaming) the numbers of dice pairs to attain the same distribution has been studied extensively in References [5–12], with additional references found in [13–17]. In Reference [18], M. Conroy

in 2018 answered many questions and provided numerous references regarding dice pairs. The above research shows that, in general, the result of simply summing a pair of dice does not produce fair (i.e., uniformly distributed) results.

Given the above, people naturally focus on whether or not there is a way to derive fairness in certain situations using a pair of dice by “sum” plus some other operations, like modular 2 (in the Japanese Cho-Han Bakuchi Playing [19]) or modular the number of players (four in typical in the popular Chinese Mahjong game). In this paper, we will discuss fairness using the two operations “sum” and “mod 4” on a pair of dice in four-person activities as there are many games involving four players in the real world where the sum of two dice is used to determine fairness. When playing mahjong, the most common rule (c.f. [20]) is that in the first hand of each round, player 1 (dealer) sitting on the East Position throws two dice to determine which player will do the second toss of the dice. If the total pips of the first toss is 5 or 9, the dealer himself throws the dice again, otherwise, the Player sitting in the South Position (neighbour of the East in the counter clock wise) will do the second toss when the sum is 2, 6, or 10, while 3, 7, or 11 for the West (the opposite of the East), and 4, 8, 12 for the North Position (neighbored to the West in the counter clock wise). Using the total number of the second toss, the starting wall is chosen clockwise. Then the dealer takes four tiles at first for himself from the initial location, and players in anti-clockwise order draw blocks of four tiles successively until all players have 12 tiles, to complete the beginning phase, the dealer takes two tiles to make a 14-tile hand, and each of other three players then draws one last tile to make a 13-tile hand. It is clear that a shifted modular four operation, namely, $((p_1 + p_2) \bmod 4) + 1$, is executed after counting the sum in both throws, here p_1, p_2 are the pips of the two dice. Indeed, it is easily seen that the method is not fair with simple analysis. For example, after the dealer tossed for the first time, the probabilities associated with the four players (at the East, the South, the West, and the North in counter-clock wise order) to do the second toss is $\frac{1}{4}, \frac{1}{4} - \frac{1}{6^2}, \frac{1}{4}$, and $\frac{1}{4} + \frac{1}{6^2}$, respectively. This shows that in the sum and modular four approach, the player sitting in the North has more chance to do the second chance and the wall in front of the West gets the highest probability for the initial draw (see Reference [2] for a detail analysis). Further, in Reference [2], Y. Huang et al. proved that, more generally, it is impossible to achieve fairness for four people using the remainder after applying “mod 4” to the sum of rolling any n standard dice, as the probability for each player is $\frac{1}{4} + \frac{\cos \frac{n\pi}{4}}{2(-3\sqrt{2})^n}, \frac{1}{4} - \frac{\sin \frac{n\pi}{4}}{2(-3\sqrt{2})^n}, \frac{1}{4} - \frac{\cos \frac{n\pi}{4}}{2(-3\sqrt{2})^n}$, and $\frac{1}{4} + \frac{\sin \frac{n\pi}{4}}{2(-3\sqrt{2})^n}$, respectively.

As noted above, we use two standard dice in Mahjong. If we appropriately modify some numbers of the dice and allow the two dice to be different, can we construct a fair dice pair using just “sum” and “mod” operations? If so, how can we modify the numbers on the dice? Moreover, how many modifications are there? In this paper, we aim to answer these questions and, in doing so, identify an interesting property for these dice pairs, which we call the mandarin duck property.

Note that, when discussing the distribution of sum of dice pairs, most references use the cyclotomic polynomial method. However, in this paper, we primarily apply Discrete Convolution and Discrete Fourier Transformation.

2. Modeling

Definition 1. If the six faces of a die are chosen from the set $\{1, 2, 3, 4, 5, 6\}$ with replacement, the die is called a *quasi-standard*. A standard die is a quasi-standard, but the converse is not true.

For a quasi-standard die, we use x_k to indicate the times that k occurs on its six faces, and we call $[x_1, x_2, x_3, x_4, x_5, x_6]$ the *quasi-distribution of the die*.

The non-descending ordered array $(n_1, n_2, n_3, n_4, n_5, n_6)$ consisting of all numbers of a quasi-standard die is called the *numbers of the die*.

The one-to-one correspondence between the quasi-distribution and the numbers of a quasi-standard die is represented as $[x_1, x_2, x_3, x_4, x_5, x_6] \leftrightarrow (n_1, n_2, n_3, n_4, n_5, n_6)$.

Notice that $x_k = 0$ indicates that k does not occur at all on any of the six faces and $x_k > 1$ representing k occurs more than one time. As an example, $[3, 0, 3, 0, 0, 0] \leftrightarrow (1, 1, 1, 3, 3, 3)$ is a correspondence relationship between the quasi-distribution of a die and its numbers. Note also that, traditionally, opposite sides of the standard die add up to seven, which implies that the 1, 2 and 3 faces share a vertex and normally the 1, 2 and 3 faces in Chinese dice are placed counterclockwise about their common vertex (dice in this design are called left-handed), while in western dice they are arranged clockwise or counterclockwise (called right-handed). But in our definition, the collocation of $n_1, n_2, \dots, n_6$ is not considered, so any quasi-standard dice that marked by 1, 2, 3, 4, 5, 6 are considered to be the same as the standard die.

Proposition 1. The array $[x_1, x_2, x_3, x_4, x_5, x_6]$ is a quasi-distribution of a quasi-standard dice if and only if x_k , with $1 \leq k \leq 6$, is non-negative integers and $\sum_{k=1}^6 x_k = 6$.

Therefore, the number of different quasi-standard dice is equal to the number of the non-negative integer solutions of equation $x_1 + x_2 + \dots + x_6 = 6$. It is easy to show that this number also equals the number of terms (equal to 462) of the polynomial $(y_1 + y_2 + \dots + y_6)^6$ after expanding. Given the above, we can also consider the quasi-distribution that satisfies Proposition 1 as a corresponding quasi-standard die.

Definition 2. For a quasi-standard die, the number X that occurs above is a random variable that appears when one throw the die. We call this a **quasi-uniform distribution** associated with the die. If the quasi-distribution of the die is $[x_1, x_2, x_3, x_4, x_5, x_6]$, then the distribution of X is

$$P(X = k) = \frac{x_k}{6}, \quad 1 \leq k \leq 6. \quad (1)$$

When $x_k = 1$, with $1 \leq k \leq 6$, the quasi-standard die is standard and the associated random variable X obeys **uniform distribution**, that is, $P(X = k) = \frac{1}{6}$, with $1 \leq k \leq 6$.

In the following, we also identify a quasi-standard die as its associated quasi-uniform distributed random variable X .

For a quasi-standard dice pair (R, B) , the two random variables R and B are always deemed to be mutually independent. We call R the red dice, and B the blue dice. Denote the quasi-distribution of R and B by $[r_1, r_2, \dots, r_6]$ and $[b_1, b_2, \dots, b_6]$, respectively, then their generating functions are

$$\Phi_R(x) = \frac{1}{6} \sum_{k=1}^6 r_k x^k, \text{ and } \Phi_B(x) = \frac{1}{6} \sum_{k=1}^6 b_k x^k.$$

Note that the properties of the generating function for discrete random variables can be found in Reference [21].

Definition 3. Let (R, B) be a pair of quasi-standard dice and the random variable F be the reminder of $R + B$ on division by 4, then

$$F \equiv (R + B) \pmod{4}. \quad (2)$$

If the distribution of F is $P(F = k) = \frac{1}{4}$, $1 \leq k \leq 4$, then we call (R, B) a **fair dice pair**. The generating function of F is $\Phi_F(x) = \frac{1}{4} \sum_{k=1}^4 x^k$.

Note that in the ring $\mathbb{Z}[4]$, Equation (2) becomes $F = R + B$.

Proposition 2. Assume k is a positive integer, X and Y are two random variables with non-negative integer value and Y takes value from $\{0, 1, \dots, k-1\}$, $\Phi_X(x)$ and $\Phi_Y(x)$ is the generating function of X and Y , respectively. Then

$$X \equiv Y \pmod{k} \iff \Phi_X(x) \equiv \Phi_Y(x) \pmod{(x^k - 1)}.$$

Proof. Let $P(X = i) = p_i$, $0 \leq i < s$ and $P(Y = j) = q_j$, $0 \leq j < k$, be the distribution of X and Y , respectively. Note that $i \equiv j \pmod{k} \iff x^i \equiv x^j \pmod{x^k - 1}$, so that $X \equiv Y \pmod{k}$ if and only if $\Phi_Y(x) = \sum_{0 \leq j < k} q_j x^j \equiv \sum_{0 \leq j < k} \sum_{\substack{0 \leq i < s \\ i \equiv j \pmod{k}}} p_i x^i = \sum_{0 \leq i < s} p_i x^i = \Phi_X(x) \pmod{(x^k - 1)}$. $\square$

Proposition 3. Equation (2) is equivalent to the following congruence relations in polynomial ring $\mathbb{Q}[x]$ over rational field $\mathbb{Q}$

$$\Phi_F(x) \equiv \Phi_R(x)\Phi_B(x) \pmod{(x^4 - 1)}. \quad (3)$$

Proof. By Proposition 2 and the independence of (R, B) , we have $\Phi_F(x) \equiv \Phi_{R+B \pmod{4}}(x) \equiv \Phi_{R+B}(x) = \Phi_R(x)\Phi_B(x) \pmod{(x^4 - 1)}$. $\square$

The Equation (3) can be represented as a discrete convolution “*” form in residue class ring $\mathbb{Q}[x]/\langle x^4 - 1 \rangle$ as

$$\Phi_F = \Phi_R * \Phi_B. \quad (4)$$

To find the fair dice pair (R, B) , we must solve Equation (3) or (4).

Let $\omega = \sqrt{-1} = \mathbf{i} \in \mathbb{C}$ be the primitive 4-th root of unity, DFT_4 be the Discrete Fourier Transformation relate to ω . By applying DFT_4 to both side of (4), we have (cf. [22], P230 and 363)

$$\text{DFT}_4(\Phi_F) = \text{DFT}_4(\Phi_R) \cdot \text{DFT}_4(\Phi_B), \quad (5)$$

where “ $\cdot$ ” denotes the pointwise multiplication of vectors. Because

$$\text{DFT}_4(\Phi_F) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathbf{i} & -1 & -\mathbf{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathbf{i} & -1 & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (6)$$

$$\text{DFT}_4(\Phi_R) = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathbf{i} & -1 & -\mathbf{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathbf{i} & -1 & \mathbf{i} \end{bmatrix} \begin{bmatrix} r_4 \\ r_1 + r_5 \\ r_2 + r_6 \\ r_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} r_6 + r_5 + r_4 + r_3 + r_2 + r_1 \\ (r_1 - r_3 + r_5)\mathbf{i} - r_2 + r_4 - r_6 \\ -r_1 + r_2 - r_3 + r_4 - r_5 + r_6 \\ (-r_1 + r_3 - r_5)\mathbf{i} - r_2 + r_4 - r_6 \end{bmatrix}, \quad (7)$$

$$\text{DFT}_4(\Phi_B) = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathbf{i} & -1 & -\mathbf{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathbf{i} & -1 & \mathbf{i} \end{bmatrix} \begin{bmatrix} b_4 \\ b_1 + b_5 \\ b_2 + b_6 \\ b_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} b_6 + b_5 + b_4 + b_3 + b_2 + b_1 \\ (b_1 - b_3 + b_5)\mathbf{i} - b_2 + b_4 - b_6 \\ -b_1 + b_2 - b_3 + b_4 - b_5 + b_6 \\ (-b_1 + b_3 - b_5)\mathbf{i} - b_2 + b_4 - b_6 \end{bmatrix}, \quad (8)$$

we have

$$\begin{bmatrix} (r_6 + r_5 + r_4 + r_3 + r_2 + r_1)(b_6 + b_5 + b_4 + b_3 + b_2 + b_1) \\ ((r_1 - r_3 + r_5)\mathbf{i} - r_2 + r_4 - r_6)((b_1 - b_3 + b_5)\mathbf{i} - b_2 + b_4 - b_6) \\ (-r_1 + r_2 - r_3 + r_4 - r_5 + r_6)(b_2 + b_4 + b_6 - b_1 - b_3 - b_5) \\ ((-r_1 + r_3 - r_5)\mathbf{i} - r_2 + r_4 - r_6)((-b_1 + b_3 - b_5)\mathbf{i} - b_2 + b_4 - b_6) \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (9)$$

Note that the second and fourth equation of (9) are conjugated, therefore, by combining Proposition 1, we can simplify it to an equivalent form as

$$\begin{bmatrix} r_6 + r_5 + r_4 + r_3 + r_2 + r_1 \\ b_6 + b_5 + b_4 + b_3 + b_2 + b_1 \\ ((r_1 - r_3 + r_5)^2 + (r_2 - r_4 + r_6)^2)((b_1 - b_3 + b_5)^2 + (b_2 - b_4 + b_6)^2) \\ (-r_1 + r_2 - r_3 + r_4 - r_5 + r_6)(b_2 + b_4 + b_6 - b_1 - b_3 - b_5) \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 0 \\ 0 \end{bmatrix}. \quad (10)$$

Here, if $(r_1 - r_3 + r_5)^2 + (r_2 - r_4 + r_6)^2$ and $-r_1 + r_2 - r_3 + r_4 - r_5 + r_6$ are both equal to 0, according to $r_6 + r_5 + r_4 + r_3 + r_2 + r_1 = 6$, we obtain $r_3 = r_4 = \frac{3}{2}$, which leads to a contradiction with r_3, r_4 are non-negative integers. Repeating this approach, it is also impossible for $(b_1 - b_3 + b_5)^2 + (b_2 - b_4 + b_6)^2$ and $b_2 + b_4 + b_6 - b_1 - b_3 - b_5$ to be both equal to 0 at the same time. Hence, (10) is equivalent to

$$\begin{bmatrix} r_6 + r_5 + r_4 + r_3 + r_2 + r_1 \\ r_1 - r_3 + r_5 \\ r_2 - r_4 + r_6 \\ b_1 + b_3 + b_5 \\ b_2 + b_4 + b_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \quad (11)$$

or

$$\begin{bmatrix} b_6 + b_5 + b_4 + b_3 + b_2 + b_1 \\ b_1 - b_3 + b_5 \\ b_2 - b_4 + b_6 \\ r_1 + r_3 + r_5 \\ r_2 + r_4 + r_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}. \quad (12)$$

By symmetry (i.e., this is equivalent to exchanging two dice in a given pair), we just need to solve (11). But (11) can easily be simplified to

$$\begin{bmatrix} r_3 + r_4 \\ r_1 - r_3 + r_5 \\ r_2 - r_4 + r_6 \\ b_1 + b_3 + b_5 \\ b_2 + b_4 + b_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}. \quad (13)$$

From (13), we observe that the constraints on the red and blue dice are independent, therefore, we can split (13) into two equations

$$\begin{bmatrix} r_3 + r_4 \\ r_1 - r_3 + r_5 \\ r_2 - r_4 + r_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad (14)$$

and

$$\begin{bmatrix} b_1 + b_3 + b_5 \\ b_2 + b_4 + b_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}. \quad (15)$$

From the above Equation (14) for the red die and Equation (15) for the blue die, we come to the main result of our paper in Theorem 1 below.

Theorem 1. A pair of quasi-standard dice is a fair dice pair if and only if the quasi-distribution of one die is a solution of (14) and the quasi-distribution of the another die is a solution of (15).

3. Solutions of the Model

3.1. Properties of Solutions

Before solving Equations (14) and (15), we discuss some properties of their solutions.

Proposition 4.

- (1) When $1 \leq i \leq 6$, we have $0 \leq r_i \leq 3$, and $0 \leq b_i \leq 3$.
- (2) At least one of r_1, r_2, r_5, r_6 is 0; and r_3, r_4 can not be 0 at the same time.
- (3) $r_3 = 0 \iff r_1 = r_5 = 0$, and $r_4 = 0 \iff r_2 = r_6 = 0$.

Proof. The properties (1) to (3) follow directly from (14) and (15). $\square$

Corollary 1. If (R, B) is a fair dice pair then the red die R must lack at least one of the number from the set $\{1, 2, 5, 6\}$.

Theorem 2 (Mandarin Duck Property). Suppose that the solutions of (14) and (15) exist, then: (1) (*compatibility*) any two differently-colored dice can construct a fair dice pair; (2) (*exclusivity*) any two same-colored dice cannot construct a fair dice pair.

Proof. (1) By Theorem 1. (2) If the quasi-distribution of a quasi-standard dice $[x_1, x_2, x_3, x_4, x_5, x_6]$ simultaneously satisfy (14) and (15), we have $x_3 = x_4 = \frac{3}{2}$, it contradicts the fact that x_3 and x_4 must both be non-negative integer. $\square$

Remark 1. This shows that, to achieve fairness, the dice pair must be constructed using two different colors. In other words, neither two red nor two blue dice can construct a fair dice pair. Theorem 2 shows that only one red die and one blue die can together form a fair dice pair, therefore, we call it a **mandarin duck fair dice pair**. Mandarin duck is a kind of bird, they always appear in pairs with one male and another female together.

3.2. All Solutions of (14) and (15)

According to the above analysis, it is easy to enumerate all solutions of (14) for the red die and (15) for the blue die. It can be summarized as shown in Tables 1 and 2, respectively.

Table 1. The 20 solutions of Equations (14) for the red dice (with numbers).

$R[1] = [0,0,0,3,0,3] \leftrightarrow (4,4,4,6,6,6)$	$R[8] = [0,2,0,3,0,1] \leftrightarrow (2,2,4,4,4,6)$	$R[15] = [1,1,2,1,1,0] \leftrightarrow (1,2,3,3,4,5)$
$R[2] = [0,0,1,2,1,2] \leftrightarrow (3,4,4,5,6,6)$	$R[9] = [0,2,1,2,1,0] \leftrightarrow (2,2,3,4,4,5)$	$R[16] = [1,2,1,2,0,0] \leftrightarrow (1,2,2,3,4,4)$
$R[3] = [0,0,2,1,2,1] \leftrightarrow (3,3,4,5,5,6)$	$R[10] = [0,3,0,3,0,0] \leftrightarrow (2,2,2,4,4,4)$	$R[17] = [2,0,2,1,0,1] \leftrightarrow (1,1,3,3,4,6)$
$R[4] = [0,0,3,0,3,0] \leftrightarrow (3,3,3,5,5,5)$	$R[11] = [1,0,1,2,0,2] \leftrightarrow (1,3,4,4,6,6)$	$R[18] = [2,0,3,0,1,0] \leftrightarrow (1,1,3,3,3,5)$
$R[5] = [0,1,0,3,0,2] \leftrightarrow (2,4,4,4,6,6)$	$R[12] = [1,0,2,1,1,1] \leftrightarrow (1,3,3,4,5,6)$	$R[19] = [2,1,2,1,0,0] \leftrightarrow (1,1,2,3,3,4)$
$R[6] = [0,1,1,2,1,1] \leftrightarrow (2,3,4,4,5,6)$	$R[13] = [1,0,3,0,2,0] \leftrightarrow (1,3,3,3,5,5)$	$R[20] = [3,0,3,0,0,0] \leftrightarrow (1,1,1,3,3,3)$
$R[7] = [0,1,2,1,2,0] \leftrightarrow (2,3,3,4,5,5)$	$R[14] = [1,1,1,2,0,1] \leftrightarrow (1,2,3,4,4,6)$	

Table 2. The 100 solutions of Equations (15) for the blue dice (with numbers).

$B[1] = [0,0,0,0,3,3] \leftrightarrow (5,5,5,6,6,6)$	$B[34] = [0,2,2,1,1,0] \leftrightarrow (2,2,3,3,4,5)$	$B[67] = [1,2,2,1,0,0] \leftrightarrow (1,2,2,3,3,4)$
$B[2] = [0,0,0,1,3,2] \leftrightarrow (4,5,5,5,6,6)$	$B[35] = [0,2,3,0,0,1] \leftrightarrow (2,2,3,3,3,6)$	$B[68] = [1,3,0,0,2,0] \leftrightarrow (1,2,2,2,5,5)$
$B[3] = [0,0,0,2,3,1] \leftrightarrow (4,4,5,5,5,6)$	$B[36] = [0,2,3,1,0,0] \leftrightarrow (2,2,3,3,3,4)$	$B[69] = [1,3,1,0,1,0] \leftrightarrow (1,2,2,2,3,5)$
$B[4] = [0,0,0,3,3,0] \leftrightarrow (4,4,4,5,5,5)$	$B[37] = [0,3,0,0,3,0] \leftrightarrow (2,2,2,5,5,5)$	$B[70] = [1,3,2,0,0,0] \leftrightarrow (1,2,2,2,3,3)$
$B[5] = [0,0,1,0,2,3] \leftrightarrow (3,5,5,6,6,6)$	$B[38] = [0,3,1,0,2,0] \leftrightarrow (2,2,2,3,5,5)$	$B[71] = [2,0,0,0,1,3] \leftrightarrow (1,1,5,6,6,6)$
$B[6] = [0,0,1,1,2,2] \leftrightarrow (3,4,5,5,6,6)$	$B[39] = [0,3,2,0,1,0] \leftrightarrow (2,2,2,3,3,5)$	$B[72] = [2,0,0,1,1,2] \leftrightarrow (1,1,4,5,6,6)$
$B[7] = [0,0,1,2,2,1] \leftrightarrow (3,4,4,5,5,6)$	$B[40] = [0,3,3,0,0,0] \leftrightarrow (2,2,2,3,3,3)$	$B[73] = [2,0,0,2,1,1] \leftrightarrow (1,1,4,4,5,6)$
$B[8] = [0,0,1,3,2,0] \leftrightarrow (3,4,4,4,5,5)$	$B[41] = [1,0,0,0,2,3] \leftrightarrow (1,5,5,6,6,6)$	$B[74] = [2,0,0,3,1,0] \leftrightarrow (1,1,4,4,4,5)$
$B[9] = [0,0,2,0,1,3] \leftrightarrow (3,3,5,6,6,6)$	$B[42] = [1,0,0,1,2,2] \leftrightarrow (1,4,5,5,6,6)$	$B[75] = [2,0,1,0,0,3] \leftrightarrow (1,1,3,6,6,6)$
$B[10] = [0,0,2,1,1,2] \leftrightarrow (3,3,4,5,6,6)$	$B[43] = [1,0,0,2,2,1] \leftrightarrow (1,4,4,5,5,6)$	$B[76] = [2,0,1,1,0,2] \leftrightarrow (1,1,3,4,6,6)$
$B[11] = [0,0,2,2,1,1] \leftrightarrow (3,3,4,4,5,6)$	$B[44] = [1,0,0,3,2,0] \leftrightarrow (1,4,4,4,5,5)$	$B[77] = [2,0,1,2,0,1] \leftrightarrow (1,1,3,4,4,6)$
$B[12] = [0,0,2,3,1,0] \leftrightarrow (3,3,4,4,4,5)$	$B[45] = [1,0,1,0,1,3] \leftrightarrow (1,3,5,6,6,6)$	$B[78] = [2,0,1,3,0,0] \leftrightarrow (1,1,3,4,4,4)$
$B[13] = [0,0,3,0,0,3] \leftrightarrow (3,3,3,6,6,6)$	$B[46] = [1,0,1,1,1,2] \leftrightarrow (1,3,4,5,6,6)$	$B[79] = [2,1,0,0,1,2] \leftrightarrow (1,1,2,5,6,6)$
$B[14] = [0,0,3,1,0,2] \leftrightarrow (3,3,3,4,6,6)$	$B[47] = [1,0,1,2,1,1] \leftrightarrow (1,3,4,4,5,6)$	$B[80] = [2,1,0,1,1,1] \leftrightarrow (1,1,2,4,5,6)$
$B[15] = [0,0,3,2,0,1] \leftrightarrow (3,3,3,4,4,6)$	$B[48] = [1,0,1,3,1,0] \leftrightarrow (1,3,4,4,4,5)$	$B[81] = [2,1,0,2,1,0] \leftrightarrow (1,1,2,4,4,5)$
$B[16] = [0,0,3,3,0,0] \leftrightarrow (3,3,3,4,4,4)$	$B[49] = [1,0,2,0,0,3] \leftrightarrow (1,3,3,6,6,6)$	$B[82] = [2,1,1,0,0,2] \leftrightarrow (1,1,2,3,6,6)$
$B[17] = [0,1,0,0,3,2] \leftrightarrow (2,5,5,5,6,6)$	$B[50] = [1,0,2,1,0,2] \leftrightarrow (1,3,3,4,6,6)$	$B[83] = [2,1,1,1,0,1] \leftrightarrow (1,1,2,3,4,6)$
$B[18] = [0,1,0,1,3,1] \leftrightarrow (2,4,5,5,5,6)$	$B[51] = [1,0,2,2,0,1] \leftrightarrow (1,3,3,4,4,6)$	$B[84] = [2,1,1,2,0,0] \leftrightarrow (1,1,2,3,4,4)$
$B[19] = [0,1,0,2,3,0] \leftrightarrow (2,4,4,5,5,5)$	$B[52] = [1,0,2,3,0,0] \leftrightarrow (1,3,3,4,4,4)$	$B[85] = [2,2,0,0,1,1] \leftrightarrow (1,1,2,2,5,6)$
$B[20] = [0,1,1,0,2,2] \leftrightarrow (2,3,5,5,6,6)$	$B[53] = [1,1,0,0,2,2] \leftrightarrow (1,2,5,5,6,6)$	$B[86] = [2,2,0,1,1,0] \leftrightarrow (1,1,2,2,4,5)$
$B[21] = [0,1,1,1,2,1] \leftrightarrow (2,3,4,5,5,6)$	$B[54] = [1,1,0,1,2,1] \leftrightarrow (1,2,4,5,5,6)$	$B[87] = [2,2,1,0,0,1] \leftrightarrow (1,1,2,2,3,6)$
$B[22] = [0,1,1,2,2,0] \leftrightarrow (2,3,4,4,5,5)$	$B[55] = [1,1,0,2,2,0] \leftrightarrow (1,2,4,4,5,5)$	$B[88] = [2,2,1,1,0,0] \leftrightarrow (1,1,2,2,3,4)$
$B[23] = [0,1,2,0,1,2] \leftrightarrow (2,3,3,5,6,6)$	$B[56] = [1,1,1,0,1,2] \leftrightarrow (1,2,3,5,6,6)$	$B[89] = [2,3,0,0,1,0] \leftrightarrow (1,1,2,2,2,5)$
$B[24] = [0,1,2,1,1,1] \leftrightarrow (2,3,3,4,5,6)$	$B[57] = [1,1,1,1,1,1] \leftrightarrow (1,2,3,4,5,6)$	$B[90] = [2,3,1,0,0,0] \leftrightarrow (1,1,2,2,2,3)$
$B[25] = [0,1,2,2,1,0] \leftrightarrow (2,3,3,4,4,5)$	$B[58] = [1,1,1,2,1,0] \leftrightarrow (1,2,3,4,4,5)$	$B[91] = [3,0,0,0,0,3] \leftrightarrow (1,1,1,6,6,6)$
$B[26] = [0,1,3,0,0,2] \leftrightarrow (2,3,3,3,6,6)$	$B[59] = [1,1,2,0,0,2] \leftrightarrow (1,2,3,3,6,6)$	$B[92] = [3,0,0,1,0,2] \leftrightarrow (1,1,1,4,6,6)$
$B[27] = [0,1,3,1,0,1] \leftrightarrow (2,3,3,3,4,6)$	$B[60] = [1,1,2,1,0,1] \leftrightarrow (1,2,3,3,4,6)$	$B[93] = [3,0,0,2,0,1] \leftrightarrow (1,1,1,4,4,6)$
$B[28] = [0,1,3,2,0,0] \leftrightarrow (2,3,3,3,4,4)$	$B[61] = [1,1,2,2,0,0] \leftrightarrow (1,2,3,3,4,4)$	$B[94] = [3,0,0,3,0,0] \leftrightarrow (1,1,1,4,4,4)$
$B[29] = [0,2,0,0,3,1] \leftrightarrow (2,2,5,5,5,6)$	$B[62] = [1,2,0,0,2,1] \leftrightarrow (1,2,2,5,5,6)$	$B[95] = [3,1,0,0,0,2] \leftrightarrow (1,1,1,2,6,6)$
$B[30] = [0,2,0,1,3,0] \leftrightarrow (2,2,4,5,5,5)$	$B[63] = [1,2,0,1,2,0] \leftrightarrow (1,2,2,4,5,5)$	$B[96] = [3,1,0,1,0,1] \leftrightarrow (1,1,1,2,4,6)$
$B[31] = [0,2,1,0,2,1] \leftrightarrow (2,2,3,5,5,6)$	$B[64] = [1,2,1,0,1,1] \leftrightarrow (1,2,2,3,5,6)$	$B[97] = [3,1,0,2,0,0] \leftrightarrow (1,1,1,2,4,4)$
$B[32] = [0,2,1,1,2,0] \leftrightarrow (2,2,3,4,5,5)$	$B[65] = [1,2,1,1,1,0] \leftrightarrow (1,2,2,3,4,5)$	$B[98] = [3,2,0,0,0,1] \leftrightarrow (1,1,1,2,2,6)$
$B[33] = [0,2,2,0,1,1] \leftrightarrow (2,2,3,3,5,6)$	$B[66] = [1,2,2,0,0,1] \leftrightarrow (1,2,2,3,3,6)$	$B[99] = [3,2,0,1,0,0] \leftrightarrow (1,1,1,2,2,4)$
		$B[100] = [3,3,0,0,0,0] \leftrightarrow (1,1,1,2,2,2)$

There are 20 solutions for (14) and 100 for (15). From these results, we have Theorem 3 below.

Theorem 3. *There are $20 \times 100 = 2000$ distinct fair dice pairs.*

Notice that from the discussion of the next paragraph of Proposition 1 there are altogether $462 \times 462 = 213,444$ different combinations of quasi-standard dice pairs, and among them there are 2000 (about 0.937%) yielding a fair distribution.

3.3. Equivalent Classes

According to Theorem 3, there are 2000 fair dice pairs. If we consider applying “mod 4” on the numbers of the dice in advance, then some solutions shown in Tables 1 and 2 can be viewed as equivalent. By applying the “mod 4” operation, number 4, 5, and 6 is congruent to 0, 1, and 2, respectively. Therefore, as an example, $(4, 4, 4, 6, 6, 6) \cong (2, 4, 4, 4, 6, 6) \cong (2, 2, 4, 4, 4, 6) \cong (2, 2, 2, 4, 4, 4)$, i.e., $R[1] \cong R[5] \cong R[8] \cong R[10]$.

On the basis of the above discussion, we classify the quasi-distributions presented in Tables 1 and 2 into equivalent classes based on the “mod 4” operation. More specifically, we use the smallest sum of numbers of the dice as the representative element in each class and then we obtain all solutions to the red and blue dice systems of equations in a compressed form, which we present in Tables 3 and 4, respectively. Here, we just list numbers of the dice and omit their quasi-distributions. Note that 0 is equivalent to 4 in

the two tables, that is, if 0 is not allowed to appear on the face of a die, then we can simply replace 0 with 4 in Tables 3 and 4 (and the subsequent tables).

Table 3. The 4 set of compressed red dice solutions based on the “mod 4” operation.

$R'[1] = (0,0,0,2,2,2)$	$R'[2] = (0,0,1,2,2,3)$	$R'[3] = (0,1,1,2,3,3)$	$R'[4] = (1,1,1,3,3,3)$
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Table 4. The 16 set of compressed blue dice solutions based on the “mod 4” operation.

$B'[1] = (0,0,0,1,1,1)$	$B'[5] = (0,0,1,1,1,2)$	$B'[9] = (0,1,1,1,2,2)$	$B'[13] = (1,1,1,2,2,2)$
$B'[2] = (0,0,0,1,1,3)$	$B'[6] = (0,0,1,1,2,3)$	$B'[10] = (0,1,1,2,2,3)$	$B'[14] = (1,1,2,2,2,3)$
$B'[3] = (0,0,0,1,3,3)$	$B'[7] = (0,0,1,2,3,3)$	$B'[11] = (0,1,2,2,3,3)$	$B'[15] = (1,2,2,2,3,3)$
$B'[4] = (0,0,0,3,3,3)$	$B'[8] = (0,0,2,3,3,3)$	$B'[12] = (0,2,2,3,3,3)$	$B'[16] = (2,2,2,3,3,3)$

From the results shown in Tables 3 and 4, we observe that there are 4 equivalent classes for the red dice and 16 for the blue dice. These equivalent classes are exactly the quasi-distributions in Tables 1 and 2 the last two components of which are 0 (i.e., 5 and 6 don't appear on the faces of the dice).

Theorem 4. There are $4 \times 16 = 64$ different equivalent classes for fair dice pairs by “mod 4” operation.

Suppose X and Y are two random variables associated with quasi-standard dice, and Φ_X, Φ_Y is the generating function of X and Y , respectively. If $X \equiv Y \pmod{4}$, i.e., $\Phi_X = \Phi_Y$ in $\mathbb{Q}[x]/\langle x^4 - 1 \rangle$, we call X and Y are equivalent.

From (14) and (15), it follows that the generating function of red and blue dice is of the equivalent form

$$\frac{1}{6} (r'_1 x + (3 - r'_1)x^2 + r'_1 x^3 + (3 - r'_1)x^4) = \frac{1}{6} x(1 + x^2)(r'_1 + (3 - r'_1)x)$$

and

$$\frac{1}{6} (b'_1 x + b'_2 x^2 + (3 - b'_1)x^3 + (3 - b'_2)x^4) = \frac{1}{6} x(1 + x)(b'_1 + (b'_2 - b'_1)x + (3 - b'_2)x^2)$$

in $\mathbb{Q}[x]/\langle x^4 - 1 \rangle$, respectively. Here, $0 \leq r'_1, b'_1, b'_2 \leq 3$.

From the above, the generating function for the red die is divisible by $1 + x^2$ but not by $1 + x$ (because $r'_1 = 3 - r'_1$ does not hold) but the generating function for the blue die is divisible by $1 + x$ but not by $1 + x^2$ (since $b'_2 - b'_1 = 0, b'_1 = 3 - b'_2$ cannot be true at the same time). These features determine the characteristics of the red and blue dice.

Two quasi-standard dice $(n_1, n_2, \dots, n_6)$ and $(n'_1, n'_2, \dots, n'_6)$ are said to be **translation-equivalent** if there exists an integer k so that $n_i = n'_i + k$ holds for $i = 1, 2, \dots, 6$. For example, the quasi-standard dice $R'[1] = (0, 0, 0, 2, 2, 2)$ and $R'[4] = (1, 1, 1, 3, 3, 3)$ in the Table 3 are apparently translation-equivalent. Using this approach, we can cluster the 4 dice in Table 3 into two classes and cluster the 16 dice in the Table 4 into four classes, using the dice with the smallest sum as the representative dice for each translation-equivalent class, as shown in Tables 5 and 6.

Table 5. translation equivalent class of red dice.

$R''[1] = (0,0,0,2,2,2)$	$R''[2] = (0,0,1,2,2,3)$
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Table 6. translation equivalent class of blue dice.

$B''[1] = (0,0,0,1,1,1)$	$B''[2] = (0,0,0,1,1,3)$	$B''[3] = (0,0,1,1,1,2)$	$B''[4] = (0,0,1,1,2,3)$
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Tables 7–14 show a detailed diagram of the eight varieties of dice. In Tables 7–18, the first column is numbers of red dice, the first row is numbers of blue dice, and the remaining elements are the remainder by mod 4 of the sum for the first row and first column.eight varieties of dice

Table 7. $(R''[1], B''[1])$.

	0	0	0	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1
2	2	2	2	3	3	3
2	2	2	2	3	3	3
2	2	2	2	3	3	3

Table 8. $(R''[1], B''[2])$.

	0	0	0	1	1	3
0	0	0	0	1	1	3
0	0	0	0	1	1	3
0	0	0	0	1	1	3
2	2	2	2	3	3	1
2	2	2	2	3	3	1
2	2	2	2	3	3	1

Table 9. $(R''[1], B''[3])$.

	0	0	1	1	1	2
0	0	0	1	1	1	2
0	0	0	1	1	1	2
0	0	0	1	1	1	2
2	2	2	3	3	3	0
2	2	2	3	3	3	0
2	2	2	3	3	3	0

Table 10. $(R''[1], B''[4])$.

	0	0	1	1	2	3
0	0	0	1	1	2	3
0	0	0	1	1	2	3
0	0	0	1	1	2	3
2	2	2	3	3	0	1
2	2	2	3	3	0	1
2	2	2	3	3	0	1

Table 11. $(R''[2], B''[1])$.

	0	0	0	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1
1	1	1	1	2	2	2
2	2	2	2	3	3	3
2	2	2	2	3	3	3
3	3	3	3	0	0	0

Table 12. $(R''[2], B''[2])$.

	0	0	0	1	1	3
0	0	0	0	1	1	3
0	0	0	0	1	1	3
1	1	1	1	2	2	0
2	2	2	2	3	3	1
2	2	2	2	3	3	1
3	3	3	3	0	0	2

Table 13. $(R''[2], B''[3])$.

	0	0	1	1	1	2
0	0	0	1	1	1	2
0	0	0	1	1	1	2
1	1	1	2	2	2	3
2	2	2	3	3	3	0
2	2	2	3	3	3	0
3	3	3	0	0	0	1

Table 14. $(R''[2], B''[4])$.

	0	0	1	1	2	3
0	0	0	1	1	2	3
0	0	0	1	1	2	3
1	1	1	2	2	3	0
2	2	2	3	3	0	1
2	2	2	3	3	0	1
3	3	3	0	0	1	2

Theorem 5. *There are $2 \times 4 = 8$ different translation-equivalent classes of fair dice pairs.*

If there exists k , with $1 \leq k \leq 3$, such that $X - Y \equiv k \pmod{4}$, then X and Y are translation equivalent. More specifically, we have $\Phi_X = x^k \Phi_Y$ in $\mathbb{Q}[x]/\langle x^4 - 1 \rangle$, where Φ_X , and Φ_Y is the generating function of X and Y , respectively.

4. The Analysis of Result

From Table 1 and 2, we note that the quasi-distributions contain components that are 0. This shows that the corresponding number does not appear on any face of the dice. It can be seen that at most, four components are 0 in the quasi-distribution of a red or blue die and at least one component is 0 in the quasi-distribution of a red die by Proposition 4. Therefore, in the quasi-distribution for a fair dice pair, it has at least one and at most eight 0's appeared. In the subsections that follow, we examine these two extreme situations.

4.1. Fair Dice Pairs Closest to the Traditional Dice Pair

In this subsection, we describe how we would make a minimal change to a pair of standard dice to obtain a fair dice pair.

As noted above, there must be at least one 0 in the quasi-distribution of a fair dice pair. From Table 2, we observe that we have the minimum 0 only in $B[57]$ for the quasi-distribution of the blue die. From Table 1, we note that there are four quasi-distributions containing one 0, that is, $R[6]$, $R[12]$, $R[14]$, and $R[15]$. We notice that $B[57] = (1, 2, 3, 4, 5, 6)$, $R[6] = (\underline{4}, 2, 3, 4, 5, 6)$, $R[12] = (1, \underline{3}, 3, 4, 5, 6)$, $R[14] = (1, 2, 3, 4, \underline{4}, 6)$, and $R[15] = (1, 2, 3, 4, 5, \underline{3})$.

Therefore, we obtain 4 fair dice pairs in total, that is, $(R[6], B[57])$, $(R[12], B[57])$, $(R[14], B[57])$, and $(R[15], B[57])$. Any one of these 4 fair dice pairs can be constructed by taking a pair of standard dice and renaming one of its 12 faces. We have just 4 renaming methods for the red(or blue) dice: 1 to 4, 2 to 3, 5 to 4, or 6 to 3. More specifically, as shown in Tables 15–18. As an example, we have $(R[15], B[57])$, in which we change one of its two 6's to 3 from a pair of standard dice.

Table 15. $(R[6], B[57])$.

	1	2	3	4	5	6
<u>4</u>	1	2	3	4	1	2
2	3	4	1	2	3	4
3	4	1	2	3	4	1
4	1	2	3	4	1	2
5	2	3	4	1	2	3
6	3	4	1	2	3	4

Table 16. $(R[12], B[57])$.

	1	2	3	4	5	6
1	2	3	4	1	2	3
<u>3</u>	4	1	2	3	4	1
3	4	1	2	3	4	1
4	1	2	3	4	1	2
5	2	3	4	1	2	3
6	3	4	1	2	3	4

Table 17. $(R[14], B[57])$.

	1	2	3	4	5	6
1	2	3	4	1	2	3
2	3	4	1	2	3	4
3	4	1	2	3	4	1
4	1	2	3	4	1	2
<u>4</u>	1	2	3	4	1	2
6	3	4	1	2	3	4

Table 18. $(R[15], B[57])$.

	1	2	3	4	5	6
1	2	3	4	1	2	3
2	3	4	1	2	3	4
3	4	1	2	3	4	1
4	1	2	3	4	1	2
5	2	3	4	1	2	3
<u>3</u>	4	1	2	3	4	1

4.2. Fair Dice Pairs with the Fewest Types of Distinct Numbers

In this subsection, we focus on the case in which the most 0 occurrences appear in the corresponding quasi-distribution.

From Table 1, there are 4 quasi-distributions for the red die with four 0's components, that is, $R[1]$, $R[4]$, $R[10]$, and $R[20]$. From Table 2, there are 9 quasi-distributions for the blue die with four 0's components, that is, $B[1]$, $B[4]$, $B[13]$, $B[16]$, $B[37]$, $B[40]$, $B[91]$, $B[94]$, and $B[100]$. Combining these two sets, we get 36 fair dice pairs with eight 0's components occurrences. In each quasi-distribution, there are only two

different numbers on each die in the fair dice pair. Further, each different number appears exactly three times. These 36 fair dice pairs are of the simplest form.

From Tables 3 and 4, we observe that there are exactly eight fair dice pairs in which the different numbers appearing on all faces are essentially the least (as shown in Tables 7–14.), that is, there are two for the red die (i.e., $(R'[1]$ and $R'[4])$) and four for the blue die (i.e., $B'[1]$, $B'[4]$, $B'[13]$, and $B'[16]$). As shown in Table 7, the fair dice pair $(R'[1], B'[4])$ (i.e., $(R''[1], B''[1])$) is the simplest fair dice pair, requiring only a sum operation.

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