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Existence and Nonexistence of Solutions to p-Laplacian Problems on Unbounded Domains

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Abstract: In this article, using a fixed point index theorem on a cone, we prove the existence and multiplicity results of positive solutions to a one-dimensional p-Laplacian problem defined on infinite intervals. We also establish the nonexistence results of nontrivial solutions to the problem.

Keywords: p-Laplacian; multiplicity of positive solutions; exterior domain

1. Introduction

In this paper, we are concerned with the following one-dimensional p-Laplacian problem defined on infinite intervals:

$$\begin{cases} \frac{1}{r^{N-1}}(r^{N-1}\varphi_p(u'))' + K(r)f(u) = 0, & r \in (R, \infty) \\ u(R) = \lim_{r\to\infty} u(r) = 0, \end{cases} \tag{1}$$

where $1 < p < N$, $\varphi_p(s) := |s|^{p-2}s$ for $s \in \mathbb{R} \setminus \{0\}$, $\varphi_p(0) := 0$, $K \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = (0, \infty)$, f is an odd and locally Lipschitz-continuous function on $\mathbb{R}$, and R is a positive parameter.

Problem (1) arises naturally in the study of radial solutions of nonlinear elliptic equations, with $g(\lambda, x, u) = K(|x|)f(u)$ and $\Omega = \mathbb{R}^N \setminus \overline{B}_R(0)$, of the form:

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(\lambda, x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

For the last several decades, there has been extensive study of Problem (2) with various assumptions for the domain Ω and the nonlinearity $g = g(\lambda, x, u)$. For example, for $p = 2, \Omega = (-1, 1)$, and $g(\lambda, x, u) = |x|^l u^p$, Tanaka ([1]) showed the existence of one positive even solution and two positive non-even solutions to problem (2) when $l(p-1) \geq 4$ and $l \geq 0$. Recently, for $p \in (1, N), \Omega = \mathbb{R}^N \setminus \overline{B}_R(0)$ and $g(\lambda, x, u) = \lambda K(|x|)f(u)$, Shivaji, Sim, and Son ([2]) proved the uniqueness of positive solution to Problem (2) for large λ under suitable additional assumptions on the reaction term f satisfying $f(0) < 0$. More recently, for $p \in (1, N), \Omega = \mathbb{R}^N \setminus \overline{B}_1(0)$ and $g(\lambda, x, u) = \lambda K(x)|u|^{p-2}u + h(x)$, Drábek, Ho, and Sarkar ([3]) investigated the Fredholm alternative for Problem (2) and also discussed the striking difference between the exterior domain and the entire space. For more references, we refer the reader to [4–13] for bounded domains and to [14–23] for unbounded domains.

By a solution u to Problem (1) with $R \in \mathbb{R}_+$, we mean $u \in C^1(R, \infty) \cap C[R, \infty)$ with $r^{N-1}\varphi_p(u') \in C^1(R, \infty)$ satisfies (1). We make a list of hypotheses that are used in this paper.

$(F0)$ there exist $0 \le \alpha < \beta \le \infty$ such that $f(u) < 0$ for $u \in (0, \alpha)$, $f(u) > 0$ for $u \in (\alpha, \beta)$ and $f(u) \le 0$ for $u > \beta$;

$(F1)$ there exists $C_0 > 0$ such that $|f(u)| \le C_0 u^{p-1}$ for $u \in \mathbb{R}_+$;

$(F2)$ $p^* K(r) + rK'(r) \le 0$ for $r \in \mathbb{R}_+$, where $p^*(= p^*(N,p)) = \frac{p(N-1)}{p-1}$;

$(F2)'$ $p^* K(r) + rK'(r) \ge 0$ for $r \in \mathbb{R}_+$;

$(F3)$ $\int_1^\infty [K(r)]^{\frac{1}{p}} dr < \infty$;

$(F4)$ $\int_1^\infty r^{p-1} K(r) dr < \infty$;

$(F5)$ there exist positive constants q_1, q_2, C_1 and C_2 such that $p < q_2 \le q_1$, $C_2 r^{-q_2} \le K(r) \le C_1 r^{-q_1}$ for $r \in (0, R_0)$, and $C_1 r^{-q_1} \le K(r) \le C_2 r^{-q_2}$ for $r \in (R_0, \infty)$. Here, $R_0 = (C_1 C_2^{-1})^{\frac{1}{q_1 - q_2}} \in \mathbb{R}_+$ if $q_1 > q_2$, and $R_0 = \infty$ if $q_1 = q_2$ and $C_2 \le C_1$.

Remark 1.

(1) *Note that* $(F2)$ *(resp.,* $(F2)'$*) holds if and only if* $\frac{d}{dr}[r^{p^*} K(r)] \le 0$ *(resp.,* ≥ 0*) for* $r \in \mathbb{R}_+$.

(2) *Assume that, for some constants* C *and* q, $K(r) = Cr^{-q}$ *for* $r \in \mathbb{R}_+$. *Then,* $(F2)$ *holds if* $q \ge p^*$; $(F2)'$ *holds if* $q \le p^*$; $(F3)$ *and* $(F4)$ *hold if* $q > p$. *Since* $p < N < p^*$, $(F2)$, $(F3)$, *and* $(F4)$ *hold if* $q \ge p^*$; $(F2)'$, $(F3)$, *and* $(F4)$ *hold if* $p < q \le p^*$. *Note that, for any* $R \in \mathbb{R}_+$, $q > N$ *if and only if* $r^{N-1} K(r) \in L^1(R, \infty)$. *Thus, in this case,* $(F2)$ *implies* $r^{N-1} K(r) \in L^1(R, \infty)$, *but* $r^{N-1} K(r)$ *may not be in* $L^1(R, \infty)$ *if* $(F2)'$ *holds.*

(3) $(F5)$ *implies* $(F3)$ *and* $(F4)$.

This paper is motivated by the recent works of Iaia ([16–19]), Joshi ([24]), and Joshi and Iaia ([20]). For $p = 2$, the existence of an infinite number of solutions with a prescribed number of zeros to Problem (1) was proven in [16–20,24], and the nonexistence of nontrivial solutions to Problem (1) was shown in [16,18,24]. The proofs in those papers are mainly based on the shooting method. In this paper, for $p \in (1, N)$, the nonexistence results of nontrivial solutions to Problem (1) are proven for sufficiently large $R > 0$, and the existence results of positive solutions to Problem (1) are established. Our approach for the existence results of positive solutions is based on a fixed point index theorem on positive cones.

2. Nonexistence of Nontrivial Solutions to Problem (1)

Let u be a solution to Problem (1). Then, it is well known that $u \in C^2([R, \infty))$ for $p \in (1, 2]$ and $u \in C^1([R, \infty)) \cap C^2([R, \infty) \setminus A)$ for $p > 2$, where $A = \{r \in [R, \infty) : u'(r) = 0\}$. Clearly, zero is a trivial solution to Problem (1), since $f(0) = 0$.

Theorem 1. *Assume that* $(F0)$,$(F1)$, $(F2)$, *and* $(F3)$ *hold. Then, there exists* $R_* > 0$ *such that Problem* (1) *has no nontrivial solutions for any* $R > R_*$.

Proof. Assume, on the contrary, that there exists a nontrivial solution u to Problem (1). By $(F1)$, if $u'(R) = 0$, then $u \equiv 0$ on $[R, \infty)$ in view of [12] (Theorem 4(δ)). Thus, $u'(R) \ne 0$. We may assume that $u'(R) > 0$, since f is an odd function. Then, there exists $M_1 > R$ such that $u'(r) > 0$ for $[R, M_1)$ and $u'(M_1) = 0$. Set:

$$E_1[u](r) = \frac{p-1}{p} \frac{|u'(r)|^p}{K(r)} + F(u(r)) \text{ for } r \in [R, \infty),$$

where $F(v) = \displaystyle\int_0^v f(s) ds$ for $v \in \mathbb{R}$. Since $u \in C^2([R, M_1))$ is a solution to Problem (1),

$$\frac{d}{dr} E_1[u](r) = -\frac{(p-1)|u'(r)|^p}{pr(K(r))^2} \left[p^* K(r) + rK'(r)\right] \text{ for } r \in [R, M_1).$$

By $(F2)$, $\frac{d}{dr}E_1[u](r) \geq 0$ for $r \in [R, M_1)$. Consequently, $E_1[u]$ is nondecreasing in $[R, M_1)$, and:

$$E_1[u](r) \leq E_1[u](M_1) = F(u(M_1)) \text{ for } r \in [R, M_1], \tag{3}$$

which implies $F(u(r)) < F(u(M_1))$ for $r \in [R, M_1]$. In particular, $0 = F(u(R)) < F(u(M_1))$. By $(F0)$, there exist $\gamma \in [\alpha, \beta)$ and $\delta \in [\beta, \infty]$ such that $F(u) < 0$ for $u \in (0, \gamma)$, $F(u) > 0$ for $u \in (\gamma, \delta)$ and $F(u) \leq 0$ for $u > \delta$. Here, $\gamma = 0$ if $\alpha = 0$, and $\gamma \in (\alpha, \delta)$ if $\alpha > 0$. Assume first that $\alpha > 0$. Since $F(u(M_1)) > 0$, $u(M_1) \in (\gamma, \delta)$, and thus, there exists a constant $F_0 > 0$ satisfying:

$$F(t) > -F_0 \text{ for all } t \in (0, u(M_1)]. \tag{4}$$

By (3) and (4), $[K(r)]^{\frac{1}{p}} \geq \sqrt[p]{\frac{p-1}{p}} \frac{u'(r)}{\sqrt[p]{F(u(M_1)) + F_0}}$ for $r \in [R, M_1]$. Then:

$$\begin{aligned}
\int_R^\infty [K(r)]^{\frac{1}{p}} dr &\geq \int_R^{M_1} [K(r)]^{\frac{1}{p}} dr \geq \sqrt[p]{\frac{p-1}{p}} \int_R^{M_1} \frac{u'(r)}{\sqrt[p]{F(u(M_1)) + F_0}} dr \\
&= \sqrt[p]{\frac{p-1}{p}} \left[\sqrt[p]{\frac{F(u(M_1))}{[u(M_1)]^p} + \frac{F_0}{[u(M_1)]^p}} \right]^{-1}.
\end{aligned}$$

By $(F1)$ and from the fact $u(M_1) \in (\gamma, \delta)$, it follows that:

$$0 < \frac{F(u(M_1))}{[u(M_1)]^p} + \frac{F_0}{[u(M_1)]^p} \leq \frac{C_0}{p} + \frac{F_0}{\gamma^p}.$$

Consequently,

$$0 < \sqrt[p]{\frac{p-1}{p}} \left[\sqrt[p]{\frac{C_0}{p} + \frac{F_0}{\gamma^p}} \right]^{-1} \leq \int_R^\infty [K(r)]^{\frac{1}{p}} dr.$$

By $(F3)$, $\int_R^\infty [K(r)]^{\frac{1}{p}} dr \to 0$ as $R \to \infty$, and thus, there exists $R_* > 0$ such that Problem (1) has no nontrivial solution for any $R > R_*$. Next, assume that $\alpha = 0$. Then, $u(M_1) \in (0, \delta)$, and (4) with $F_0 = 0$ holds. Then, one can easily prove the desired result, by arguments similar to those in the proof for the case $\alpha > 0$. Thus, the proof is complete. □

Using transformation $v(t) = u(r)$ with $t = r^{\frac{p-N}{p-1}}$, (1) can be rewritten equivalently as follows:

$$\begin{cases} (\varphi_p(v'))' + h(t) f(v) = 0, & t \in (0, R^{\frac{p-N}{p-1}}), \\ v(0) = v(R^{\frac{p-N}{p-1}}) = 0, \end{cases} \tag{5}$$

where $h(t) = \left(\frac{p-1}{N-p}\right)^p t^{\frac{p(N-1)}{p-N}} K(t^{\frac{p-1}{p-N}}) = \left(\frac{p-1}{N-p}\right)^p r^{p^*} K(r)$.

For the sake of convenience, we make a list of classes of the weight h as follows:

- $\mathcal{A} = \{h \in C(\mathbb{R}_+, \mathbb{R}_+) : \int_0^1 s^{p-1} h(s) ds < \infty\}$;
- $\mathcal{B} = \{h \in C(\mathbb{R}_+, \mathbb{R}_+) : \int_0^1 \varphi_p^{-1} \left(\int_s^1 h(\tau) d\tau \right) ds < \infty\}$;
- $\mathcal{C} = \{h \in C(\mathbb{R}_+, \mathbb{R}_+) : \int_0^1 s^\theta h(s) ds < \infty \text{ for some } \theta \in (0, p-1)\}$;
- $\mathcal{D} = \{h \in C(\mathbb{R}_+, \mathbb{R}_+) : \int_0^1 [h(s)]^{\frac{1}{p}} ds < \infty\}$.

Remark 2.

(1) *It is well known that* $L^1(0,1) \cap C(\mathbb{R}_+, \mathbb{R}_+) \subsetneq \mathcal{C} \subsetneq \mathcal{A} \cap \mathcal{B}$, $\mathcal{A} \subsetneq \mathcal{B}$ *for* $p \in (1,2)$, $\mathcal{B} \subsetneq \mathcal{A}$ *for* $p > 2$, *and* $\mathcal{A} = \mathcal{B}$ *for* $p = 2$ *(see, e.g., [25]). It is obvious that* $h \in \mathcal{C}$, *provided* $h(t) = t^{-\rho}$ *for some* $\rho < p$. *Moreover,* $\mathcal{C} \subseteq \mathcal{D}$. *Indeed, let* $h \in \mathcal{C}$, *and let* $p' := \frac{p}{p-1} \in (1, \infty)$. *Then,* $\theta\left(-\frac{p'}{p}\right) > -1$, *and by the Hölder inequality,*

$$\int_0^1 [h(t)]^{\frac{1}{p}} dt = \int_0^1 (t^\theta h(t))^{\frac{1}{p}} (t^\theta)^{-\frac{1}{p}} dt \le \left[\int_0^1 t^\theta h(t) dt\right]^{\frac{1}{p}} \left[\int_0^1 (t^\theta)^{-\frac{p'}{p}} dt\right]^{\frac{1}{p'}} < \infty.$$

(2) *Assume that* $h(t) = \left(\frac{p-1}{N-p}\right)^p r^{p^*} K(r)$ *with* $t = r^{\frac{p-N}{p-1}}$. *Then,* $(F3)$ *holds if and only if* $h \in \mathcal{D}$, *and* $(F4)$ *holds if and only if* $h \in \mathcal{A}$.

(3) *In [6] (Theorem 2.4), the* C^1*-regularity of solutions was proven, provided* $h \in \mathcal{A} \cap \mathcal{B}$ *and* $f(s) > 0$ *for* $s > 0$. *However, if* $p > 2$ *and* $h \in \mathcal{A} \setminus \mathcal{B}$, *the solutions to (5) may not be in* $C^1[0, R^{\frac{p-N}{p-1}}]$ *(see, e.g., [6] (Example 2.7)).*

Lemma 1. *Assume that* $(F0), (F1)$ *and* $(F4)$ *hold. Let* v *be a nontrivial solution to (5). Then, there exists* $\delta > 0$ *such that* $v(t) \neq 0$ *for* $t \in (0, \delta)$, *and* $v'(0) \in [-\infty, 0) \cup (0, \infty]$.

Proof. Let v be a nontrivial solution to Problem (5). Then, $v \in C[0, R^{\frac{p-N}{p-1}}] \cap C^1(0, R^{\frac{p-N}{p-1}}]$. First, we prove that there exists $\delta > 0$ such that $v(t) \neq 0$ for $t \in (0, \delta)$. Assume, on the contrary, that there exists a strictly decreasing sequence (t_n) satisfying $v(t_n) = 0$ and $t_n \to 0$ as $n \to \infty$. Multiplying the first equation in (5) by v and integrating it over (t_{n+1}, t_n), by $(F1)$,

$$\int_{t_{n+1}}^{t_n} |v'(s)|^p ds = \int_{t_{n+1}}^{t_n} h(s) f(v(s)) v(s) ds \le C_0 \int_{t_{n+1}}^{t_n} h(s) |v(s)|^p ds. \tag{6}$$

By the Hölder inequality, for $s \in (t_{n+1}, t_n)$,

$$|v(s)| \le \int_{t_{n+1}}^{s} |v'(\tau)| d\tau \le s^{\frac{p-1}{p}} \left(\int_{t_{n+1}}^{s} |v'(\tau)|^p d\tau\right)^{\frac{1}{p}}. \tag{7}$$

Substituting (7) into the integrand on the right-hand side in (6),

$$\int_{t_{n+1}}^{t_n} |v'(s)|^p ds \le C_0 \int_{t_{n+1}}^{t_n} s^{p-1} h(s) ds \int_{t_{n+1}}^{t_n} |v'(s)|^p ds.$$

By $(F4)$, $h \in \mathcal{A}$, and there exists $N > 0$ such that $C_0 \int_0^{t_N} s^{p-1} h(s) ds < \frac{1}{2}$. For any $n \ge N$,

$$\int_{t_{n+1}}^{t_n} |v'(s)|^p ds < \frac{1}{2} \int_{t_{n+1}}^{t_n} |v'(s)|^p ds.$$

which implies $v'(s) = 0$ for $s \in [t_{n+1}, t_n]$ for all $n \ge N$. Consequently, by (7), $v \equiv 0$ on $[0, t_N]$. Since $h \in C[t_N, R^{\frac{p-N}{p-1}}]$, by Grönwall's inequality, it can be easily proven that $v \equiv 0$ on $[0, R^{\frac{p-N}{p-1}}]$, which contradicts the fact that v is a nontrivial solution to Problem (5).

Since f is an odd function, we may assume that, for some $\delta > 0$, $v(t) > 0$ for $t \in (0, \delta)$. We prove $v'(0) \in (0, \infty]$ in order to complete the proof. Since v is a solution to Problem (5), by $(F0)$, v' is a monotonic function in $(0, \delta_1)$ for some $\delta_1 \in (0, \delta)$. Then, $\lim_{t \to 0+} v'(t) \in [0, \infty]$. Assume, on the contrary, that $v'(0) = 0$. By L'Hôpital's rule, $v'(0) = \lim_{t \to 0+} \frac{v(t)}{t} = \lim_{t \to 0+} v'(t)$, and thus, $v \in C^1[0, R^{\frac{p-N}{p-1}}]$

satisfying $v'(0) = 0$. Define $w : [0, R^{\frac{p-N}{p-1}}] \to \mathbb{R}$ by $w(0) = 0$ and $w(t) = \frac{v(t)}{t}$ for $t \in (0, R^{\frac{p-N}{p-1}}]$. Then $w \in C[0, R^{\frac{p-N}{p-1}}]$, since $v'(0) = 0$. For $t \in (0, R^{\frac{p-N}{p-1}}]$,

$$\begin{aligned} |w(t)| &= \left| \frac{1}{t} \int_0^t \varphi_p^{-1} \left(\int_0^s h(\tau) f(v(\tau)) d\tau \right) ds \right| \leq \frac{1}{t} \int_0^t \varphi_p^{-1} \left(\int_0^s h(\tau) |f(v(\tau))| d\tau \right) ds \\ &\leq \varphi_p^{-1} \left(\int_0^t h(\tau) |f(v(\tau))| d\tau \right). \end{aligned}$$

By $(F1)$, $|w(t)|^{p-1} \leq C_0 \int_0^t h(\tau)|v(\tau)|^{p-1} d\tau = C_0 \int_0^t \tau^{p-1} h(\tau) |w(\tau)|^{p-1} d\tau$ for $t \in (0, R^{\frac{p-N}{p-1}}]$. By Grönwall's inequality, $|w(t)|^{p-1} = 0$ on $[0, R^{\frac{p-N}{p-1}}]$, and consequently, $v \equiv 0$ on $[0, R^{\frac{p-N}{p-1}}]$, which contradicts the fact that v is a nontrivial solution to problem (5). Thus, the proof is complete. □

Theorem 2. *Assume that* $(F0)$,$(F1)$, $(F2)'$, $(F3)$, *and* $(F4)$ *hold. Then, there exists* $R_* > 0$ *such that for any* $R > R_*$, *Problem* (5) *(or equivalently* (1)*) has no nontrivial solutions.*

Proof. Let v be a nontrivial solution to Problem (5). Then, $v \in C[0, R^{\frac{p-N}{p-1}}] \cap C^1(0, R^{\frac{p-N}{p-1}}]$. By Lemma 1, we may assume that $v'(0) \in (0, \infty]$ and $v(t) > 0$ for $t \in (0, \delta)$, since f is an odd function. Then, there exists $M_2 \in (0, R^{\frac{p-N}{p-1}})$ such that $v'(t) > 0$ for $(0, M_2)$ and $v'(M_2) = 0$. Thus, $v \in C^2(0, M_2)$. Set:

$$E_2[v](t) = \frac{p-1}{p} \frac{|v'(t)|^p}{h(t)} + F(v(t)),\ t \in (0, R^{\frac{p-N}{p-1}}].$$

From the facts $\dfrac{dr}{dt} = \dfrac{p-1}{p-N} \dfrac{r}{t} < 0$ and $\dfrac{dh}{dt} = \left(\dfrac{p-1}{N-p} \right)^p \dfrac{d}{dr} \left[r^{p^*} K(r) \right] \dfrac{dr}{dt}$, it follows that $(F2)'$ holds if and only if $h' \leq 0$ on $\mathbb{R}_+$. Since $v \in C^2(0, M_2)$, by $(F2)'$,

$$\frac{d}{dt} E_2[v](t) = -\frac{p-1}{p} \frac{|v'(t)|^p h'(t)}{[h(t)]^2} \geq 0 \text{ for } t \in (0, M_2),$$

and thus, $E_2[v]$ is nondecreasing in $(0, M_2)$. By an argument similar to those in the proof of Theorem 1, the proof is complete. □

3. Existence of Positive Solutions to Problem (1)

Let $R \in \mathbb{R}_+$ be given. Using transformation $w(t) = u(r)$ with $t = (R^{-1} r)^{\frac{p-N}{p-1}}$, (1) can be rewritten equivalently as follows:

$$\begin{cases} (\varphi_p(w'))' + h_R(t) f(w) = 0,\ t \in (0,1), \\ w(0) = w(1) = 0, \end{cases} \tag{8}$$

where $h_R(t) = \left(\dfrac{p-1}{N-p} \right)^p R^p t^{\frac{p(N-1)}{p-N}} K(R t^{\frac{p-1}{p-N}}) \in C((0,1], \mathbb{R}_+)$.

For convenience, we denote $(F0)$ with $\alpha = 0$ and $\beta = \infty$ by $(F0)'$, i.e.,

$(F0)'$ $f(u) > 0$ for $u \in (0, \infty)$.

Throughout this section, we assume $(F0)'$ and $(F5)$ hold, unless otherwise stated.

Remark 3. *Assume that* $(F5)$ *holds and that* $R \in (0, R_0)$. *Then,* $(F5)$ *implies that:*

$$C_2 R^{p-q_2} k_1(t) \leq h_R(t) \leq C_1 R^{p-q_1} k_2(t) \text{ for } t \in (0,1], \tag{9}$$

where $k_i(t) = \left(\dfrac{p-1}{N-p} \right)^p t^{\frac{p(N-1)-q_i(p-1)}{p-N}}$ *for* $i = 1, 2$. *Since* $p < q_2 \leq q_1$,

$$\frac{p(N-1)-q_1(p-1)}{p-N} \geq \frac{p(N-1)-q_2(p-1)}{p-N} > -p,$$

so that $h_R \in \mathcal{C} \subsetneq \mathcal{A} \cap \mathcal{B} \cap \mathcal{D}$.

Denote $X = (C_0[0,1], \|\cdot\|_\infty)$, where $C_0[0,1] = \{w \in C[0,1] : w(0) = w(1) = 0\}$ and $\|w\|_\infty = \max_{t\in[0,1]} |w(t)|$ for $w \in C_0[0,1]$. Then, X is a Banach space, and $\mathcal{K} = \{w \in X : w$ is a nonnegative and concave function$\}$ is a positive cone in X. For $r > 0$, we define $\mathcal{K}_r = \{w \in \mathcal{K} : \|w\|_\infty < r\}$, $\partial\mathcal{K}_r = \{w \in \mathcal{K} : \|w\|_\infty = r\}$, and $\overline{\mathcal{K}}_r = \mathcal{K}_r \cup \partial\mathcal{K}_r$. For $w \in \mathcal{K}$, it is well known that, for any $\delta \in (0,1/2)$, $w(t) \geq \delta\|w\|_\infty$ for all $t \in [\delta, 1-\delta]$ by the concavity of w on $[0,1]$ (see, e.g., [26] (Lemma 1)).

The following well-known result on the fixed point index is crucial in this section:

Lemma 2 ([27,28])**.** *Assume that, for some $r > 0$, $T : \overline{\mathcal{K}}_r \to \mathcal{K}$ is completely continuous, i.e., compact and continuous on $\overline{\mathcal{K}}_r$. Then, the following results hold:*

(i) if $\|Tx\|_\infty > \|x\|_\infty$ for $x \in \partial\mathcal{K}_r$, then $i(T, \mathcal{K}_r, \mathcal{K}) = 0$;
(ii) if $\|Tx\|_\infty < \|x\|_\infty$ for $x \in \partial\mathcal{K}_r$, then $i(T, \mathcal{K}_r, \mathcal{K}) = 1$.

Let $R \in \mathbb{R}_+$ be given. Define $T_R : \mathcal{K} \to \mathcal{K}$ by, for $w \in \mathcal{K}$,

$$T_R(w)(t) = \begin{cases} \int_0^t \varphi_p^{-1}\left(\int_s^{A_{R,w}} h_R(\tau) f(w(\tau)) d\tau\right) ds, & 0 \leq t \leq A_{R,w}, \\ \int_t^1 \varphi_p^{-1}\left(\int_{A_{R,w}}^s h_R(\tau) f(w(\tau)) d\tau\right) ds, & A_{R,w} \leq t \leq 1, \end{cases}$$

where $A_{R,w}$ is a constant satisfying:

$$\int_0^{A_{R,w}} \varphi_p^{-1}\left(\int_s^{A_{R,w}} h_R(\tau) f(w(\tau)) d\tau\right) ds = \int_{A_{R,w}}^1 \varphi_p^{-1}\left(\int_{A_{R,w}}^s h_R(\tau) f(w(\tau)) d\tau\right) ds.$$

Since $h_R \in \mathcal{B}$, it is well known that T_R is well defined, $T_R(\mathcal{K}) \subseteq \mathcal{K}$, and T_R is completely continuous on $\mathcal{K}$ (see, e.g., [4] (Lemma 3)). Clearly, $T_R(A_{R,w}) = \|T_R(w)\|_\infty$ for all $R \in \mathbb{R}_+$ and all $w \in \mathcal{K}$. It can be easily seen that (8) has a positive solution w if and only if T_R has a fixed point w in $\mathcal{K} \setminus \{0\}$.

Let $\underline{f}(m) = \min\{f(y) : \frac{1}{4}m \leq y \leq m\}$ and $\overline{f}(m) = \max\{f(y) : 0 \leq y \leq m\}$ for $m \in \mathbb{R}_+$. Define continuous functions $R_1, R_2 : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$R_1(m) = \frac{m^{p-1}}{\underline{f}(m) A_1^{p-1}} \text{ and } R_2(m) = \frac{m^{p-1}}{\overline{f}(m) A_2^{p-1}} \text{ for } m \in \mathbb{R}_+.$$

Here:

$$A_1 = \min\left\{\int_{\frac{1}{4}}^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} k_1(\tau) d\tau\right) ds, \int_{\frac{1}{2}}^{\frac{3}{4}} \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s k_1(\tau) d\tau\right) ds\right\}$$

and:

$$A_2 = \max\left\{\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} k_2(\tau) d\tau\right) ds, \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s k_2(\tau) d\tau\right) ds\right\}.$$

Remark 4. *It is easily verified that $(\overline{f})_c = 0$ if $f_c = 0$, and $(\underline{f})_c = \infty$ if $f_c = \infty$. Here, $f_c := \lim_{s\to+c} \frac{f(s)}{s^{p-1}}$ for $c \in \{0, \infty\}$. Consequently, $\lim_{m\to+c} R_2(m) = \infty$ if $f_c = 0$, and $\lim_{m\to+c} R_1(m) = 0$ if $f_c = \infty$. Since $R_1(m) > R_2(m) > 0$ for all $m \in \mathbb{R}_+$, for $i = 1, 2$, $\lim_{m\to+c} R_i(m) = \infty$ if $f_c = 0$, and $\lim_{m\to+c} R_i(m) = 0$ if $f_c = \infty$ for $c \in \{0, \infty\}$.*

Lemma 3. *Assume that $(F0)'$ and $(F5)$ hold. Let $m \in \mathbb{R}_+$ be fixed. Then, for any $R \in (0, R_0)$ satisfying $C_2 R^{p-q_2} > R_1(m)$,*

$$i(T_R, \mathcal{K}_m, \mathcal{K}) = 0. \tag{10}$$

Proof. Let $R \in (0, R_0)$ satisfying $C_2 R^{p-q_2} > R_1(m)$ be fixed, and let $w \in \partial\mathcal{K}_m$. Then, $\frac{1}{4}m \le w(t) \le m$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, and:

$$f(w(t)) \ge \underline{f}(m) = \frac{m^{p-1}}{R_1(m) A_1^{p-1}} \text{ for } t \in [\frac{1}{4}, \frac{3}{4}]. \tag{11}$$

We have two cases: either $(i) A_{R,w} \in [\frac{1}{2}, 1)$ or (ii) $A_{R,w} \in (0, \frac{1}{2})$. We only consider the case (i), since the case (ii) can be dealt with in a similar manner. Since $C_2 R^{p-q_2} > R_1(m)$, from (9) and (11), it follows that:

$$\begin{aligned}
\|T_R(w)\|_\infty &= T_R(w)(A_{R,w}) = \int_0^{A_{R,w}} \varphi_p^{-1}\left(\int_s^{A_{R,w}} h_R(\tau) f(w(\tau)) d\tau\right) ds \\
&\ge \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} C_2 R^{p-q_2} k_1(\tau) \frac{m^{p-1}}{R_1(m) A_1^{p-1}} d\tau\right) ds \\
&> \frac{m}{A_1} \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} k_1(\tau) d\tau\right) ds \ge m = \|w\|_\infty.
\end{aligned}$$

By Lemma 2, (10) holds for any $R \in (0, R_0)$ satisfying $C_2 R^{p-q_2} > R_1(m)$. Thus, the proof is complete. □

Lemma 4. *Assume that* $(F0)'$ *and* $(F5)$ *hold. Let* $m \in \mathbb{R}_+$ *be fixed. Then, for any* $R \in (0, R_0)$ *satisfying* $C_1 R^{p-q_1} < R_2(m)$,

$$i(T_R, \mathcal{K}_m, \mathcal{K}) = 1. \tag{12}$$

Proof. Let $R \in (0, R_0)$ satisfying $C_1 R^{p-q_1} < R_2(m)$ be fixed, and let $w \in \partial\mathcal{K}_m$. Then:

$$f(w(t)) \le \overline{f}(m) = \frac{m^{p-1}}{R_2(m) A_2^{p-1}} \text{ for } t \in [0, 1]. \tag{13}$$

We only consider $A_{R,w} \in (0, \frac{1}{2})$, since the case $A_{R,w} \in [\frac{1}{2}, 1)$ can be dealt with in a similar manner. Since $C_1 R^{p-q_1} < R_2(m)$, from (9) and (13), it follows that:

$$\begin{aligned}
\|T_R(w)\|_\infty &= T_R(u)(A_{R,w}) = \int_0^{A_{R,u}} \varphi_p^{-1}\left(\int_s^{A_{R,u}} h_R(\tau) f(w(\tau)) d\tau\right) ds \\
&\le \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} C_1 R^{p-q_1} k_2(\tau) \frac{m^{p-1}}{R_2(m) A_2^{p-1}} d\tau\right) ds \\
&< \frac{m}{A_2} \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} k_2(\tau) d\tau\right) ds \le m = \|w\|_\infty.
\end{aligned}$$

By Lemma 2, (12) holds for any $R \in (0, R_0)$ satisfying $C_1 R^{p-q_1} < R_2(m)$, and thus, the proof is complete. □

By Lemmas 3 and 4, the result that (8) (or equivalently (1)) has arbitrarily many positive solutions can be obtained. For example, we have the following Theorems 3–8. Since the proofs are similar, we only give the proof of Theorem 6 in detail.

Theorem 3. *Assume that* $(F0)'$ *and* $(F5)$ *and that there exist* $R \in (0, R_0)$, m_1, *and* m_2 *such that* $0 < m_1 < m_2$ *(resp.,* $0 < m_2 < m_1$*),* $C_2 R^{p-q_2} > R_1(m_1)$ *and* $C_1 R^{p-q_1} < R_2(m_2)$*. Then, (8) has a positive solution* w *satisfying* $m_1 < \|w\|_\infty < m_2$ *(resp.,* $m_2 < \|w\|_\infty < m_1$*).*

Theorem 4. *Assume that* $(F0)'$ *and* $(F5)$ *and that there exist* $R \in (0, R_0)$, m_1, m_2 *and* M_1 *(resp.,* M_2*) such that* $0 < m_1 < m_2 < M_1$ *(resp.,* $0 < m_2 < m_1 < M_2$*),* $C_2 R^{p-q_2} > R_1(m_1)$, $C_1 R^{p-q_1} < R_2(m_2)$, *and*

$C_2R^{p-q_2} > R_1(M_1)$ *(resp.,* $C_1R^{p-q_1} < R_2(M_2)$*). Then,* (8) *has two positive solutions* w_1, w_2 *satisfying* $m_1 < \|w_1\|_\infty < m_2 < \|w_2\|_\infty < M_1$ *(resp.,* $m_2 < \|w_1\|_\infty < m_1 < \|w_2\|_\infty < M_2$*).*

Theorem 5. *Assume that* $(F0)'$ *and* $(F5)$ *hold and that there exist* $R \in (0, R_0)$, m_1, m_2, M_1 *and* M_2 *such that* $0 < m_2 < m_1 < M_2 < M_1$ *(resp.,* $0 < m_1 < m_2 < M_1 < M_2$*),* $C_2R^{p-q_2} > R_1(m_1)$, $C_2R^{p-q_2} > R_1(M_1)$, $C_1R^{p-q_1} < R_2(M_2)$*, and* $C_1R^{p-q_1} < R_2(m_2)$*. Then,* (8) *has three positive solutions* w_1, w_2, w_3 *satisfying* $m_2 < \|w_1\|_\infty < m_1 < \|w_2\|_\infty < M_2 < \|w_3\|_\infty < M_1$ *(resp.,* $m_1 < \|w_1\|_\infty < m_2 < \|w_2\|_\infty < M_1 < \|w_3\|_\infty < M_2$*).*

Theorem 6. *Assume that* $(F0)'$ *and* $(F5)$ *hold and that* $f_0 = f_\infty = 0$*. Then, there exists* $R^* > 0$ *such that for any* $R \in (0, R^*)$*,* (8) *has two positive solutions.*

Proof. From $f_0 = f_\infty = 0$, it follows that $\lim_{m\to 0^+} R_1(m) = \lim_{m\to\infty} R_1(m) = \infty$. Then, there exists $m_1^* \in \mathbb{R}_+$ satisfying $R_1(m_1^*) = \min\{R_1(m) : m \in \mathbb{R}_+\} \in \mathbb{R}_+$. Set $R^* = \min\{R_0, (\frac{R_1(m_1^*)}{C_2})^{\frac{1}{p-q_2}}\}$. For any $R \in (0, R^*)$, there exist $m_1 = m_1(R)$ and $M_1 = M_1(R)$ such that $0 < m_1 < m_1^* < M_1$ and $C_2R^{p-q_2} > R_1(m_1) = R_1(M_1)$. By Lemma 3,

$$\text{for any } R \in (0, R^*),\ i(T_R, \mathcal{K}_m, \mathcal{K}) = 0 \text{ for } m \in \{m_1, M_1\}. \tag{14}$$

On the other hand, since $f_0 = f_\infty = 0$, $\lim_{m\to 0^+} R_2(m) = \lim_{m\to\infty} R_2(m) = \infty$. For any $R \in (0, R^*)$, there exist $m_1' \in (0, m_1)$ and $M_1' \in (M_1, \infty)$ such that $C_1R^{p-q_1} < R_2(m_1') = R_2(M_1')$. By Lemma 4,

$$\text{for any } R \in (0, R^*),\ i(T_R, \mathcal{K}_m, \mathcal{K}) = 1 \text{ for } m \in \{m_1', M_1'\} \tag{15}$$

Then, by (14) and (15) and the additivity property of the fixed point index, for any $R \in (0, R^*)$,

$$i(T_R, \mathcal{K}_{m_1} \setminus \overline{\mathcal{K}}_{m_1'}, \mathcal{K}) = -1 \text{ and } i(T_R, \mathcal{K}_{M_1'} \setminus \overline{\mathcal{K}}_{M_1}, \mathcal{K}) = 1.$$

In view of the solution property of the fixed point index, for any $R \in (0, R^*)$, there exist $u_1 \in \mathcal{K}_{m_1} \setminus \overline{\mathcal{K}}_{m_1'}$ and $u_2 \in \mathcal{K}_{M_1'} \setminus \overline{\mathcal{K}}_{M_1}$ such that $T_R(u_i) = u_i$ for $i = 1, 2$. Thus, (8) has two positive solutions for any $R \in (0, R^*)$. □

Note that if either (i) $f_0 = 0$ and $f_\infty \in \mathbb{R}_+$ or (ii) $f_0 \in \mathbb{R}_+$ and $f_\infty = 0$, then there exists $\epsilon > 0$ satisfying $R_1(m) > \epsilon$ for all $m \in \mathbb{R}_+$. By an argument similar to those in the proof of Theorem 6, we have the following theorem:

Theorem 7. *Assume that* $(F0)'$ *and* $(F5)$ *hold and that either* (i) $f_0 = 0$ *and* $f_\infty \in \mathbb{R}_+$ *or* (ii) $f_0 \in \mathbb{R}_+$ *and* $f_\infty = 0$*. Then, there exists* $R^* > 0$ *such that* (8) *has a positive solution for any* $R \in (0, R^*)$*.*

If $f_0 = 0$ and $f_\infty = \infty$ (resp., $f_0 = \infty$ and $f_\infty = 0$), by Remark 4, $\lim_{m\to+0} R_2(m) = \infty$ and $\lim_{m\to+\infty} R_1(m) = 0$ (resp., $\lim_{m\to+0} R_1(m) = 0$ and $\lim_{m\to+\infty} R_2(m) = \infty$). Then, for any $R \in \mathbb{R}_+$, there exist m_1, m_2 satisfying $0 < m_2 < m_1$ (resp., $0 < m_1 < m_2$), $R_2(m_2) > C_1R^{p-q_1}$, and $R_1(m_1) < C_2R^{p-q_2}$. In view of Theorem 3, we have the following theorem:

Theorem 8. *Assume that* $(F0)'$ *and* $(F5)$ *hold and that either* (i) $f_0 = 0$ *and* $f_\infty = \infty$ *or* (ii) $f_0 = \infty$ *and* $f_\infty = 0$*. Then,* (8) *has a positive solution for all* $R \in \mathbb{R}_+$*.*

In the results so far, we assumed that f is positive for all $u > 0$, since it always satisfies $(F0)'$. If we assume that f has a positive falling zero instead of $(F0)'$, i.e., f satisfies the following:

$(F0)''$ there exists $\beta \in \mathbb{R}_+$ such that $f(u) > 0$ for $u \in (0, \beta)$ and $f(u) < 0$ for $u \in (\beta, \infty)$, then $\lim_{m\to\beta^-} R_2(m) = \infty$, so that we can obtain results similar to Theorems 3–8 above as follows:

Theorem 9. *Assume that $(F0)''$ and $(F5)$ and that there exist $R \in (0, R_0)$ and m_1 such that $0 < m_1 < \beta$ and $C_2R^{p-q_2} > R_1(m_1)$. Then, (8) has a positive solution w satisfying $m_1 < \|w\|_\infty < \beta$.*

Theorem 10. *Assume that $(F0)''$ and $(F5)$ and that there exist $R \in (0, R_0)$, m_1 and m_2 such that $0 < m_2 < m_1 < \beta$, $C_2R^{p-q_2} > R_1(m_1)$ and $C_1R^{p-q_1} < R_2(m_2)$. Then, (8) has two positive solutions w_1, w_2 satisfying $m_2 < \|w_1\|_\infty < m_1 < \|w_2\|_\infty < \beta$.*

Theorem 11. *Assume that $(F0)''$ and $(F5)$ hold and that there exist $R \in (0, R_0)$, m_1, m_2 and M_1 such that $0 < m_1 < m_2 < M_1 < \beta$, $C_2R^{p-q_2} > R_1(m_1)$, $C_2R^{p-q_2} > R_1(M_1)$, and $C_1R^{p-q_1} < R_2(m_2)$. Then, (8) has three positive solutions w_1, w_2, w_3 satisfying $m_1 < \|w_1\|_\infty < m_2 < \|w_2\|_\infty < M_1 < \|w_3\|_\infty < \beta$.*

Theorem 12. *Assume that $(F0)''$ and $(F5)$ hold and that $f_0 = 0$. Then, there exists $R^* > 0$ such that for any $R \in (0, R^*)$, (8) has two positive solutions.*

Theorem 13. *Assume that $(F0)''$ and $(F5)$ hold and that $f_0 \in \mathbb{R}_+$. Then, there exists $R^* > 0$ such that (8) has a positive solution for any $R \in (0, R^*)$.*

Theorem 14. *Assume that $(F0)''$ and $(F5)$ hold and that $f_0 = \infty$. Then, (8) has a positive solution for all $R \in \mathbb{R}_+$.*

Finally, the examples to illustrate the results obtained in this paper are given.

Example 1.

(1) *Let $f(s) = \begin{cases} s^{\alpha_1}, & \text{for } s \in [0,1), \\ s^{\alpha_2}, & \text{for } s \in [1,\infty), \end{cases}$ and let $K(r) = r^{-q}$ for $r \in \mathbb{R}_+$, where $p \in (1, N)$, $q > p$, and $0 < \alpha_2 < p - 1 < \alpha_1$. Then, $(F0)'$, $(F2)$ (or $(F2)'$), $(F5)$, and $f_0 = f_\infty = 0$ are satisfied. By Theorem 1 (or Theorem 2) and Theorem 6, there exist positive constants R_* and R^* such that (1) has two positive solutions for $R \in (0, R^*)$, and it has no nontrivial solutions for $R > R_*$.*

(2) *Let $K(r) = r^{-3}$ for $r \in \mathbb{R}_+$, and let $p = N - 1 = 2$. Then, in the assumption $(F5)$ and in (9), $q_1 = q_2 = 3$, $C_1 = C_2 = 1$, $R_0 = \infty$, and $k_1(t) = k_2(t) = t^{-1} \in \mathcal{C} \setminus L^1(0,1)$. By direct calculation, $A_1 = \frac{1}{4}(1 - \ln 2)$ and $A_2 = \frac{1}{2}$. Let:*

$$f(s) = \begin{cases} s^2, & \text{for } s \in [0, 2^4 A_1^2), \\ 2^6 A_1^3 s^{\frac{1}{2}}, & \text{for } s \in [2^4 A_1^2, 2^8), \\ 2^{-6} A_1^3 s^2, & \text{for } s \in [2^8, \infty). \end{cases}$$

Then, $f_0 = 0$ and $f_\infty = \infty$. Thus, by Remark 4, $\lim_{m\to+0} R_i(m) = \infty$ and $\lim_{m\to+\infty} R_i(m) = 0$ for $i = 1, 2$. Moreover, since $\overline{f}(m) = f(m)$ and $\underline{f}(m) = f(m/4)$, $R_2(m)$ (resp., $R_1(m)$) is decreasing in $(0, 2^4A_1^2)$ (resp., $(0, 2^6A_1^2)$), increasing in $(2^4A_1^2, 2^8)$ (resp., $(2^6A_1^2, 2^{10})$), and decreasing in $(2^8, \infty)$ (resp., $(2^{10}, \infty)$). Since $R_1(2^6A_1^2) = (4A_1^3)^{-1} < (2A_1^3)^{-1} = R_2(2^8)$, for each $R \in (2A_1^3, 4A_1^3)$, there exist $m_2 < 2^4A_1^2 < m_1 < 2^6A_1^2 < M_2 < 2^8 < M_1$ satisfying $R^{-1} < R_2(m_2)$, $R^{-1} > R_1(m_1)$, $R^{-1} < R_2(M_2)$ and $R^{-1} > R_1(M_1)$. Consequently, by Theorem 5 and Theorem 8, Problem (1) has three positive solutions for $R \in (2A_1^3, 4A_1^3)$, and it has a positive solution for all $R \in \mathbb{R}_+$.

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