Article

Characterizations of Regular Ordered Semigroups by $(\in, \in \vee (k^*, q_k))$ -Fuzzy Quasi-Ideals

Ahsan Mahboob ^{*,†}, Abdus Salam [†] , Md. Firoj Ali [†] and Noor Mohammad Khan [†]

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India;

salam.abdus92@gmail.com (A.S.); ali.firoj97@gmail.com (M.F.A.); nm_khan123@yahoo.co.in (N.M.K.)

* Correspondence: khanahsan56@gmail.com

† All authors contributed equally to this work.

Received: 20 March 2019; Accepted: 29 April 2019; Published: 5 May 2019

Abstract: In this paper, some properties of the (k^*, k) -lower part of $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideals are obtained. Then, we characterize regular ordered semigroups in terms of its $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideals, $(\in, \in \vee (k^*, q_k))$ -fuzzy generalized bi-ideals, $(\in, \in \vee (k^*, q_k))$ -fuzzy left ideals and $(\in, \in \vee (k^*, q_k))$ -fuzzy right ideals, and an equivalent condition for $(\in, \in \vee (k^*, q_k))$ -fuzzy left (resp. right) ideals is obtained. Finally, the existence theorems for an $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal as well as for the minimality of an $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal of an ordered semigroup are provided.

Keywords: ordered semigroups; fuzzy subsets; $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideals

1. Introduction

Since the concept of fuzzy sets was introduced by Zadeh in 1965 [1], the theories of fuzzy sets and fuzzy systems developed rapidly. The study of fuzzy algebraic structures started in the pioneering paper of Rosenfeld [2] in 1971. Rosenfeld introduced the notion of fuzzy groups and showed that many results in groups can be extended in an elementary manner to develop the theory of fuzzy groups. Since then, the literature on various fuzzy algebraic concepts has been growing very rapidly. Kuroki [3] introduced fuzzy sets in semigroup theory. Fuzzy (left, right) ideals and fuzzy bi-ideals in semigroups were introduced and studied by Kuroki [3,4]. As an extension of the concept of a fuzzy bi-ideal, Kuroki [5] introduced the concept of a fuzzy generalized bi-ideal of a semigroup and characterized different classes of ordered semigroups in terms of fuzzy generalized bi-ideals. Furthermore, interior ideals and a semiprime subset in a semigroup were extended [6] to introduce the concept of fuzzy interior ideals and the fuzzy semiprimality of a fuzzy set in a semigroup, and then characterize different classes of a semigroup in terms of fuzzy semiprime interior ideals.

Chang [7] was the first to initiate and extend the concept of general theory of fuzzy sets to topological spaces. After that, several authors [8–13] had investigated the different topological properties of a set in terms of its fuzzy subsets.

Kehayopulu and Tsingelis [14] were first to initiate the study of fuzzy sets in ordered semigroups. They also introduced the concept of fuzzy bi-ideals in ordered semigroups [15] and characterized bi-ideals, left and right simple, the completely regular and the strongly regular ordered semigroups in terms of fuzzy bi-ideals. Kazanci and Yamak [16] introduced and investigated some important properties of $(\in, \in \vee q)$ -fuzzy bi-ideals of a semigroup. Jun et al. [17] generalized the concept of fuzzy bi-ideals of ordered semigroups by introducing the concept of (α, β) -fuzzy generalized bi-ideals in ordered semigroups and characterized regular and intra-regular ordered semigroups in terms of $(\in, \in \vee q)$ -fuzzy generalized bi-ideals. Thereafter in 2012, Tang characterized ordered semigroups by $(\in, \in \vee q)$ -fuzzy ideals [18]. The concept of $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideals in ordered semigroups was introduced by Khan et al. [19]. Shabir et al. [20] characterized the

regular semigroups by $(\in, \in \vee q_k)$ -fuzzy ideals. Recently, Khan et al. [21] generalized the notion of $(\in, \in \vee q_k)$ -fuzzy left ideals, $(\in, \in \vee q_k)$ -fuzzy right ideals, and $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideals by introducing the notions of $(\in, \in \vee (k^*, q_k))$ -fuzzy left ideals, $(\in, \in \vee (k^*, q_k))$ -fuzzy right ideals, and $(\in, \in \vee (k^*, q_k))$ -fuzzy generalized bi-ideals of ordered semigroups, respectively. They also introduced (k^*, k) -lower parts of these generalized fuzzy ideals.

A generalized notion of a one-sided ideal is a notion of a quasi-ideal. In 1953, Steinfeld introduced quasi-ideals for rings [22] and, in 1956, for semigroups [23]. After that, quasi-ideals were widely studied in different algebraic structures by several authors. In his study of fuzzy ideals in semigroups, Kuroki [4] investigated some properties of fuzzy quasi-ideals of semigroups [6] and characterized different classes of semigroups in terms of fuzzy semiprime quasi-ideals. The concept of fuzzy quasi-ideals in ordered semigroups had been considered by Kehayoupu [24]. Shabir and Khan [25] characterized left (resp. right) simple and completely regular ordered semigroups in terms of fuzzy quasi-ideals. They also defined semiprime fuzzy quasi-ideals of ordered semigroups and characterized completely regular ordered semigroups in terms of fuzzy semiprime quasi-ideals.

Motivated by the above, in the present paper we introduce, as a generalization of the notion of the k -lower part of an $(\in, \in \vee q_k)$ -fuzzy quasi-ideal defined in an ordered semigroup, the notion of the (k^*, k) -lower part of an $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal in an ordered semigroup. Our first aim is to provide some different characterizations of regular ordered semigroups in terms of $(\in, \in \vee (k^*, q_k))$ -fuzzy left ideals, $(\in, \in \vee (k^*, q_k))$ -fuzzy right ideals, $(\in, \in \vee (k^*, q_k))$ -fuzzy generalized bi-ideals, and $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideals. Secondly, we present the relationships between $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideals and similar types of fuzzy left/right ideals, while the last section offers concluding remarks and some ideas for future work on the topic.

2. Preliminaries

In this section, we give some basic definitions and results that are needed to develop our paper. For more details and unexplained notions, readers may consult [17,21].

Our study is based on the notion of the ordered semigroup, that is, a semigroup $(S, \cdot)$ endowed with a partial order $\leq$ which is compatible with the $\cdot$ operation, i.e., for each $x, y \in S$, the relation $x \leq y$ implies that $zx \leq zy$ and $xz \leq yz$ for all $z \in S$.

For a subset A of an ordered semigroup S , we denote $(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}$. If $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$. For any non-empty subsets A and B of an ordered semigroup S , the following properties hold: (1) $A \subseteq (A]$; (2) $((A]) = (A]$; (3) if $A \subseteq B$, then $(A] \subseteq (B]$; (4) $(A](B] \subseteq (AB]$; and (5) $((A])(B]) = (AB]$.

All of these notions have also been transferred into the fuzzy sets theory. A non-empty subset A of an ordered semigroup S is called a subsemigroup of S if for all $x, y \in A$, $xy \in A$; while A is called a left (resp. right) ideal of S if it satisfies the conditions $SA \subseteq A$ ($AS \subseteq A$) and $(A] \subseteq A$. In addition A is called an ideal of S if it is both a left and a right ideal of S . A subsemigroup A of S is called a bi-ideal of S if $ASA \subseteq A$ and $(A] \subseteq A$. A non-empty subset Q of S is called a quasi-ideal of S if $(QS] \cap (SQ] \subseteq Q$ and $(Q] \subseteq Q$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A mapping η from S to the real closed interval $[0, 1]$ is called a fuzzy subset of S . We denote by η_A the characteristic function of the subset A of S , which is defined by

$$\eta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Moreover, the identical function $1(x) = 1$, for all $x \in S$, is a fuzzy subset of S . Let η and ξ be two fuzzy subsets of S . Then, for each $x \in S$, the intersection, union, and composition between these two fuzzy subsets are defined as

$$\begin{aligned}
(\eta \cap \xi)(x) &= \min\{\eta(x), \xi(x)\} = \eta(x) \wedge \xi(x) \\
(\eta \cup \xi)(x) &= \max\{\eta(x), \xi(x)\} = \eta(x) \vee \xi(x) \\
(\eta \circ g)(x) &= \begin{cases} \bigvee_{(y,z) \in A_x} \{\eta(y) \wedge \xi(z)\} & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset, \end{cases}
\end{aligned}$$

where $A_x = \{(y, z) \in S \times S \mid x \leq yz\}$. Moreover, on the set of all fuzzy subsets of S , one may define an order relation $\preceq$ by $\eta \preceq \xi \Leftrightarrow \eta(x) \leq \xi(x)$ for all $x \in S$.

If η, ξ are fuzzy subsets of S such that $\eta \preceq \xi$, then $\eta \circ \lambda \preceq \xi \circ \lambda$ and $\lambda \circ \eta \preceq \lambda \circ \xi$ for every fuzzy subset λ of S .

Now we recall the basic properties of fuzzy ideals on ordered semigroups, in accordance with [15,25]. A fuzzy subset η of an ordered semigroup S is called a fuzzy subsemigroup of S if $\eta(xy) \geq \min\{\eta(x), \eta(y)\}$ for all $x, y \in S$, while it is called a fuzzy left (resp. right) ideal of S if it satisfies the following two relations: (1) $x \leq y \Rightarrow \eta(x) \geq \eta(y)$ and (2) $\eta(xy) \geq \eta(y)$ (resp. $\eta(xy) \geq \eta(x)$) for all $x, y \in S$. η is called a fuzzy ideal of S if it is both a fuzzy left and right ideal of S . A fuzzy subsemigroup η of S is called a fuzzy bi-ideal of S if there are (1) $x \leq y \Rightarrow \eta(x) \geq \eta(y)$ and (2) $\eta(xyz) \geq \min\{\eta(x), \eta(z)\}$ for all $x, y, z \in S$.

For more details on fuzzy ideals and their related notions in ordered semigroups, the reader is referred to [26–32].

Let S be an ordered semigroup, $a \in S$ and $u \in (0, 1]$. An ordered fuzzy point a_u of S is defined by

$$a_u(x) = \begin{cases} u, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

For any fuzzy subset η of S , we denote $a_u \subseteq \eta$ by $a_u \in \eta$. Thus, $a_u \in \eta$ if and only if $\eta(a) \geq u$.

In the following, let $(S, \cdot, \leq)$ be an ordered semigroup, and $0 \leq k < k^* \leq 1$.

Definition 1 ([21]). The ordered fuzzy point a_u of S , for any $k^* \in (0, 1]$, is said to be (k^*, q) -quasi-coincident with a fuzzy subset η of S , denoted as $a_u(k^*, q)\eta$, if

$$\eta(a) + u > k^*.$$

Let $0 \leq k < k^* \leq 1$. For any ordered fuzzy point x_u , we say that

- (1) $x_u(k^*, q_k)\eta$ if $\eta(x) + u + k > k^*$;
- (2) $x_u \in \vee(k^*, q_k)\eta$ if $x_u \in \eta$ or $x_u(k^*, q_k)\eta$;
- (3) $x_u \bar{\alpha}\eta$ if $x_u \alpha\eta$ does not hold for $\alpha \in \{(k^*, q_k), \in \vee(k^*, q_k)\}$.

A fuzzy subset η of S is called an $(\in, \in \vee(k^*, q_k))$ -fuzzy subsemigroup [21] if $x_u \in \eta$ and $y_v \in \eta$ implies $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)\eta$ for all $u, v \in (0, 1]$ and $x, y \in S$. It is called an $(\in, \in \vee(k^*, q_k))$ -fuzzy left (resp. right) [21] ideal of S if $x \leq y$, $y_u \in \eta$ implies that $x_u \in \vee(k^*, q_k)\eta$, and $x \in S$, $y_u \in \eta$ implies $(xy)_u \in \vee(k^*, q_k)\eta$ (resp. $(yx)_u \in \vee(k^*, q_k)\eta$) for all $u \in (0, 1]$ and $x, y \in S$. If it is both an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal and $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal, then it is called an $(\in, \in \vee(k^*, q_k))$ -fuzzy ideal [21]. A fuzzy subset η of S is called an $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal [21] of S if $x \leq y$, $y_u \in \eta$ implies $x_u \in \vee(k^*, q_k)\eta$, and $x_u \in \eta$, $z_v \in \eta$ implies $(xyz)_{\min\{u,v\}} \in \vee(k^*, q_k)\eta$ for all $u, v \in (0, 1]$ and $x, y, z \in S$.

Theorem 1 ([21]). Let η be a fuzzy subset of S . Then η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S if and only if

- (1) $r \leq s$ implies that $\eta(r) \geq \min\{\eta(s), \frac{k^*-k}{2}\}$,
- (2) $\eta(rs) \geq \min\{\eta(s), \frac{k^*-k}{2}\}$,

for all $r, s \in S$.

Theorem 2 ([21]). Let η be a fuzzy subset of S . Then η is an $(\in, \in \vee (k^*, q_k))$ -fuzzy right ideal of S if and only if

- (1) $r \leq s$ implies that $\eta(r) \geq \min \left\{ \eta(s), \frac{k^* - k}{2} \right\}$,
- (2) $\eta(rs) \geq \min \left\{ \eta(r), \frac{k^* - k}{2} \right\}$,

for all $r, s \in S$.

Theorem 3 ([21]). Let η be a fuzzy subset of S . Then η is an $(\in, \in \vee (k^*, q_k))$ -fuzzy generalized bi-ideal of S if and only if

- (1) $r \leq s$ implies that $\eta(r) \geq \min \left\{ \eta(s), \frac{k^* - k}{2} \right\}$,
- (2) $\eta(rsw) \geq \min \left\{ \eta(r), \eta(w), \frac{k^* - k}{2} \right\}$,

for all $r, s, w \in S$.

Definition 2 ([21]). For any fuzzy subset η of S , the (k^*, k) -lower part $\underline{\eta}_k^{k^*}$ of η is defined as follows:

$$\underline{\eta}_k^{k^*}(a) = \min \left\{ \eta(a), \frac{k^* - k}{2} \right\}$$

for all $a \in S$ and $0 \leq k < k^* \leq 1$. Clearly, $\underline{\eta}_k^{k^*}$ is a fuzzy subset of S .

For any subset $T (\neq \emptyset)$ of S and fuzzy subset η of S , $(\underline{\eta}_T)_k^{k^*}$, the (k^*, k) -lower part of the characteristic function η_T will be denoted by $(\underline{\eta}_k^{k^*})_T$ in the sequel.

Lemma 1 ([21]). The (k^*, k) -lower part $(\underline{\eta}_k^{k^*})_T$ of the characteristic function η_T of any subset $T (\neq \emptyset)$ of S is an $(\in, \in \vee (k^*, q_k))$ -fuzzy left (resp. right) ideal of S if and only if T is a left (resp. right) ideal of S .

Lemma 2 ([21]). The (k^*, k) -lower part $(\underline{\eta}_k^{k^*})_T$ of the characteristic function η_T of any subset $T (\neq \emptyset)$ of S is an $(\in, \in \vee (k^*, q_k))$ -fuzzy generalized bi-ideal of S if and only if T is a generalized bi-ideal of S .

Lemma 3 ([33]). The following assertions are equivalent in S :

- (1) S is regular;
- (2) $B = (BSB)$ for every bi-ideal B of S ;
- (3) $Q = (QSQ)$ for every quasi-ideal Q of S .

Lemma 4 ([33]). The following assertions are equivalent in S :

- (1) S is regular;
- (2) $A \cap B = (AB)$ for every right-ideal A and left ideal B of S ;
- (3) $R(u) \cap L(u) = (R(u)L(u))$ for every $u \in S$.

Lemma 5 ([32]). The following assertions are equivalent in S :

- (1) S is regular;
- (2) $A \cap B = (AB)$ for every generalized bi-ideal A and left ideal B of S ;
- (3) $B(u) \cap L(u) = (B(u)L(u))$ for every $u \in S$.

Theorem 4 ([21]). The following assertions are equivalent in S :

- (1) S is regular;
- (2) $\eta(\circ)_k^{k^*} \xi \supseteq \eta(\cap)_k^{k^*} \xi$ for each $(\in, \in \vee (k^*, q_k))$ -fuzzy generalized bi-ideal η and for each $(\in, \in \vee (k^*, q_k))$ -fuzzy left ideal ξ of S .

3. $(\in, \in \vee (k^*, q_k))$ -Fuzzy Quasi-Ideals in Regular Ordered Semigroups

Definition 3 ([34]). A fuzzy subset η of an ordered semigroup S is called an $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal of S if

- (1) $a \leq b$, and $b_u \in \eta$ imply that $a_u \in \vee (k^*, q_k)\eta$, and
- (2) $a \leq by$, $a \leq cz$, and $b_u, c_v \in \eta$ imply that $a_{\min\{u,v\}} \in \vee (k^*, q_k)\eta$

for all $a, b, c, y, z \in S$ and $u, v \in (0, 1]$.

Lemma 6 ([34]). Every $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal of S is an $(\in, \in \vee (k^*, q_k))$ -fuzzy bi-ideal of S .

Example 1. Let $S = \{w, x, y, z\}$. Define a binary operation " $\cdot$ " and order " $\leq$ " in the following way:

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	w	w	w
y	w	w	w	x
z	w	w	x	y

$$\leq := \{(w, w), (x, x), (y, y), (z, z), (w, x)\}.$$

Then $(S, \cdot, \leq)$ is an ordered semigroup. Now define a fuzzy subset η on S as follows:

$$\eta(a) = \begin{cases} 0.2 & \text{if } a \in \{w, x\}, \\ 0 & \text{if } a \in \{y, z\}. \end{cases}$$

Take $k^* = 0.3$ and $k = 0.1$. It is routine to verify that η is an $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal of S .

Theorem 5 ([34]). A fuzzy subset η of an ordered semigroup S is an $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal of S if and only if

- (1) $a \leq b$ implies that $\eta(a) \geq \min\{\eta(b), \frac{k^*-k}{2}\}$ and
- (2) $(\eta \circ)_k^{k^*} 1) (\cap)_k^{k^*} (1 \circ)_k^{k^*} \eta) \preceq \eta$,

for each $a, b \in S$.

Lemma 7 ([21]). Let H and J be any non-empty subsets of S . Then

- (1) $\eta_H(\cap)_k^{k^*} \eta_J = (\eta_k^{k^*})_{H \cap J}$;
- (2) $\eta_H(\cup)_k^{k^*} \eta_J = (\eta_k^{k^*})_{H \cup J}$;
- (3) $\eta_H(\circ)_k^{k^*} \eta_J = (\eta_k^{k^*})_{(HJ)}$.

In the following result, a correspondence between quasi-ideals and the (k^*, k) -lower part of the $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideals of an ordered semigroup is discussed.

Lemma 8. The (k^*, k) -lower part $(\eta_k^{k^*})_T$ of the characteristic function η_T of a subset $T (\neq \emptyset)$ of S is an $(\in, \in \vee (k^*, q_k))$ -fuzzy quasi-ideal of S if and only if T is a quasi-ideal of S .

Proof. $(\Rightarrow)$ Let $r, s \in S$, $r \leq s$ and $u \in (0, 1]$ be such that $s_u \in (\eta_k^{k^*})_T$. As $\eta_T(r) = \begin{cases} \frac{k^*-k}{2}, & \text{if } r \in T; \\ 0, & \text{if } r \notin T, \end{cases}$ $s \in T$, then $(\eta_k^{k^*})_T(s) \geq u$. As T is a quasi-ideal of S and $r \leq s \in T$, we have $r \in T$. Thus $\eta_T(r) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $\eta_T(r) \geq u$. So we have $r_u \in (\eta_k^{k^*})_T$. If $u > \frac{k^*-k}{2}$, then $\eta_T(r) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. Hence, $r_u(k^*, q_k)(\eta_k^{k^*})_T$. Therefore, $r_u \in \vee (k^*, q_k)(\eta_k^{k^*})_T$.

Let $r, s, t, p, q \in S$ and $u, v \in (0, 1]$ be such that $r \leq sp, r \leq qt$ and $s_u \in (\eta_k^{k*})_T$ and $t_v \in (\eta_k^{k*})_T$. Then $s, t \in T$, $(\eta_k^{k*})_T(z) \geq u$ and $(\eta_k^{k*})_T(z) \geq v$. Since T is a quasi-ideal of S , we have $r \in T$. Thus, $\eta_T(r) \geq \frac{k^*-k}{2}$. If $\min\{u, v\} \leq \frac{k^*-k}{2}$, then $\eta_T(r) \geq \min\{u, v\}$. Therefore, $(r)_{\min\{u, v\}} \in (\eta_k^{k*})_T$. Again, if $\min\{u, v\} > \frac{k^*-k}{2}$, then $\eta_T(r) + \min\{u, v\} > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. Thus, $(r)_{\min\{u, v\}}(k^*, q_k)(\eta_k^{k*})_T$. Therefore, $(r)_{\min\{u, v\}} \in \vee(k^*, q_k)(\eta_k^{k*})_T$. Hence, $(\eta_k^{k*})_T$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S .

($\Leftarrow$) Let $(\eta_k^{k*})_T$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S . Then by Theorem 5, $((\eta_k^{k*})_T (\circ)_k^{k*} 1) (\cap)_k^{k*} (1 (\circ)_k^{k*} (\eta_k^{k*})_T) \preceq (\eta_k^{k*})_T$. By Lemma 7(3), $(\eta_k^{k*})_{(TS]} (\cap)_k^{k*} (\eta_k^{k*})_{(ST]} \preceq (\eta_k^{k*})_T$. Again by Lemma 7(2), $(\eta_k^{k*})_{(TS] \cap (ST]} \preceq (\eta_k^{k*})_T$. Therefore $(TS] \cap (ST] \subseteq T$. Let now $s \in S$ and $t \in T$ such that $s \leq t$. As $(\eta_k^{k*})_T$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S , $(\eta_k^{k*})_T(s) \geq \min\{(\eta_k^{k*})_T(t), \frac{k^*-k}{2}\}$. Since $t \in T$, we have $(\eta_k^{k*})_T = \frac{k^*-k}{2}$. So $(\eta_k^{k*})_T(s) \geq \frac{k^*-k}{2}$, it follows that $(\eta_k^{k*})_T(s) = \frac{k^*-k}{2}$. Thus $s \in T$. Hence T is a quasi-ideal of S . $\square$

Lemma 9. If η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S , then η_k^{k*} is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S .

Proof. Let $r, s \in S$, and $r \leq s$. Since η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S and $r \leq s$, we have $\eta(r) \geq \min\{\eta(s), \frac{k^*-k}{2}\}$. Therefore, $\min\{\eta(r), \frac{k^*-k}{2}\} \geq \min\{\eta(s), \frac{k^*-k}{2}\}$, and hence, $(\eta_k^{k*})(r) \geq \{(\eta_k^{k*})(s), \frac{k^*-k}{2}\}$.

Next, we suppose that $r, s, z, p, q \in S$ such that $r \leq sp$ and $r \leq qz$, and we have $\eta(r) \geq \min\{\eta(s), \eta(z), \frac{k^*-k}{2}\}$. Then $\min\{\eta(r), \frac{k^*-k}{2}\} \geq \min\{\eta(s), \eta(z), \frac{k^*-k}{2}\}$, and so $(\eta_k^{k*})(r) \geq \min\{(\eta_k^{k*})(s), (\eta_k^{k*})(z), \frac{k^*-k}{2}\}$. Hence, η_k^{k*} is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S . $\square$

Recall the fuzzy sets from [21] defined as follows:

$$\begin{aligned}(\eta(\cap)_k^{k*} \xi)(a) &= \min\left\{(\eta \cap \xi)(a), \frac{k^*-k}{2}\right\}, \\(\eta(\cup)_k^{k*} \xi)(a) &= \min\left\{(\eta \cup \xi)(a), \frac{k^*-k}{2}\right\}, \\(\eta(\circ)_k^{k*} \xi)(a) &= \min\left\{(\eta \circ \xi)(a), \frac{k^*-k}{2}\right\},\end{aligned}$$

for all $a \in S$ and $0 \leq k < k^* \leq 1$.

These fuzzy sets help us to obtain different characterizations of $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideals of ordered semigroups. Their immediate properties are expressed in the following results.

Theorem 6. An ordered semigroup S is regular $\Leftrightarrow \eta_k^{k*} \preceq \eta(\circ)_k^{k*} 1(\circ)_k^{k*} \eta$ for each fuzzy subset η of S .

Proof. ($\Rightarrow$) Take any $a \in S$. Then, there exists $r \in S$ such that $a \leq ara$. Thus, $(ar, a) \in A_a$. Now, we have

$$\begin{aligned}&(\eta(\circ)_k^{k*} 1(\circ)_k^{k*} \eta)(a) \\&= \bigvee_{(p,q) \in A_a} \min\{(\eta(\circ)_k^{k*} 1)(p), \eta(q), \frac{k^*-k}{2}\} \\&= \bigvee_{(p,q) \in A_a} \min\left\{\min\{(\eta \circ 1)(p), \frac{k^*-k}{2}\}, \eta(q), \frac{k^*-k}{2}\right\} \\&= \bigvee_{(p,q) \in A_a} \min\left\{(\eta \circ 1)(p), \eta(q), \frac{k^*-k}{2}\right\} \\&\geq \min\left\{(\eta \circ 1)(ar), \eta(a), \frac{k^*-k}{2}\right\}\end{aligned}$$

$$\begin{aligned} &\geq \min \left\{ \min \{ \eta(a), 1(r) \}, \eta(a), \frac{k^* - k}{2} \right\} \\ &= \underline{\eta}_k^{k^*}(a). \end{aligned}$$

Therefore $\underline{\eta}_k^{k^*} \preceq \eta(\circ)_k^{k^*} 1(\circ)_k^{k^*} \eta$.

($\Leftarrow$) Assume that $a \in S$. As $(\underline{\eta}_k^{k^*})_a$ is a fuzzy subset of S , $\frac{k^* - k}{2} = (\underline{\eta}_k^{k^*})_a(a) \leq ((\underline{\eta}_k^{k^*})_a(\circ)_k^{k^*} 1(\circ)_k^{k^*} (\underline{\eta}_k^{k^*})_a)(a)$ implies that

$$((\underline{\eta}_k^{k^*})_a(\circ)_k^{k^*} 1(\circ)_k^{k^*} (\underline{\eta}_k^{k^*})_a)(a) \neq 0.$$

Therefore, there exists $(r, s) \in A_a$ such that $((\underline{\eta}_k^{k^*})_a(\circ)_k^{k^*} 1)(r) \neq 0$ and $((\underline{\eta}_k^{k^*})_a)(s) \neq 0$, which implies that $(\eta_a \circ 1)(r) \neq 0$ and $(\eta_a)(s) \neq 0$. As $(\eta_a \circ \overline{1})(r) \neq 0$, $\exists (u, v) \in A_r$ such that $\eta_a(u) \neq 0$. Therefore, $a \leq rs \leq uvs = ava$, as required. $\square$

Theorem 7. The following assertions are equivalent in S :

- (1) S is regular;
- (2) $\underline{\eta}_k^{k^*} = \eta(\circ)_k^{k^*} 1(\circ)_k^{k^*} \eta$ or $\underline{\eta}_k^{k^*} = \underline{\eta}_k^{k^*}(\circ)_k^{k^*} 1(\circ)_k^{k^*} \underline{\eta}_k^{k^*}$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal η of S ;
- (3) $\underline{\xi}_k^{k^*} = \xi(\circ)_k^{k^*} 1(\circ)_k^{k^*} \xi$ or $\underline{\xi}_k^{k^*} = \underline{\xi}_k^{k^*}(\circ)_k^{k^*} 1(\circ)_k^{k^*} \underline{\xi}_k^{k^*}$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal ξ of S .

Proof. (1) $\Rightarrow$ (2). Suppose that η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal of an ordered semigroup S . Then we have

$$\begin{aligned} &(\eta(\circ)_k^{k^*} 1(\circ)_k^{k^*} \eta)(a) \\ &= \bigvee_{(r,s) \in A_a} \min \{ (\eta(\circ)_k^{k^*} 1)(r), \eta(s), \frac{k^* - k}{2} \} \\ &= \bigvee_{(r,s) \in A_a} \min \left\{ \min \{ (\eta \circ 1)(r), \frac{k^* - k}{2} \}, \eta(s), \frac{k^* - k}{2} \right\} \\ &= \bigvee_{(r,s) \in A_a} \min \left\{ (\eta \circ 1)(r), \eta(s), \frac{k^* - k}{2} \right\} \\ &= \bigvee_{(r,s) \in A_a} \min \left\{ \bigvee_{(p,q) \in A_r} \min \{ \eta(p), 1(q) \}, \eta(s), \frac{k^* - k}{2} \right\} \\ &= \bigvee_{(r,s) \in A_a} \bigvee_{(p,q) \in A_r} \min \left\{ \eta(p), \eta(s), \frac{k^* - k}{2} \right\} \\ &\leq \bigvee_{(pq,s) \in A_a} \min \left\{ \eta(p), \eta(s), \frac{k^* - k}{2}, \frac{k^* - k}{2} \right\} \\ &= \bigvee_{(pq,s) \in A_a} \min \left\{ \{ \min \{ \eta(p), \eta(s), \frac{k^* - k}{2} \}, \frac{k^* - k}{2} \} \right\} \\ &\leq \bigvee_{(pq,s) \in A_a} \min \left\{ \eta(pqs), \frac{k^* - k}{2} \right\} \\ &\leq \min \left\{ \eta(a), \frac{k^* - k}{2} \right\} \\ &= \underline{\eta}_k^{k^*}(a). \end{aligned}$$

As S is regular, by Theorem 6, $\underline{\eta}_k^{k*} \preceq \eta(\circ)_k^{k*} 1(\circ)_k^{k*} \eta$. Hence, $\underline{\eta}_k^{k*} = \eta(\circ)_k^{k*} 1(\circ)_k^{k*} \eta$.

(2) $\Rightarrow$ (3). Since $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideals are $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideals, by (2) it follows that $\underline{\zeta}_k^{k*} = \zeta(\circ)_k^{k*} 1(\circ)_k^{k*} \zeta$ or $\underline{\zeta}_k^{k*} = \underline{\zeta}_k^{k*}(\circ)_k^{k*} 1(\circ)_k^{k*} \underline{\zeta}_k^{k*}$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal ζ of S .

(3) $\Rightarrow$ (1). Suppose that T is a quasi-ideal of S . By Lemma 8, $(\underline{\eta}_k^{k*})_T$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S . Then by (3), we have

$$(\underline{\eta}_k^{k*})_T = (\underline{\eta}_k^{k*})_T(\circ)_k^{k*} 1(\circ)_k^{k*} (\underline{\eta}_k^{k*})_T = (\underline{\eta}_k^{k*})_{(TST)}.$$

Therefore $T = (TST)$. Hence, by Lemma 3, S is a regular ordered semigroup. $\square$

Theorem 8. The following assertions are equivalent in S :

- (1) S is regular;
- (2) $\eta(\circ)_k^{k*} \zeta = \eta(\cap)_k^{k*} \zeta$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal η and $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal ζ of S .

Proof. (1) $\Rightarrow$ (2). Suppose that η and ζ are an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal and an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S , respectively, and $a \in S$. Then there exists $r \in S$ such that $a \leq ara$. It follows that $(ar, a) \in A_a$. Then we have

$$\begin{aligned} (\eta(\circ)_k^{k*} \zeta)(a) &= \bigvee_{(y,z) \in A_a} \min\{\underline{\eta}_k^{k*}(y), \underline{\zeta}_k^{k*}(z)\} \\ &\geq \min\{\underline{\eta}_k^{k*}(ar), \underline{\zeta}_k^{k*}(a)\} \\ &= \min\{\eta(ar), \underline{\zeta}_k^{k*}(a), \frac{k^* - k}{2}\} \\ &\geq \min\{\eta(a), \underline{\zeta}_k^{k*}(a), \frac{k^* - k}{2}\} \\ &= \min\{\underline{\eta}_k^{k*}(a), \underline{\zeta}_k^{k*}(a)\} \\ &= (\eta(\cap)_k^{k*} \zeta)(a). \end{aligned}$$

Thus $\eta(\cap)_k^{k*} \zeta \subseteq \eta(\circ)_k^{k*} \zeta$. Inverse inclusion is obvious. Therefore $\eta(\cap)_k^{k*} \zeta = \eta(\circ)_k^{k*} \zeta$.

(2) $\Rightarrow$ (1). Suppose that A and B are a left ideal and a right ideal of S . Take any $x \in A \cap B$. Then $x \in A$ and $x \in B$. As A is a left ideal and B is a right ideal of S , by Lemma 1, $(\underline{\eta}_k^{k*})_A$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal, and $(\underline{\eta}_k^{k*})_B$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal of S . Thus, we have $(\eta_B(\circ)_k^{k*} \eta_A)(x) \geq (\eta_B(\cap)_k^{k*} \eta_A)(x) = \min\{(\underline{\eta}_k^{k*})_B(x), (\underline{\eta}_k^{k*})_A(x)\}$. Since $x \in A$, $x \in B$, $(\underline{\eta}_k^{k*})_A = \frac{k^* - k}{2}$, and $(\underline{\eta}_k^{k*})_B = \frac{k^* - k}{2}$, then $(\eta_B(\cap)_k^{k*} \eta_A)(x) = \frac{k^* - k}{2}$. Therefore, $(\eta_B(\circ)_k^{k*} \eta_A)(x) \geq \frac{k^* - k}{2}$. Since $(\eta_B(\circ)_k^{k*} \eta_A)(x) \leq \frac{k^* - k}{2}$, therefore $(\eta_B(\circ)_k^{k*} \eta_A)(x) = \frac{k^* - k}{2}$. By Lemma 7(3), we have

$$(\eta_B(\circ)_k^{k*} \eta_A)(x) = (\underline{\eta}_k^{k*})_{(BA)}(x).$$

Therefore, $(\underline{\eta}_k^{k*})_{(BA)}(x) = \frac{k^* - k}{2}$, and so $x \in (BA)$. It follows that $B \cap A \subseteq (BA)$. In addition, $(BA) \subseteq B \cap A$. Thus $B \cap A = (BA)$. Hence, by Lemma 4, S is regular. $\square$

Lemma 10. An ordered semigroup S is regular $\Leftrightarrow C \cap D \subseteq (CD)$ for each right ideal C and for each generalized bi-ideal D of S .

Proof. ($\Rightarrow$) As S is regular, we have

$$(C \cap D) \subseteq ((C \cap D)S(C \cap D)) \subseteq (CSD) \subseteq (CD).$$

Therefore, $C \cap D \subseteq (CD)$.

($\Leftarrow$) Take any $u \in S$. Then $B(u)$ and $R(u)$ are a generalized bi-ideal and right ideal of S , respectively. By hypothesis, we have $B(u) = B(u) \cap S \subseteq (SB(u)) \subseteq (S(u \cup uSu)) \subseteq (Su \cup SuSu) = (Su)$. Similarly, $R(u) \subseteq (uS)$. As (uS) and (Su) are a right ideal and generalized bi-ideal of S , respectively, again by hypothesis we have

$$\{u\} \subseteq R(u) \cap B(u) \subseteq (uS) \cap (Su) \subseteq ((uS)(Su)) \subseteq (uSu).$$

Hence S is regular. $\square$

Theorem 9. The following assertions are equivalent in S :

- (1) S is regular;
- (2) $\eta(\circ)_k^{k*} \xi \supseteq \eta(\cap)_k^{k*} \xi$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal η and for each $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal ξ of S .

Proof. (1) $\Rightarrow$ (2). Suppose that η and ξ are an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal and an $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal of S , respectively. Take any $a \in S$, then there exists $r \in S$ such that $a \leq ara$ implies $(a, ra) \in A_a$. Then we have

$$\begin{aligned} (\eta(\circ)_k^{k*} \xi)(a) &= \bigvee_{(y,z) \in A_a} \min\{\eta_k^{k*}(y), \xi_k^{k*}(z)\} \\ &\geq \min\{\eta_k^{k*}(ar), \xi_k^{k*}(a)\} \\ &= \min\{\eta(ar), \xi_k^{k*}(a), \frac{k^* - k}{2}\} \\ &\geq \min\{\eta(a), \xi_k^{k*}(a), \frac{k^* - k}{2}\} \\ &= \min\{\eta_k^{k*}(a), \xi_k^{k*}(a)\} \\ &= (\eta(\cap)_k^{k*} \xi)(a). \end{aligned}$$

Thus, $\eta(\cap)_k^{k*} \xi \subseteq \eta(\circ)_k^{k*} \xi$.

(2) $\Rightarrow$ (1). Suppose that C is a right ideal and D is a generalized bi-ideal of S . Take any $x \in C \cap D$. As C is a right ideal and D is a generalized bi-ideal of S , by Lemmas 1 and 2, $(\eta_k^{k*})_C$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal and $(\eta_k^{k*})_D$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal of S . Therefore, we have $(\eta_C(\circ)_k^{k*} \eta_D)(x) \geq (\eta_C(\cap)_k^{k*} \eta_D)(x) = \min\{(\eta_k^{k*})_C(x), (\eta_k^{k*})_D(x)\}$. Since $x \in C$ and $x \in D$, $(\eta_k^{k*})_D = \frac{k^* - k}{2}$ and $(\eta_k^{k*})_C = \frac{k^* - k}{2}$. Thus, $(\eta_C(\cap)_k^{k*} \eta_D)(x) = \frac{k^* - k}{2}$. Therefore, $(\eta_C(\circ)_k^{k*} \eta_D)(x) \geq \frac{k^* - k}{2}$. As $(\eta_C(\circ)_k^{k*} \eta_D)(x) \leq \frac{k^* - k}{2}$, $(\eta_C(\circ)_k^{k*} \eta_D)(x) = \frac{k^* - k}{2}$. Now we have

$$(\eta_C(\circ)_k^{k*} \eta_D)(x) = (\eta_k^{k*})_{(CD)}(x).$$

Therefore $(\eta_k^{k*})_{(CD)}(x) = \frac{k^* - k}{2}$, and so $x \in (CD)$. Thus, $C \cap D \subseteq (CD)$. Hence, by Lemma 10, S is regular. $\square$

Lemma 11. An ordered semigroup S is regular $\Leftrightarrow H \cap K \subseteq (HK)$ for each quasi-ideal H and for each left ideal K of S .

Proof. ($\Rightarrow$) The direct part follows from Lemma 5 because each quasi-ideal of S is a generalized bi-ideal of S .

($\Leftarrow$) Let $u \in S$. As $Q(u)$ is a quasi-ideal and S is a left ideal of S , we have

$$Q(u) = Q(u) \cap S \subseteq (Q(u)S] \subseteq ((Q(u)S] \subseteq (uS].$$

Similarly, $L(u) \subseteq (Su]$. As $(uS]$ and $(Su]$ are a quasi-ideal and left ideal of S , respectively, by hypothesis we have

$$\{u\} \subseteq Q(u) \cap L(u) \subseteq (uS] \cap (Su] \subseteq ((uS](Su)] \subseteq (uSu].$$

Hence S is regular. $\square$

Lemma 12. An ordered semigroup S is regular $\Leftrightarrow H \cap K \subseteq (HK]$ for each right ideal H and for each quasi ideal K of S .

Proof. Along similar lines to the proof of Lemma 11. $\square$

Theorem 10. The following assertions are equivalent in S :

- (1) S is regular;
- (2) $\eta(\circ)_k^{k*} \xi \supseteq \eta(\cap)_k^{k*} \xi$ for any $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal η and any $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal ξ of S .

Proof. (1) $\Rightarrow$ (2). Suppose that η and ξ are an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal and an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S . Take any $a \in S$, and as S is regular, there exists $r \in S$ such that $a \leq ara$. It follows that $(a, ra) \in A_a$. Then we have

$$\begin{aligned} (\eta(\circ)_k^{k*} \xi)(a) &= \bigvee_{(y,z) \in A_a} \min\{\eta_k^{k*}(y), \xi_k^{k*}(z)\} \\ &\geq \min\{\eta_k^{k*}(a), \xi_k^{k*}(ra)\} \\ &= \min\{\eta_k^{k*}(a), \xi(ra), \frac{k^* - k}{2}\} \\ &\geq \min\{\eta_k^{k*}(a), \xi(a), \frac{k^* - k}{2}\} \\ &= (\eta(\cap)_k^{k*} \xi)(a). \end{aligned}$$

Thus $\eta(\cap)_k^{k*} \xi \subseteq \eta(\circ)_k^{k*} \xi$.

(2) $\Rightarrow$ (1). Suppose that C is a left ideal and D is a quasi ideal of S . Take any $x \in C \cap D$. Since D is a quasi-ideal and C is a left ideal of S , by Lemmas 9 and 7, $(\eta_k^{k*})_D$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal, and $(\eta_k^{k*})_C$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S . As such, we have $(\eta_D(\circ)_k^{k*} \eta_C)(x) \geq (\eta_D(\cap)_k^{k*} \eta_C)(x) = \min\{(\eta_k^{k*})_D(x), (\eta_k^{k*})_C(x)\}$. Since $x \in D$ and $x \in C$, $(\eta_k^{k*})_D = \frac{k^* - k}{2}$ and $(\eta_k^{k*})_C = \frac{k^* - k}{2}$. Thus $(\eta_D(\cap)_k^{k*} \eta_C)(x) = \frac{k^* - k}{2}$. Therefore $(\eta_D(\circ)_k^{k*} \eta_C)(x) \geq \frac{k^* - k}{2}$. As $(\eta_D(\circ)_k^{k*} \eta_C)(x) \leq \frac{k^* - k}{2}$, $(\eta_D(\circ)_k^{k*} \eta_C)(x) = \frac{k^* - k}{2}$. By Lemma 7(3), we have

$$(\eta_D(\circ)_k^{k*} \eta_C)(x) = (\eta_k^{k*})_{(DC]}(x).$$

Therefore, $(\eta_k^{k*})_{(DC]}(x) = \frac{k^* - k}{2}$, and so $x \in (DC]$. This implies that $D \cap C \subseteq (DC]$. Hence, by Lemma 11, $\bar{S}$ is regular. $\square$

Similarly, we may prove the following theorem:

Theorem 11. The following assertions are equivalent in S :

- (1) S is regular;
- (2) $\eta(\circ)_k^{k^*} \xi \supseteq \eta(\cap)_k^{k^*} \xi$ for any $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal η and any $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal ξ of S .

Corollary 1. The following assertions are equivalent in S :

- (1) S is regular;
- (2) $B \cap L \subseteq (BL]$ for each generalized bi-ideal B and for each left ideal L of S ;
- (3) $Q \cap L \subseteq (QL]$ for each quasi-ideal Q and for each left ideal L of S ;
- (4) $R \cap B \subseteq (RB]$ for each right ideal R and for each generalized bi-ideal B of S ;
- (5) $R \cap Q \subseteq (RQ]$ for each right ideal R and for each quasi-ideal Q of S ;

Corollary 2. The following assertions are equivalent in S :

- (1) S is regular;
- (2) $\eta(\circ)_k^{k^*} \xi \supseteq \eta(\cap)_k^{k^*} \xi$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal η and for each $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal ξ of S ;
- (3) $\eta(\circ)_k^{k^*} \xi \supseteq \eta(\cap)_k^{k^*} \xi$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal η and for each $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal ξ of S ;
- (4) $\eta(\circ)_k^{k^*} \xi \supseteq \eta(\cap)_k^{k^*} \xi$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy right-ideal η and for each $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal ξ of S ;
- (5) $\eta(\circ)_k^{k^*} \xi \supseteq \eta(\cap)_k^{k^*} \xi$ for each $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal η and for each $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideal ξ of S .

The next result provides a characterization of the notion of the $(\in, \in \vee(k^*, q_k))$ -fuzzy left (resp. right) ideal of an ordered semigroup.

Lemma 13. Let η be any fuzzy subset of an ordered semigroup S . Then η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal (resp. right ideal) of $S \Leftrightarrow$

- (1) $a \leq b$ implies that $\eta(a) \geq \min\{\eta(b), \frac{k^* - k}{2}\}$, and
- (2) $1(\circ)_k^{k^*} \eta \subseteq \eta$ (resp. $\eta(\circ)_k^{k^*} 1 \subseteq \eta$)

for each $a, b \in S$.

Proof. $(\Rightarrow)$ Suppose that η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S , and $a \in S$. The condition (1) follows by Theorem 1. If $A_a = \emptyset$, then $1(\circ)_k^{k^*} \eta \subseteq \eta$. Suppose that $(y, z) \in A_a$. Then we have

$$\begin{aligned}
 (1(\circ)_k^{k^*} \eta)(a) &= \min\{(1 \circ \eta)(a), \frac{k^* - k}{2}\} \\
 &= \min\left\{\bigvee_{(y,z) \in A_a} \min\{1(y), \eta(z)\}, \frac{k^* - k}{2}\right\} \\
 &= \min\left\{\bigvee_{(y,z) \in A_a} \min\left\{\eta(z), \frac{k^* - k}{2}\right\}\right\} \\
 &\leq \min\left\{\bigvee_{(y,z) \in A_a} \min\{\eta(yz)\}\right\} \\
 &\leq \min\left\{\eta(a), \frac{k^* - k}{2}\right\} \\
 &\leq \eta(a).
 \end{aligned}$$

Therefore, $1(\circ)_k^{k^*} \eta \subseteq \eta$.

($\Leftarrow$) Assume that $1(\circ)_k^{k^*} \eta \subseteq \eta$ and $a, b \in S$. Since $ab \leq ab$, it follows that $(a, b) \in A_{ab}$. Then we have

$$\begin{aligned} \eta(ab) &\geq (1(\circ)_k^{k^*} \eta)(ab) \\ &= \bigvee_{(u,v) \in A_{ab}} \min \left\{ 1(p), \eta(q), \frac{k^* - k}{2} \right\} \\ &\geq \min \left\{ 1(a), \eta(b), \frac{k^* - k}{2} \right\} \\ &= \min \left\{ \eta(b), \frac{k^* - k}{2} \right\}. \end{aligned}$$

Hence, by Theorem 1, η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal. $\square$

The next two results provide the existence of the notion of an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal and minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of an ordered semigroup.

Theorem 12. Let η be a fuzzy subset of S such that $\eta(a) \geq \eta(b)$ for each $a, b \in S$ with $a \leq b$. Then η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of $S \Leftrightarrow$ there exists an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal g and an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal h of S such that $\eta = g(\cap)_k^{k^*} h$.

Proof. ($\Rightarrow$) Let η be any $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S . Let $g = \eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta$. To show that g is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S , first we show that for each $a, b \in S$ such that $a \leq b$, we have $(1(\circ)_k^{k^*} \eta)(a) \geq (1(\circ)_k^{k^*} \eta)(b)$. Now

$$\begin{aligned} (1(\circ)_k^{k^*} \eta)(b) &= \min \left\{ (1 \circ \eta)(b), \frac{k^* - k}{2} \right\} \\ &= \min \left\{ \bigvee_{(y,z) \in A_b} \min \{ 1(y), \eta(z) \}, \frac{k^* - k}{2} \right\} \\ &= \min \left\{ \bigvee_{(y,z) \in A_b} \min \left\{ \eta(z), \frac{k^* - k}{2} \right\} \right\}. \end{aligned}$$

It is sufficient to show that for each $(y, z) \in A_b$, it holds that

$$(1(\circ)_k^{k^*} \eta)(a) \geq \min \left\{ \eta(z), \frac{k^* - k}{2} \right\}.$$

Let $(y, z) \in A_b$. As $a \leq b$, and $a \leq b \leq yz$, then we have

$$\begin{aligned} (1(\circ)_k^{k^*} \eta)(a) &= \min \left\{ (1 \circ \eta)(a), \frac{k^* - k}{2} \right\} \\ &= \min \left\{ \bigvee_{(y,z) \in A_a} \min \{ 1(y), \eta(z) \}, \frac{k^* - k}{2} \right\} \\ &= \min \left\{ \min \{ 1(y), \eta(z) \}, \frac{k^* - k}{2} \right\} \\ &= \min \left\{ \eta(z), \frac{k^* - k}{2} \right\}. \end{aligned}$$

Therefore, $(1(\circ)_k^{k^*} \eta)(a) \geq (1(\circ)_k^{k^*} \eta)(b)$. Since $\eta(a) \geq \eta(b)$, then $(\eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta)(a) \geq (\eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta)(b)$ i.e., $g(a) \geq g(b)$.

Next, we show that $1(\circ)_k^{k^*} g \subseteq g$. Since $ab \leq ab$, it follows that $(a, b) \in A_{ab}$. Then we have

$$\begin{aligned} (1(\circ)_k^{k^*} \eta)(ab) &= \min\left\{(1 \circ \eta)(ab), \frac{k^* - k}{2}\right\} \\ &= \min\left\{\bigvee_{(r,s) \in A_{ab}} \min\{1(r), \eta(s)\}, \frac{k^* - k}{2}\right\} \\ &\geq \min\left\{\min\{1(a), \eta(b)\}, \frac{k^* - k}{2}\right\} \\ &= \min\left\{\eta(b), \frac{k^* - k}{2}\right\}. \end{aligned} \quad (1)$$

Similarly,

$$(1(\circ)_k^{k^*} \eta)(ab) \geq \min\left\{1(\circ)_k^{k^*} \eta(b), \frac{k^* - k}{2}\right\}. \quad (2)$$

From (1) and (2), it follows that, $(\eta(\cup)_k^{k^*} (1(\circ)_k^{k^*} \eta))(ab) \geq (\eta(\cup)_k^{k^*} (1(\circ)_k^{k^*} \eta))(b)$. Thus, $1(\circ)_k^{k^*} g \subseteq g$. Therefore, g is an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S . Similarly, $h = \eta(\cup)_k^{k^*} \eta(\circ)_k^{k^*} 1$ is an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal of S . Now, for any $u \in S$, we have

$$\begin{aligned} \eta(u) &\leq (g(\cap)_k^{k^*} h)(u) \\ &= \min\left\{(\eta(\cup)_k^{k^*} \eta(\circ)_k^{k^*} 1)(u) \cap (\eta(\cup)_k^{k^*} 1 \circ \eta)(u), \frac{k^* - k}{2}\right\} \\ &= \min\left\{\min\{\eta(u) \cup (\eta(\circ)_k^{k^*} 1)(u), \frac{k^* - k}{2}\} \cap \min\{\eta(u) \cup (1(\circ)_k^{k^*} \eta)(u), \frac{k^* - k}{2}\}, \frac{k^* - k}{2}\right\} \\ &\leq \min\left\{\eta(u), \frac{k^* - k}{2}\right\} \cup \min\left\{(\eta(\circ)_k^{k^*} 1)(u) \cap (1(\circ)_k^{k^*} \eta)(u), \frac{k^* - k}{2}\right\} \\ &\leq \min\left\{\eta(u), \frac{k^* - k}{2}\right\} \cap \eta(u) \\ &= \eta(u), \end{aligned}$$

and so, $\eta = g(\cap)_k^{k^*} h$.

($\Leftarrow$) Suppose that η is a fuzzy subset of S such that $\eta = g(\cap)_k^{k^*} h$ for an $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal g and an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal h of S . Let $r, s \in S$ such that $r \leq s$. Then clearly $\eta(r) \geq \min\{\eta(s), \frac{k^* - k}{2}\}$. If $(\eta(\circ)_k^{k^*} 1) (\cap)_k^{k^*} (1(\circ)_k^{k^*} \eta)(a) = 0$, for any $a \in S$, then $(\eta(\circ)_k^{k^*} 1) (\cap)_k^{k^*} (1(\circ)_k^{k^*} \eta) \preceq \eta$. Suppose now that, for an arbitrary $a \in S$, there is $(\eta(\circ)_k^{k^*} 1) (\cap)_k^{k^*} (1(\circ)_k^{k^*} \eta)(a) \neq 0$. Then we have

$$\begin{aligned} &(\eta(\circ)_k^{k^*} 1) (\cap)_k^{k^*} (1(\circ)_k^{k^*} \eta)(a) \\ &= ((g(\cap)_k^{k^*} h)(\circ)_k^{k^*} 1) (\cap)_k^{k^*} (1(\circ)_k^{k^*} (g(\cap)_k^{k^*} h))(a) \\ &= \min\left\{((g(\cap)_k^{k^*} h)(\circ)_k^{k^*} 1)(a) \cap (1(\circ)_k^{k^*} (g(\cap)_k^{k^*} h)(a)), \frac{k^* - k}{2}\right\}. \end{aligned} \quad (1)$$

Now we have

$$\begin{aligned} ((g(\cap)_k^{k^*} h)(\circ)_k^{k^*} 1)(a) &= \min\left\{((g(\cap)_k^{k^*} h) \circ 1)(a), \frac{k^* - k}{2}\right\} \\ &= \min\left\{\bigvee_{(x,y) \in A_a} \min\{(g(\cap)_k^{k^*} h)(x), 1(y)\}, \frac{k^* - k}{2}\right\} \\ &= \min\left\{\bigvee_{(x,y) \in A_a} (g(\cap)_k^{k^*} h)(x), \frac{k^* - k}{2}\right\} \end{aligned}$$

$$\begin{aligned}
&= \min \left\{ \bigvee_{(x,y) \in A_a} \min \left\{ (g \cap h)(x), \frac{k^* - k}{2} \right\}, \frac{k^* - k}{2} \right\} \\
&\leq \min \left\{ \bigvee_{(x,y) \in A_a} \min \left\{ h(x), \frac{k^* - k}{2} \right\}, \frac{k^* - k}{2} \right\} \\
&\leq \min \left\{ \bigvee_{(x,y) \in A_a} h(xy), \frac{k^* - k}{2} \right\} \\
&= \bigvee_{(x,y) \in A_a} \min \left\{ h(xy), \frac{k^* - k}{2} \right\} \\
&\leq h(a).
\end{aligned}$$

Similarly, $(1(\circ)_k^{k^*} (g(\cap)_k^{k^*} h)(a) \leq g(a)$. Therefore by (1),

$$\begin{aligned}
&((\eta(\circ)_k^{k^*} 1) (\cap)_k^{k^*} (1(\circ)_k^{k^*} \eta))(a) \\
&\leq \min \left\{ g(a) \cap h(a), \frac{k^* - k}{2} \right\} \\
&= (g(\cap)_k^{k^*} h)(a) \\
&= \eta(a).
\end{aligned}$$

Hence, by Theorem 5, η is an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S . $\square$

Definition 4. An $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal f of S is said to be minimal if there does not exist any $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal g of S such that $g \preceq f$.

Theorem 13. Let η be an $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S such that $\eta(a) \geq \eta(b)$ for each a, b with $a \leq b$. Then η is a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of $S \Leftrightarrow$ there exists a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal g and a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal h of S such that $\eta = g(\cap)_k^{k^*} h$.

Proof. $(\Rightarrow)$ By Theorem 12, $\eta = (\eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta)(\cap)_k^{k^*} \eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta$. Now we show that $\eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta$ is a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S , so let g be any $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S such that $g \preceq \eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta$. Now we have $g(\cap)_k^{k^*} (\eta(\cup)_k^{k^*} \eta(\circ)_k^{k^*} 1) \preceq (\eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta)(\cap)_k^{k^*} (\eta(\cup)_k^{k^*} \eta(\circ)_k^{k^*} 1) \preceq \eta$. Since η is a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S , $g(\cap)_k^{k^*} (\eta(\cup)_k^{k^*} \eta(\circ)_k^{k^*} 1) = \eta$. Therefore, $\eta \preceq g$ implies $\eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta \preceq g(\cup)_k^{k^*} 1(\circ)_k^{k^*} g = g$. Thus, $\eta(\cup)_k^{k^*} 1(\circ)_k^{k^*} \eta$ is a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S . Similarly, $\eta(\cup)_k^{k^*} \eta(\circ)_k^{k^*} 1$ is a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal of S .

$(\Leftarrow)$ Assume that $\eta = g(\cap)_k^{k^*} h$ for a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal g and a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal h of S . Therefore, $\eta \subseteq g$ and $\eta \preceq h$. Let μ be any $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S such that $\mu \preceq \eta$. Then $\mu(\circ)_k^{k^*} 1 \preceq \eta(\circ)_k^{k^*} 1 \preceq g(\circ)_k^{k^*} 1 \preceq g$ and $1(\circ)_k^{k^*} \mu \preceq 1(\circ)_k^{k^*} \eta \preceq 1(\circ)_k^{k^*} h \preceq h$. As $\mu(\circ)_k^{k^*} 1$ and $1(\circ)_k^{k^*} \mu$ are an $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideal and $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideal of S , by minimality of g and h , $1(\circ)_k^{k^*} \mu = g$ and $\mu(\circ)_k^{k^*} 1 = h$. Therefore, $\eta = g(\cap)_k^{k^*} h \preceq 1(\circ)_k^{k^*} \mu(\cap)_k^{k^*} \mu(\circ)_k^{k^*} 1 \preceq \mu$. Thus, $\mu = \eta$. Hence η is a minimal $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideal of S . $\square$

4. Conclusions and Future Work

The aim of the present paper is to enhance the understanding of ordered semigroups and regular ordered semigroups by considering the structural influence of $(\in, \in \vee q_k)$ -fuzzy quasi-ideals. In this view, we obtain several characterizations of regular ordered semigroups in terms of $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideals, $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideals, $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideals, and $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideals. In addition, we characterize the minimal of $(\in, \in$

$\vee(k^*, q_k)$ -fuzzy quasi-ideals in terms of $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideals and $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideals. Following are the particular cases of the present paper:

One may also conclude that:

- (1) If we put $k^* = 1$, then most of the results of this paper reduce in the setting of $(\in, \in \vee q_k)$ -fuzzy quasi-ideals.
- (2) If we put $k^* = 1$ and $k = 0$, then most of the results of this paper reduce in the setting of $(\in, \in \vee q)$ -fuzzy quasi-ideals.

It is hoped that the properties of $(\in, \in \vee(k^*, q_k))$ -fuzzy right ideals, $(\in, \in \vee(k^*, q_k))$ -fuzzy left ideals, $(\in, \in \vee(k^*, q_k))$ -fuzzy quasi-ideals, and $(\in, \in \vee(k^*, q_k))$ -fuzzy generalized bi-ideals may prove to be instrumental for characterizing different classes of ordered semigroups such as regular ordered semigroups, intra-regular ordered semigroups, and semisimple ordered semigroups.

Author Contributions: All authors contributed equally.

Funding: This research received no external funding.

Acknowledgments: We are very thankful to the reviewers for careful detailed reading and helpful comments/suggestions that improve the overall presentation of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control.* **1965**, *8*, 338–353. [\[CrossRef\]](#)
2. Rosenfeld, A. Fuzzy groups. *J. Math. Anal. Appl.* **1971**, *35*, 512–517. [\[CrossRef\]](#)
3. Kuroki, N. Fuzzy bi-ideals in semigroups. *Comment. Math. Univ. St. Pauli.* **1979**, *28*, 17–21.
4. Kuroki, N. On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets Syst.* **1981**, *5*, 203–215. [\[CrossRef\]](#)
5. Kuroki, N. Fuzzy generalized bi-ideals in semigroups. *Inf. Sci.* **1992**, *66*, 235–243. [\[CrossRef\]](#)
6. Kuroki, N. Fuzzy semiprime ideals in semigroups. *Fuzzy Sets Syst.* **1982**, *8*, 71–79.
7. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, *24*, 182–190. [\[CrossRef\]](#)
8. Bordbar, H.; Cristea, I.; Novak, M. Height of hyperideals in Noetherian Krasner hyperrings. *Univ. Polit. Buch. Sci. Bull. Ser. A Appl. Math. Phys.* **2017**, *79*, 31–42.
9. Chakrabarty, M.K.; Ahsanullah, T.M.G. Fuzzy topology on fuzzy sets and tolerance topology. *Fuzzy Sets Syst.* **1992**, *45*, 103–108. [\[CrossRef\]](#)
10. Cristea, I.; Hoskova, S. Fuzzy topological hypergroupoids. *Iran. J. Fuzzy Syst.* **2009**, *6*, 13–21.
11. Cristea, I.; Zhan, J.M. Lower and upper fuzzy topological subhypergroups. *Acta Math. Sin. (Engl. Ser.)* **2013**, *29*, 315–330. [\[CrossRef\]](#)
12. Onasanya, B.O.; Mayerova, S.H. Some topological and algebraic properties of alpha-level subsets' topology of a fuzzy subset. *An. St. Univ. Ovidius Constanta* **2018**, *26*, 213–227. [\[CrossRef\]](#)
13. Shojaei, H.; Ameri, R.; Mayerova, S.H. Pre-semihyperadditive categories. *An. St. Univ. Ovidius Constanta* **2019**, *27*, 269–288. [\[CrossRef\]](#)
14. Kehayopulu, N.; Tsingelis, M. Fuzzy sets in ordered groupoids. *Semigroup Forum* **2002**, *65*, 128–132. [\[CrossRef\]](#)
15. Kehayopulu, N.; Tsingelis, M. Fuzzy bi-ideals in ordered semigroups. *Inf. Sci.* **2005**, *171*, 13–28. [\[CrossRef\]](#)
16. Kazanci, O.; Yamak, S. Generalized fuzzy bi-ideals of semigroup. *Soft Comput.* **2008**, *12*, 1119–1124. [\[CrossRef\]](#)
17. Jun, Y.B.; Khan, A.; Shabir, M. Ordered semigroup characterized by their $(\in, \in \vee q)$ -fuzzy bi-ideals. *Bull. Malays. Math. Soc.* **2009**, *32*, 391–408.
18. Tang, J. Characterization of ordered semigroups by of $(\in, \in \vee q)$ -fuzzy ideals. *World Acad. Sci. Eng. Technol.* **2012**, *6*, 518–530.
19. Khan, A.; Jun, Y.B.; Sarmin, N.H.; Khan, F.M. Ordered semigroups characterized by $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideals. *Neural Comput. Appl.* **2012**, *21*, 121–132. [\[CrossRef\]](#)
20. Shabir, M.; Jun, Y.B.; Nawaz, Y. Characterizations of regular semigroups by $(\in, \in \vee q_k)$ -fuzzy ideals. *Comput. Math. Appl.* **2010**, *59*, 539–549. [\[CrossRef\]](#)

21. Khan, N.M.; Davvaz, B.; Khan, M.A. Ordered semigroups characterized in terms of generalized fuzzy ideals. *J. Intell. Fuzzy Syst.* **2017**, *32*, 1045–1057. [[CrossRef](#)]
22. Steinfeld, O. On the ideals quotients and prime ideals. *Acta Math. Acad. Sci. Hung.* **1953**, *4*, 289–298. [[CrossRef](#)]
23. Steinfeld, O. Über die quasiideale von halbgruppen. *Publ. Math. Debrecen* **1956**, *4*, 262–275.
24. Kehayopulu, N.; Tsingelis, M. Regular ordered semigroups in terms of fuzzy subsets. *Inf. Sci.* **2006**, *176*, 3675–3693. [[CrossRef](#)]
25. Shabir, M.; Khan, A. Fuzzy quasi-ideals of ordered semigroups. *Bull. Malays. Math. Sci. Soc.* **2011**, *34*, 87–102.
26. Ali, M.I.; Mahmood, T.; Hussain, A. A study of generalized roughness in $(\in, \in \vee_k)$ -fuzzy filters of ordered semigroups. *J. Taibah Univ. Sci.* **2018**, *12*, 163–172. [[CrossRef](#)]
27. Ameri, R.; Kordi, A.; Mayerova, S.H. Multiplicative hyperring of fractions and coprime hyperideals. *An. St. Univ. Ovidius Constanta* **2017**, *25*, 5–23. [[CrossRef](#)]
28. Khan, A.; Shabir, M.; Jun, Y.B. Intuitionistic fuzzy quasi-ideals of ordered semigroups. *Russ. Math.* **2010**, *54*, 59–71. [[CrossRef](#)]
29. Khan, A.; Sarmin, N.H.; Davvaz, B.; Khan, F.M. New types of fuzzy bi-ideals in ordered semigroups. *Neural Comput. Appl.* **2012**, *21*, 295–305. [[CrossRef](#)]
30. Khan, F.M.; Yufeng, N.; Khan, H.U.; Khan, B.M. Ordered semigroups based on $(\in, \in \vee q_k^\delta)$ -fuzzy ideals. *Adv. Fuzzy Syst.* **2018**, *2018*, 5304514.
31. Kuka, S.; Hila, K.; Naka, K. Application of L-fuzzy sets in m-ary semigroups. *J. Intell. Fuzzy Syst.* **2018**, *34*, 4031–4040. [[CrossRef](#)]
32. Shabir, M.; Khan, A. Characterizations of ordered semigroups by the properties of their fuzzy generalized bi-ideals. *New Math. Nat. Comput.* **2008**, *4*, 161–175. [[CrossRef](#)]
33. Xie, X.Y.; Tang, J. Regular and intra-regular ordered semigroups in terms of fuzzy subsets. *Iran. J. Fuzzy Syst.* **2010**, *7*, 121–140.
34. Cristea, I.; Mahboob, A.; Khan, N.M. A new type of fuzzy quasi-ideals of ordered semigroups. *Neural Comput. Appl.* **2019**, submitted.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).