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Some Metrical Properties of Lattice Graphs of Finite Groups

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Abstract: This paper is concerned with the combinatorial facts of the lattice graphs of $\mathbb{Z}_{p_1\times p_2\times\cdots\times p_m}$, $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}}$, and $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3^1}$. We show that the lattice graph of $\mathbb{Z}_{p_1\times p_2\times\cdots\times p_m}$ is realizable as a convex polytope. We also show that the diameter of the lattice graph of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times\cdots\times p_r^{m_r}}$ is $\sum\limits_{i=1}^{r} m_i$ and its girth is 4.

Keywords: finite group; lattice graph; convex polytope; diameter; girth

1. Introduction

The relation between the structure of a group and the structure of its subgroups constitutes an important domain of research in both group theory and graph theory. The topic has enjoyed a rapid development starting with the first half of the twentieth century.

The main object of this paper is to study the interplay of group-theoretic properties of a group G with graph-theoretic properties of its lattice graph *L(G)*. Every group has a corresponding lattice graph, which can be finite or infinite depending upon the order of the group. This study helps illuminate the structure of the set $H(G)$ of subgroups of G.

The *lattice graph* $L(G)$ of a finite cyclic group G is obtained as follows: Each vertex of $L(G)$ corresponds to an element of $H(G)$, and two vertices corresponding to two elements H_1, H_2 of $H(G)$ are connected by an edge if and only if $H_1 \leq H_2$ and that there is no element K of $H(G)$ such that $H_1 \lneqq K \lneqq H_2$ (see [1,2]), thus $\leq$ is used when H_1 is proper maximal subgroup of H_2. The notation $\leq$ is used as subgroup. Degree of a vertex is the number of edges attached to that vertex. A vertex is defined to be even or odd if its degree is even or odd. The *degree vector* of a graph is the sequence of degrees of its vertices arranged in non-increasing order [3]. The *diameter* $\mathrm{diam}(G)$ of a connected graph G is the maximum distance among vertices of G [4,5]. The *girth* $g(G)$ of a graph G is the length of a smallest cycle in G, and is infinity if G is acyclic [6]. It is a fact that the lattice of subgroups of a given group can rarely be drawn without its edges crossing [1,2]. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings of its edges among the drawings of G in the plane. A graph is considered Eulerian if there exists a Eulerian path in which we can start at a vertex, traverse through every edge only once, and return to the same vertex where we started. A connected graph G is *Eulerian* if each vertex has even degree, and is semi-Eulerian if it has exactly two vertices with odd degrees [7].

A *polytope* is a finite region of $\mathbb{R}^n$ enclosed by a finite number of hyper-planes. A polytope is called *convex* if its points form a convex subset of $\mathbb{R}^n$. Combinatorial aspects of the groups can be computed using its lattice graphs [8,9]. The authors discussed some finite simple groups of low rank in [8]. Tarnauceanu introduced new arithmetic method of counting the subgroups of a finite Abelian group in [10]. The author of [11] described the finite groups Ghaving $|G|-1$ cyclic subgroups. Saeedi and Farrokhi [12] computed factorization number of some finite groups. Tarnauceanu [13] characterized elementary Abelian two-groups. Tarnauceanu and Toth discussed cyclicity degree of finite groups in [14]. Tarnauceanu [15] discussed finite groups with dismantlable subgroup lattices.

In the present article, we are interested in the lattices of finite groups. We demonstrate that the lattice graph of $\mathbb{Z}_{p_1\times p_2\times\cdots\times p_m}$ can be viewed as a convex polytope, and that the diameter of the lattice graph of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times\cdots\times p_r^{m_r}}$ is $\sum\limits_{i=1}^{r} m_i$ and its girth is 4. We also compute many other properties of these lattices. We are interested developing some combinatorial invariants of the groups coming from their lattice graphs. The main motivation comes from the fact that finite graphs are relatively easier to handle than the finite groups. One is interested in capturing facts of the group while studying the graph associated to it. This study ultimately relates some parameters of the graphs with some parameters of the groups. Several authors computed some combinatorial aspects of finite graphs such as diameter, girth and radius but, for the lattice graph, similar questions are still open and need to be addressed. This article can be considered as a step forward in this direction. We, naturally, pose problems about the exact values of these parameters for more general classes of groups such as S_n, A_n, and sporadic groups.

2. The Results

In this section, we give the combinatorial results about the lattice graphs of $\mathbb{Z}_{p_1\times p_2\times\cdots\times p_m}$, $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}}$, and $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$. In the end, we give the general results about $L(\mathbb{Z}n)$.

2.1. $L(\mathbb{Z}_{p_1\times p_2\times\cdots\times p_m})$

Theorem 1. *The lattice graph of $\mathbb{Z}_n, n = p_1\times p_2\times\cdots\times p_m$ is realizable as a convex polytope with 2^m vertices.*

Proof. We prove it using all possible maximal chains of subgroups of $\mathbb{Z}_n$. It is clear that $\mathbb{Z}_n$ is finite cyclic group, thus, for each possible divisor of n, there exist a unique subgroup of this group, up to isomorphism. This leads us to the set of all subgroups of $\mathbb{Z}_n$; $V(G) = \{<1>, <p_1>, <p_2>, <p_3>, \ldots, <p_{m-1}>, <p_m>, <p_1\times p_2>, <p_1\times p_3>, \ldots, <p_1\times p_m>, <p_1\times p_2\times p_3>, \ldots, <p_1\times p_2\times p_m>, \ldots, <p_1\times p_2\times p_3\times, \ldots, \times p_m>\}$. Now, the maximal chains: The subgroup $H_0 = <0>$ is contained immediately in m subgroups of $\mathbb{Z}_n$: $H_{1_1} = <p_1\times p_2\times\cdots\times p_{m-1}>, H_{1_2} = <p_1\times p_2\times\cdots\times p_{m-2}\times p_m>, H_{1_m} = <p_2\times p_3\times\cdots\times p_m>$, as shown in Figure 1.

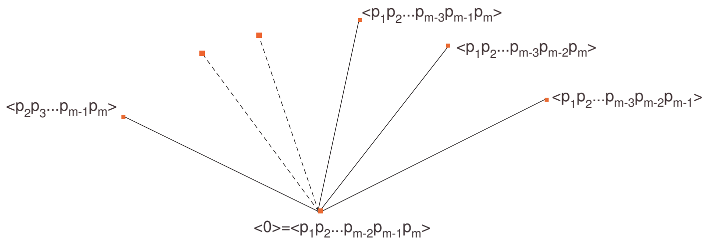

Figure 1. First step graph.

Each H_{1_i} is contained in $m-1$ subgroups of $\mathbb{Z}_n$: For instance, $H_{1_1} = <p_1\times p_2\times\ldots\times p_{m-1}>$ is contained in $H_{2_1} = <p_1\times p_2\times\ldots\times p_{m-2}>, H_{2_2} = <p_1\times p_2\times\ldots\times p_{m-3}\times p_{m-1}>, \ldots, H_{2_{m-1}} = <p_2\times p_3\times\ldots\times p_{m-1}>$ (see Figure 2).

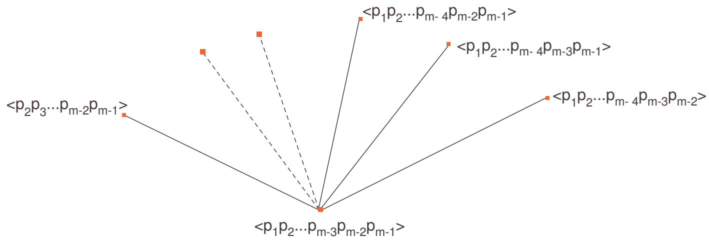

Figure 2. Intermediate step Graph.

Similarly, every H_{2_j} is contained in $m-2$ subgroups of $\mathbb{Z}_n$: $H_{2_1} = < p_1 \times p_2 \times \ldots \times p_{m-2} >$ is contained in H_{3_r} subgroups of $\mathbb{Z}_n$ (see Figure 3).

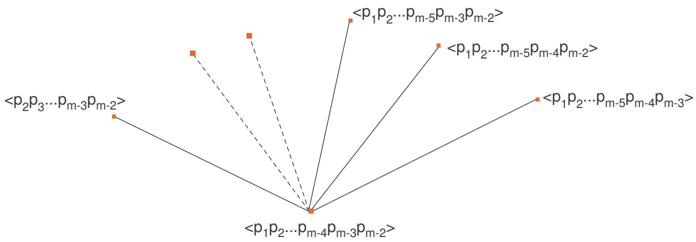

Figure 3. Intermediate step Graph.

The process will continue until we receive $< p_1 >, < p_2 > \ldots < p_m >$ at the second last stage. Now, each one of these is contained in $< 1 >$ = $\mathbb{Z}_n$ and the process is finished (see Figure 4).

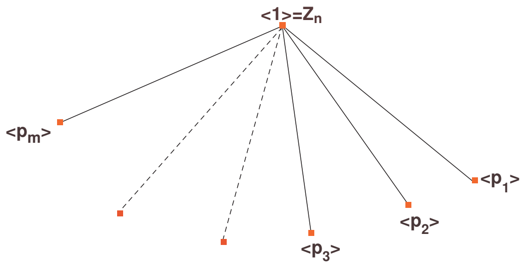

Figure 4. Last step Graph.

It is clear that there are m possibilities of subgroups of $\mathbb{Z}_n$ containing trivial group, $m-1$ possibilities of each of H_{1_i} to be contained in other subgroups of $\mathbb{Z}_n$, $m-2$ possibilities for each of H_{2_j} to be contained in next subgroups and so on. Thus, by the product rule, the number of all possibilities of maximal chains is $(m)(m-1)(m-2)\cdots 3.2.1 = m!$. Now, we put all these chains of subgroup in the plane such that each subgroup is identified to itself occurring in all these series. Thus, the same subgroups that appear in more than one series, appear only as a single vertex of the lattice graph in the plane. This lattice graph starts of at the identity and finishes at $\mathbb{Z}_n$ because these subgroups appear in all series. This may have crossings. If we imagine this gluing in $\mathbb{R}^3$ avoiding all crossings in higher dimension, we get a convex polytope with 2^m vertices, one vertex corresponding to each element of $H(\mathbb{Z}_n)$. For instance, the cases for $m = 3, 4, 5$ are shown in Figures 5–7, respectively. $\square$

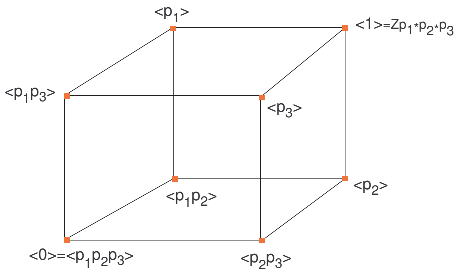

Figure 5. $L(\mathbb{Z}_{p_1 \times p_2 \times p_3})$.

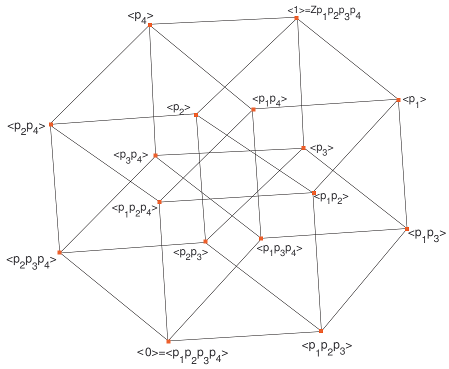

Figure 6. $L(\mathbb{Z}_{p_1 \times p_2 \times p_3 \times p_4})$.

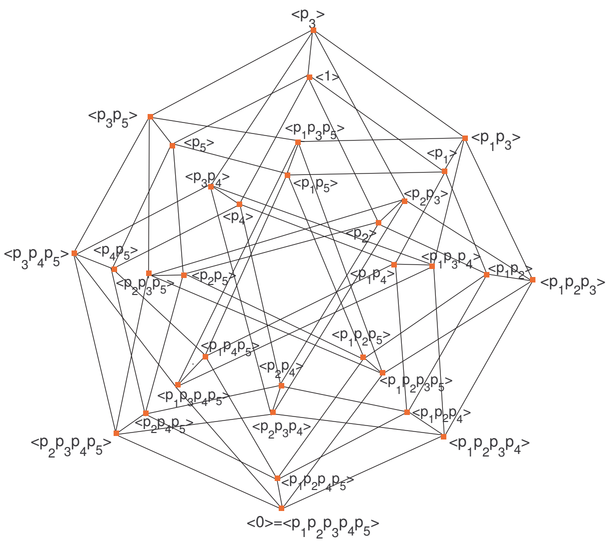

Figure 7. $L(\mathbb{Z}_{p_1 \times p_2 \times p_3 \times p_4 \times p_5})$.

Corollary 1.
(a) *The number of maximal subgroups of* $\mathbb{Z}_n$ *is* m.
(b) *The length of each maximal series is* $m + 1$.
(c) *The diameter of* $L(\mathbb{Z}_n)$ *is* m.
(d) $L(\mathbb{Z}_n)$ *is* m*-regular.*

Proof.
(a) It is clear that $\mathbb{Z}_n$ is cyclic, thus, for each divisor, we have a unique subgroup. Prime divisor p_i of n yields maximum quotient, thus these numbers correspond to the maximal subgroups of $\mathbb{Z}_n$, which are $< p_1 >, < p_2 >, < p_3 >, \ldots, < p_{m-1} >, < p_m >$.
(b) This can be shown by simply counting the number of vertices of $L(\mathbb{Z}_n)$ along a path from $< 0 >$

to $< 1 >$; see, for instance, a typical series: $< 0 > \subseteq < p_1 \times p_2 \times \ldots \times p_{m-1} > \subseteq \ldots \subseteq < p_1 \times p_2 > \subseteq < p_1 > \subseteq < 1 > = Z_n$.

(**c**) diam$L(\mathbb{Z}_n)$ is one less than the length of a maximal series, which is the length of each path from $< 0 >$ to $< 1 >$.

(**d**) At the first stage, the degree of the vertex H_0 is m because it is adjacent to m vertices H_{1_i}, $i = 1, 2, \ldots, m$ in the second stage, as shown in the construction of $L(\mathbb{Z}_n)$.

The degree of each vertex at the second stage is m as each vertex H_{1_i} is adjacent to $m-1$ vertices lying higher to it and to one vertex lying below it. For instance, $H_{1_1} = < p_1 \times p_2 \times \ldots p_{m-1} >$ is attached with $H_{2_1} = < p_1 \times p_2 \times \ldots \times p_{m-2} >, H_{2_2} = < p_1 \times p_2 \times \ldots \times p_{m-3} \times p_{m-1} >, \ldots, H_{2_{m-1}} = < p_2 \times p_3 \times \ldots \times p_{m-1} >$. Similarly, $deg(H_{2_j})$ is m. Continuing this process, we receive that the vertices corresponding to $< p_1 >, < p_2 >, \ldots, < p_m >$ are adjacent to the vertex corresponding to $< 1 >$, giving the degree of the last vertex m too. Thus, each vertex of $L(\mathbb{Z}_n)$ has degree m, and the proof is finished. □

Remark 1. *It should be remarked that the lattice graph of Z_n is a small-world network in which nearly all nodes are not neighbors of one another, but the neighbors of any given node are likely to be neighbors of each other and most nodes can be reached from every other node by a small number of steps. Thus, it is a connected graph [16].*

2.2. $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$

The construction of $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$ is given by using all maximal chains from the set $H(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}) = \{< 1 >, < p_1 >, < p_1^2 >, \ldots, < p_1^{m_1-1} >, < p_1^{m_1} >, < p_2 >, < p_2^2 >, \ldots, < p_2^{m_2-1} >, < p_2^{m_2} >, < p_1 \times p_2 >, \ldots, < p_1^{m_1} \times p_2 >, \ldots, < p_1^{m_1} \times p_2^{m_2} >\}$. The subgroup $< 0 > = < p_1^{m_1} \times p_2^{m_2} >$ is contained immediately in two subgroups $< p_1^{m_1} \times p_2^{m_2-1} >$ and $< p_1^{m_1-1} \times p_2^{m_2} >$ which are further contained in $< p_1^{m_1-1} \times p_2^{m_2-1} >$. The subgroup $< p_1^{m_1-1} \times p_2^{m_2} >$ is contained in $< p_1^{m_1-2} \times p_2^{m_2} >$, which is contained in next two subgroups, $< p_1^{m_1-2} \times p_2^{m_2-1} >$ and $< p_1^{m_1-3} \times p_2^{m_2} >$. $< p_1^{m_1-3} \times p_2^{m_2} >$ is contained in other two subgroups of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$. The process will continue until we receive a subgroup $< p_1 \times p_2^{m_2} >$ that is contained in next two subgroups, $< p_1 \times p_2^{m_2} >$ and $< p_2^{m_2} >$. Both are further contained in $< p_2^{m_2-1} >$, which itself is contained in $< p_2^{m_2-2} >$. This process is continued until we obtain a subgroup $< p_2 >$, which is continued in $< 1 > = \mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$, and thus we receive a series of subgroups of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$; for instance, $< 0 > \subseteq < p_1^{m_1-1} \times p_2^{m_2} > \subseteq < p_1^{m_1-2} \times p_2^{m_2} > \subseteq \ldots \subseteq < p_2^{m_2} > \subseteq < p_2^{m_2-1} > \subseteq < p_2^{m_2-2} > \subseteq \ldots \subseteq < p_2 > \subseteq < 1 >$. We obtain all other series of subgroups similarly. Now, gluing all maximal series, we obtain the lattice graph of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$, as shown in Figure 8.

From the construction, we reach at the results:

Proposition 1.
*(**a**) $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$ is planar.*
*(**b**) Length of maximal series of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$ is $(m_1 + m_2 + 1)$.*
*(**c**) The diameter of lattice graph of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$ is $m_1 + m_2$.*

Proof.
(**a**) This is obvious from the construction of $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$.
(**b**) Again, this is obvious; see a typical maximal series: $< 0 > \subseteq < p_1^{m_1-1} \times p_2^{m_2} > \subseteq < p_1^{m_1-2} \times p_2^{m_2} > \subseteq \ldots \subseteq < p_2^{m_2} > \subseteq < p_2^{m_2-1} > \subseteq < p_2^{m_2-2} > \subseteq \ldots \subseteq < p_2 > \subseteq < 1 >$.
(**c**) In this case, the diameter is exactly one less than the number of elements in a maximal series. □

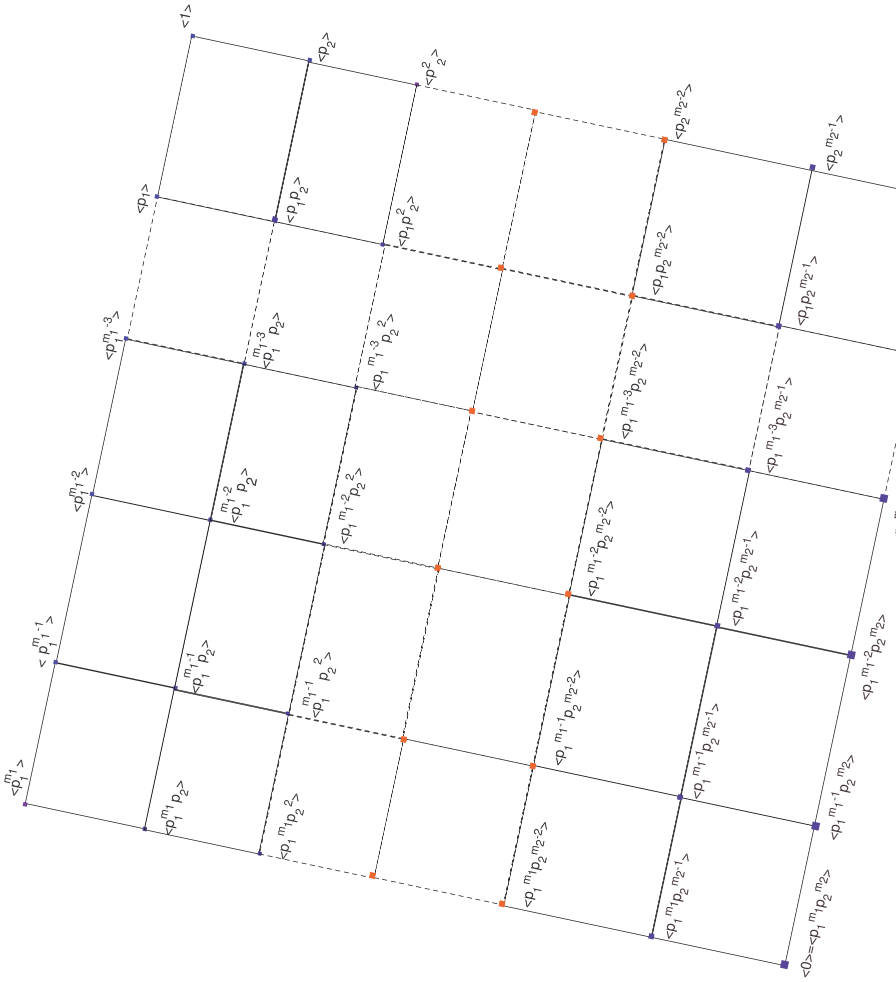

Figure 8. $L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}})$.

2.3. On the Lattice Graph of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$

The set of subgroups of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$ is $H(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}) = \{(< p_1^{m_1}\times p_2^{m_2}\times p_3 >, < p_1^{m_1-1}\times p_2^{m_2}\times p_3 >, \ldots, < p_1\times p_2^{m_2}\times p_3 >, < p_2^{m_2}\times p_3 >\} \cup \{< p_1^{m_1}\times p_2^{m_2} >, < p_1^{m_1-1}\times p_2^{m_2} >, \ldots, < p_1^2\times p_2^{m_2} >, < p_1\times p_2^{m_2} >, < p_2^{m_2} >\} \cup \{< p_1^{m_1}\times p_2^{m_2-1}\times p_3 >, < p_1^{m_1-1}\times p_2^{m_2-1}\times p_3 >, \ldots, < p_1\times p_2^{m_2-1}\times p_3 >, < p_2^{m_2-1}\times p_3 >\} \cup \ldots \cup \{< p_1^{m_1}\times p_3 >, < p_1^{m_1-1}\times p_3 >, \ldots, < p_1\times p_3 >, < p_3 >\} \cup \{< p_1^{m_1} >, < p_1^{m_1-1} >, \ldots, < p_1^2 >, < p_1 >, < 1 >)\}$. The subgroup $< 0 > = < p_1^{m_1}\times p_2^{m_2}\times p_3 >$ is contained immediately in three subgroups $< p_1^{m_1}\times p_2^{m_2-1}\times p_3 >$, $< p_1^{m_1-1}\times p_2^{m_2}\times p_3 >$ and $< p_1^{m_1}\times p_2^{m_2} >$. The first two of these are contained in $< p_1^{m_1-1}\times p_2^{m_2-1}\times p_3 >$. The subgroup $< p_1^{m_1-1}\times p_2^{m_2}\times p_3 >$ is contained in $< p_1^{m_1-2}\times p_2^{m_2}\times p_3 >$ which is further contained in two subgroups $< p_1^{m_1-2}\times p_2^{m_2-1}\times p_3 >$ and $< p_1^{m_1-3}\times p_2^{m_2}\times p_3 >$. The subgroup $< p_1^{m_1-3}\times p_2^{m_2}\times p_3 >$ is contained in other two subgroups of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$. The process will continue till we receive a subgroup $< p_1\times p_2^{m_2}\times p_3 >$. It is contained in three subgroups $< p_1\times p_2^{m_2} >$, $< p_1\times p_2^{m_2-1}\times p_3 >$ and $< p_2^{m_2}\times p_3 >$. The subgroup $< p_2^{m_2}\times p_3 >$ is contained in $< p_2^{m_2} >$. It is contained in $< p_2^{m_2-1} >$. Continuing this process till we obtain $< p_2 >$ which is contained in $< 1 > = \mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$, and we will receive series of subgroups of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$; for instance, $< 0 > \subseteq < p_1^{m_1-1}\times p_2^{m_2}\times p_3 > \subseteq < p_1^{m_1-2}\times p_2^{m_2}\times p_3 > \subseteq \ldots \subseteq < p_2^{m_2}\times p_3 > \subseteq < p_2^{m_2} > \subseteq < p_2^{m_2-1} > \subseteq \ldots \subseteq < p_2 > \subseteq < 1 >$. Similarly, all the remaining subgroups will form other series of subgroups. At the end, gluing the vertices occurring in all maximal series, we obtain the required graph, as shown in Figure 9.

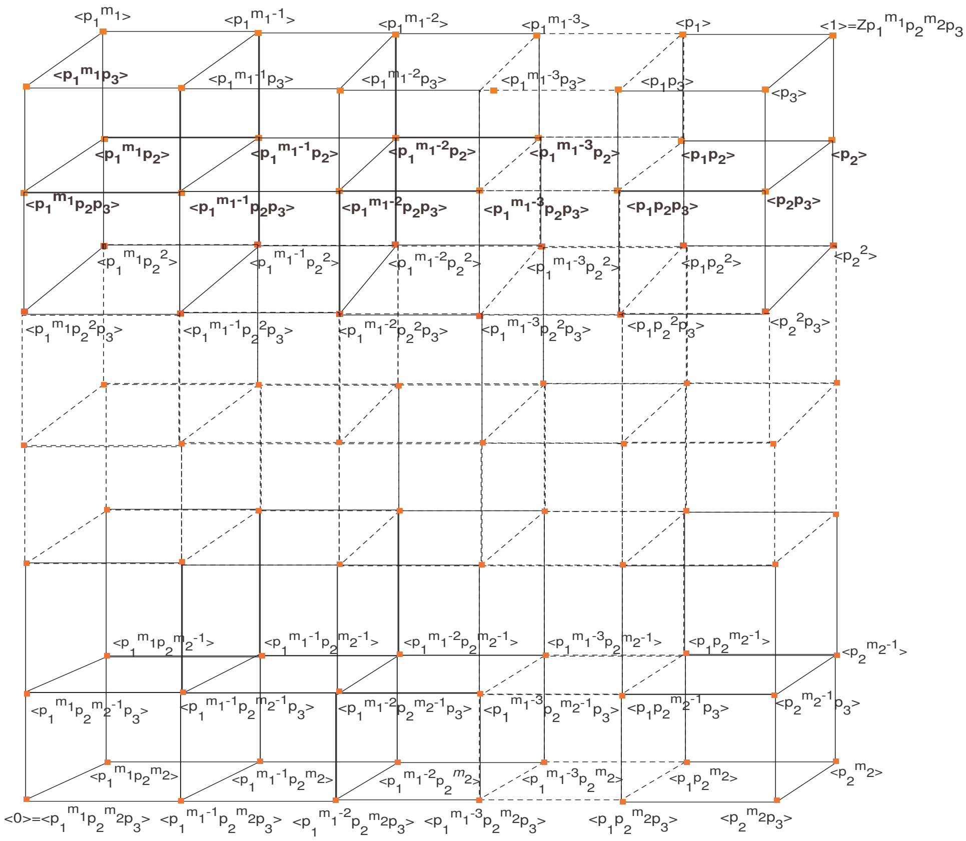

Figure 9. $L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3})$.

Proposition 2. *The crossing number of* $L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3})$ *is* $2m_1m_2$.

Proof. The proof is clear from the Table 1:

Table 1. Crossing numbers.

m_1, m_2	$1,1$	$2,1$	$3,1$	$\dots$	$m_1,1$	$m_1,2$	$m_1,3$	$\dots$	m_1,m_2
$cr(G)$	2	4	6	$\dots$	$2m_1$	$4m_1$	$6m_1$	$\dots$	$2(m_1)(m_2)$

□

The following proposition relates the number of crossings and number of subgroups of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$.

Proposition 3. $crL(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}) = |V(L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}))| - 2(m_1+m_2+1)$.

Proof. We proceed as follows: $|V(L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}))| = 2(m_1+1)(m_2+1) = 2(m_1m_2+m_1+m_2+1) = 2m_1m_2 + 2(m_1+m_2+1)$. However, as in Proposition 2.4, $(crL(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3})) = 2(m_1)(m_2)$ we have, $|V(L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}))| = \mathrm{cr}L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}) + 2(m_1+m_2+1)$, which leads to the given result. □

Similarly, the following corollary provides a link between number of crossings and number of subgroups of $\mathbb{Z}_{p_1^{m_1}\times p_2\times p_3}$.

Corollary 2. $\mathrm{cr}L(\mathbb{Z}_{p_1^{m_1}\times p_2\times p_3}) = \frac{|V(L(\mathbb{Z}_{p_1^{m_1}\times p_2\times p_3})|}{2} - 2$.

Proof. The result follows immediately from the relation $|V(L(\mathbb{Z}_{p_1^{m_1}\times p_2\times p_3}))| = 4(m_1+1) = 2(2m_1+2) = 2(\mathrm{cr}L(\mathbb{Z}_{p_1^{m_1}\times p_2\times p_3})+2)$. □

Proposition 4. *The length of maximal series of* $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$ *is* m_1+m_2+2.

Proof. A typical maximal series of subgroups of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$ is $<0> \subseteq < p_1^{m_1-1}\times p_2^{m_2}\times p_3 > \subseteq < p_1^{m_1-2}\times p_2^{m_2}\times p_3 > \subseteq \ldots \subseteq < p_2^{m_2}\times p_3 > \subseteq < p_2^{m_2} > \subseteq < p_2^{m_2-1} > \subseteq < p_2^{m_2-2} > \subseteq \ldots \subseteq < p_2 > \subseteq < 1 >$, which consists of m_1+m_2+2. Since each maximal series has the same number of elements, we are done. □

Proposition 5. *The diameter of* $L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3})$ *is* m_1+m_2+1.

Proof. The diameter is one less than the length of the maximal series of $L(\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3})$. □

Proposition 6. *Girth of the lattice graphs of* $\mathbb{Z}_n$, $n = p_1\times p_2\times \ldots\times p_{m-1}\times p_m$, $m\neq 1$, $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}}$ *and* $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$ *is* 4.

Proof. If the degree of each vertex of a graph is at least2, then then there exists a cycle. It is clear that the degree of each vertex of L($\mathbb{Z}_n$) is 2 except $m=1$, thus there exists a cycle and the length of shortest cycle is 4. Thus, the girth of L($\mathbb{Z}_n$) is 4 except $m=1$. Similarly, the girth of $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}}$ and $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3}$ is 4. □

Finally, the information about the diameter and girth is enclosed in the most general result:

Theorem 2 (The Main Theorem)**.** *If G is the group* $\mathbb{Z}_{p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}}\times p_r^{m_r}}$, *then:*

(a) *The length of the maximal series of G is* $1+\sum\limits_{i=1}^{r} m_i$.

(b) *The diameter of* $L(G)$ *is* $\sum\limits_{i=1}^{r} m_i$.

(c) $g\big(L(G)\big) = \begin{cases} \infty & \text{when } r=1 \\ 4 & \text{when } r\neq 1 \end{cases}$

Proof.
(a) The set of subgroups of G is $H(G) = \{< p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}}\times p_r^{m_r} >, < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}}\times p_r^{m_r-1} >, \ldots, < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}}\times p_r >\}\cup\{< p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}-1}\times p_r >, < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}-2}\times p_r >, \ldots, < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}\times p_r >\}\cup\ldots\cup\{< p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3-1}\times\ldots\times p_{r-1}\times p_r >, < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3-2}\times\ldots\times p_{r-1}\times p_r >, \ldots, < p_1^{m_1}\times p_2^{m_2}\times p_3\times\ldots\times p_{r-1}\times p_r >\}\cup\{< p_1^{m_1}\times p_2^{m_2-1}\times p_3\times\ldots\times p_{r-1}\times p_r >, \ldots, < p_1^{m_1}\times p_2\times p_3\times\ldots\times p_{r-1}\times p_r >\}\cup\{< p_1^{m_1-1}\times p_2\times p_3\times\ldots\times p_{r-1}\times p_r >, \ldots, < p_1\times p_2\times p_3\times\ldots\times p_{r-1}\times p_r >\}\cup\{< p_1\times p_2\times p_3\times\ldots\times p_{r-1} >, \ldots, < p_1\times p_2 >, < p_1 >, < 1 >\}$. This set consists of $(m_1+1)(m_2+1)(m_3+1)\ldots(m_{r-1}+1)(m_r+1)$ elements; this number is actually the number of divisors of the order of G. Now, take elements of $H(G)$ and form a (maximal) series from $<0>$ to $<1>$. A typical series of such kind is:

$$\begin{aligned}
&<0> = < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}}\times p_r^{m_r} > \\
&\subseteq < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}}\times p_r^{m_r-1} > \\
&\subseteq \ldots \subseteq < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}}\times p_r > \\
&\subseteq < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}-1}\times p_r > \\
&\subseteq < p_1^{m_1}\times p_2^{m_2}\times p_3^{m_3}\times\ldots\times p_{r-1}^{m_{r-1}-2}\times p_r >
\end{aligned}$$

$\subseteq \ldots \subseteq < p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3} \times \ldots \times p_{r-1} \times p_r >$
$\subseteq \ldots \subseteq < p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3-1} \times \ldots \times p_{r-1} \times p_r >$
$\subseteq < p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3-2} \times \ldots \times p_{r-1} \times p_r >$
$\subseteq \ldots \subseteq < p_1^{m_1} \times p_2^{m_2} \times p_3 \times \ldots \times p_{r-1} \times p_r >$
$\subseteq < p_1^{m_1} \times p_2^{m_2-1} \times p_3 \times \ldots \times p_{r-1} \times p_r >$
$\subseteq \ldots < p_1^{m_1} \times p_2 \times p_3 \times \ldots \times p_{r-1} \times p_r >$
$\subseteq < p_1^{m_1-1} \times p_2 \times p_3 \times \ldots \times p_{r-1} \times p_r >$
$\subseteq \ldots \subseteq < p_1 \times p_2 \times p_3 \times \ldots \times p_{r-1} \times p_r >$
$\subseteq < p_1 \times p_2 \times p_3 \times \ldots \times p_{r-1} >$
$\subseteq \ldots \subseteq < p_1 \times p_2 > \subseteq < p_1 > \subseteq < 1 >$.

This maximal series contains $m_r + m_{r-1} - 1 + m_{r-2} - 1 + \ldots + m_3 - 1 + m_2 - 1 + m_1 - 1 + r = m_1 + m_2 + m_3 + \ldots + m_{r-1} + m_r + (-1)(r-1) + r = \sum_{i=1}^{r} m_i + 1$ subgroups. Since each maximal series contains exactly the same number of subgroups, the length of maximal series is $1 + \sum_{i=1}^{r} m_i$.

(b) It now follows that the diameter of $L(G)$ is $\sum_{i=1}^{r} m_i$, which, in our case, is actually one less than the length of maximal series of G. This completes the proof.
(c) **Case I**. ($r \neq 1$)

Consider two typical maximal series of subgroups of G:

$< 0 > = < p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq < p_1^{m_1-1} \times p_2^{m_2} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq < p_1^{m_1-1} \times p_2^{m_2-1} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq < p_1^{m_1-1} \times p_2^{m_2-1} \times p_3^{m_3-1} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq \ldots \subseteq < p_1^{m_1-1} \times p_2^{m_2-1} \times p_3^{m_3-1} \times \ldots \times p_{r-1}^{m_{r-1}-1} \times p_r^{m_r-1} > \subseteq < p_1^{m_1-2} \times p_2^{m_2-1} \times p_3^{m_3-1} \times \ldots \times p_{r-1}^{m_{r-1}-1} \times p_r^{m_r-1} > \subseteq \ldots \subseteq < p_1 \times p_2 \times p_3 \times \ldots \times p_{r-1} \times p_r > \subseteq \ldots \subseteq < p_1 p_2 > \subseteq < p_1 > \subseteq < 1 >$.

and

$< 0 > = < p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq < p_1^{m_1} \times p_2^{m_2-1} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq < p_1^{m_1-1} \times_2^{m_2-1} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq < p_1^{m_1-1} \times p_2^{m_2-1} \times p_3^{m_3-1} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} > \subseteq \ldots \subseteq < p_1^{m_1-1} \times p_2^{m_2-1} \times p_3^{m_3-1} \times \ldots \times p_{r-1}^{m_{r-1}-1} \times p_r^{m_r-1} > \subseteq < p_1^{m_1-1} \times p_2^{m_2-2} \times p_3^{m_3-1} \times \ldots \times p_{r-1}^{m_{r-1}-1} \times p_r^{m_r-1} > \subseteq \ldots \subseteq < p_1 \times p_2 \times p_3 \times \ldots \times p_{r-1} \times p_r > \subseteq \ldots \subseteq < p_1 p_2 > \subseteq < p_1 > \subseteq < 1 >$.

On gluing these maximal series in this a way, the subgroup $< 0 >$ is contained immediately in two subgroups: $H_1 = < p_1^{m_1-1} \times p_2^{m_2} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} >$ and $H_2 = < p_1^{m_1} \times p_2^{m_2-1} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} >$. Both these are further contained in $H_3 = < p_1^{m_1-1} \times p_2^{m_2-1} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r} >$, and we obtain a closed path $< 0 > \rightarrow H_1 \rightarrow H_3 \rightarrow H_2 \rightarrow < 0 >$. We can split this closed path in two closed paths, $< 0 > \rightarrow H_1 \rightarrow H_2 \rightarrow < 0 >$ and $H_1 \rightarrow H_3 \rightarrow H_2 \rightarrow H_1$. For if $H_1 \subseteq H_2$, we receive a contradiction that there does not exist a closed path of length 3.

Similarly, gluing the vertices occurring in all maximal series, we obtain a graph in which one vertex is joined at least with two vertices which are further joined with another vertex, but these two vertices are not joined with each other. This confirms that the length of shortest cycle is 4.

Case II. ($r = 1$)

In this case, since we receive just a finite path, $g(L(G)) = \infty$. $\square$

Theorem 3.
(a) *$L(\mathbb{Z}_{p_1 \times p_2 \times \cdots \times p_m})$ is Eulerian if m is even and is non-Eulerian if $m (> 1)$ is odd.*
(b) *$L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$ is non-Eulerian if $m_1 \neq 1, 2$ and $m_2 \neq 1$.*
(c) *$L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3})$ is non-Eulerian.*

Proof. The proof is given in the following Table 2:

Table 2. Degree vectors and graph types.

Lattice Graphs	m	Degree Vector	Graph Type
L($\mathbb{Z}_n$), $n = p_1 \times p_2 \times \ldots \times p_{m-1} \times p_m$	1	V_1 = [1 1]	Semi-Eulerian
	2	V_2 = [2 2 2 2]	Eulerian
	3	V_3 = [3 3 ... 8 times 3]	Non-eulerian
	4	V_4 = [4 4 ... 16 times 4]	Eulerian
	$\vdots$	$\vdots$	$\vdots$
	m	V_m = [m m ... 2^m times m]	Eulerian if $m = even$ Non-Eulerian if $m = odd$
	m_1, m_2		
L($\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$)	$1, 1$	V'_1 = [2 2 2 2]	Eulerian
	$2, 1$	V'_2 = [3 3 2 2 2 2]	Semi-Eulerian
	$3, 1$	V'_3 = [3 3 3 3 2 2 2 2]	Non-Eulerian
	$4, 1$	V'_4 = [3 3 3 3 3 3 2 2 2 2]	Non-Eulerian
	$\vdots$	$\vdots$	$\vdots$
	$m_1, 1$	V'_r = [3 3 ... $(2m_1 - 2)$ times 3 2 2 2 2]	Non-Eulerian
	$m_1, 2$	V'_s = [4 4 ... $(m_1 - 1)$ times 4 3 3 ... $(2m_1)$ times 3 2 2 2 2]	Non-Eulerian
	$\vdots$	$\vdots$	$\vdots$
	m_1, m_2	V'_t = [4 4 4 ...$((m_2 - 1) \times (m_1 - 1))$ times 4 3 3...($2m_1$ $+2m_2 - 4$) times 3 2 2 2 2]	Non-Eulerian When $m_1 \neq 1, 2$ and $m_2 \neq 1$
L($\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3}$)	$1, 1$	V''_1 = [3 3 3 3 3 3 3 3]	Non-Eulerian
	$2, 1$	V''_2 = [4 4 4 4 3 3 3 3 3 3 3 3]	Non-Eulerian
	$\vdots$	$\vdots$	$\vdots$
	$m_1, 1$	V''_r = [4 4 ... $(4m_1 - 4)$ times 4 3 3 3 3 3 3 3 3]	Non-Eulerian
	$\vdots$	$\vdots$	$\vdots$
	m_1, m_2	V''_t = [5 5 ... $((m_2 - 1) \times (2m_1 - 2))$ times 5 4 4 4 ... $(4m_1 + 4m_2 - 8)$ times 4 ... 3 3 3 3 3 3 3 3]	Non-Eulerian $\forall m_1, m_2$

□

3. Conclusions

In this article, we discuss some metrical aspects of the lattice graphs of some families of finite groups. We obtain the diameter and girth as well as many other aspects of these lattice graphs. Along with many other results, the following are the main contributions of this article.

Theorem 4 (The Main Theorem)**.** *If G is the group $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r}}$, then:*

(a) The length of the maximal series of G is $1 + \sum_{i=1}^{r} m_i$.

(b) The diameter of $L(G)$ is $\sum_{i=1}^{r} m_i$.

(c) $g(L(G)) = \begin{cases} \infty & when\ r = 1 \\ 4 & when\ r \neq 1 \end{cases}$

and

Theorem 5.
(a) *$L(\mathbb{Z}_{p_1 \times p_2 \times \cdots \times p_m})$ is Eulerian if m is even and is non-Eulerian if $m(>1)$ is odd.*
(b) *$L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$ is non-Eulerian if $m_1 \neq 1, 2$ and $m_2 \neq 1$.*
(c) *$L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3})$ is non-Eulerian.*

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