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Common Fixed Point Results for Rational $(\alpha, \beta)_{\varphi}$-$m\omega$ Contractions in Complete Quasi Metric Spaces

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Abstract: The ω-distance mapping is one of the important tools that can be used to get new contractions in fixed point theory. The aim of this paper is to use the concept of modified ω-distance mapping to introduce the notion of rational $(\alpha, \beta)_{\varphi}$-$m\omega$ contraction. We utilize our new notion to construct and formulate many fixed point results for a pair of two mappings defined on a nonempty set A. Our results modify many existing known results. In addition, we support our work by an example.

Keywords: fixed point; quasi metric; contraction mappings

1. Introduction

The Banach contraction principle [1] is one of the most famous results in the setting of fixed point theory. Subsequently, many generalizations and modifications were studied in many directions by many authors; see [2–17].

Wilson [18] defined the quasi metric space as follows:

Definition 1. *[18] For $A \neq \emptyset$, let $q : A \times A \to [0, +\infty)$ satisfy:*

(i) $q(s_1, s_2) = 0$ *if and only if* $s_1 = s_2$ *and*
(ii) $q(s_1, s_2) \leq q(s_1, s_3) + q(s_3, s_2)$ *for all* $s_1, s_2, s_3 \in A$.

Then, we call q a quasi metric on A and (A, q) a quasi metric space.

Define $q_m : A \times A \to [0, +\infty)$ by:

$$q_m(j_1, j_2) = \max\{q(j_1, j_2), q(j_2, j_1)\}.$$

Then, (q_m, A) is a metric space.

From now on, we let (A, q) stand for a quasi metric space and (j_k) stand for a sequence in (A, q).

Definition 2. *[15,19] We say that a sequence (j_n) in A converges to $j \in A$ iff* $\lim_{k\to\infty} q(j_k, j) = \lim_{k\to\infty} q(j, j_k) = 0$.

Definition 3. *[19]*

(i) *We call (j_k) left Cauchy if for each $\gamma > 0$, there exists a positive integer i such that $q(j_k, j_l) \leq \gamma$ for all $k \geq l > i$.*

(ii) *We call (j_k) right Cauchy if for each $\gamma > 0$, there exists a positive integer i such that $q(j_k, j_l) \leq \gamma$ for all $l \geq k > i$.*

Definition 4. *[15,19] We call (j_k) Cauchy if for each $\gamma > 0$, there exists a positive integer i such that $q(j_k, j_l) \leq \gamma$ for all $l, k > i$.*

Note that (j_k) is Cauchy if (j_k) is a right and left Cauchy.

Definition 5. *[19] (A, q) is complete iff every Cauchy sequence in A is convergent.*

Alegre and Marin [20] introduced the notion of modified ω-distance mappings as follows:

Definition 6. *[20] A function $p : A \times A \to [0, +\infty)$ is called modified ω-distance (mω-distance) on (A, q) if p satisfies:*

(W1) $p(c_1, c_2) \leq p(c_1, c_3) + p(c_3, c_2)$ *for all* $c_1, c_2, c_3 \in A$,
(W2) *for any* $c \in A$, $p(c, .) : A \to [0, +\infty)$ *is lower semi-continuous, and*
(mW3) *for each* $\varepsilon > 0$, *there exists* $\mu > 0$ *such that if* $p(c_1, c_2) \leq \mu$ *and* $p(c_2, c_3) \leq \mu$, *then* $q(c_1, c_3) \leq \varepsilon$ *for all* $c_1, c_2, c_3 \in A$.

Definition 7. *[20] An mω-distance function $p : A \times A \to [0, \infty)$ is called strong mω-distance on (A, q) if:*

(mW2) *for any* $c \in A$, *then* $p(., c) : A \to [0, \infty)$ *is lower semi-continuous.*

Remark 1. *[20] Every quasi metric q on A is an mω-distance.*

Lemma 1. *[7] Let p be an mω-distance on (A, q). Let (j_k) be a sequence in A and $(\zeta_k), (\xi_k)$ be two nonnegative sequences converging to zero. Then, we have:*

(i) *If* $p(j_k, j_l) \leq \zeta_k$ *for any* $k, l \in \mathbb{N}$ *with* $l \geq k$, *then* (j_k) *is right Cauchy.*
(ii) *If* $p(j_k, j_l) \leq \xi_l$ *for any* $k, l \in \mathbb{N}$ *with* $k \geq l$, *then* (j_k) *is left Cauchy.*

Remark 2. *[7] From Lemma 1, we conclude that if $\lim_{k,l \to \infty} p(j_k, j_l) = 0$, then (j_k) is Cauchy in (A, q).*

Definition 8. *[21] A function $\varphi : [0, \infty) \to [0, \infty)$ is called a c-comparison if φ satisfies:*

(i) φ *is nondecreasing and*
(ii) $\sum_{k=0}^{\infty} \varphi^k(\mu) < \infty$ *for all* $\mu \geq 0$.

Remark 3. *If φ is a c-comparison, then $\varphi(\mu) < \mu \ \forall \mu > 0$ and $\varphi(0) = 0$.*

In the rest of this paper, by α and β, we mean functions from $A \times A$ into $[0, +\infty)$.

Definition 9. *[22] A mapping $h_1 : A \to A$ is α-admissible if for $i, j \in A$,*

$$\alpha(i, j) \geq 1 \Longrightarrow \alpha(h_1 i, h_1 j) \geq 1.$$

Definition 10. *[23] A mapping $h_1 : A \to A$ is called triangular α-admissible if:*

(i) h_1 *is* α*-admissible,*
(ii) *For all* $i, j, k \in A$,

$$\alpha(i, j) \geq 1 \text{ and } \alpha(j, k) \geq 1 \Longrightarrow \alpha(i, k) \geq 1.$$

Definition 11. *[9] For the two mappings $h_1, h_2 : A \to A$, we say that (h_1, h_2) is α-admissible if for all $i, j \in A$,*

$$\alpha(i, j) \geq 1 \Longrightarrow \alpha(h_1 i, h_2 j) \geq 1 \text{ and } \alpha(h_2 i, h_1 j) \geq 1.$$

Definition 12. *[24] For the two mappings $h_1, h_2 : A \to A$, we say that (h_1, h_2) is α, β-admissible if for all $i, j \in A$,*

$$\alpha(i, j) \geq \beta(i, j) \Longrightarrow \alpha(h_1 i, h_2 j) \geq \beta(h_1 i, h_2 j) \text{ and } \alpha(h_2 i, h_1 j) \geq \beta(h_2 i, h_1 j).$$

For more work on contractions involving admissible conditions, see [25–27].

2. Main Results

We start with the following definition.

Definition 13. *For the two mappings $h_1, h_2 : A \to A$, we call the pair (h_1, h_2) α, β-triangular admissible if:*

(i) (h_1, h_2) *is α, β-admissible, and*
(ii) *if $s_1, s_2, s_3 \in A$, $\alpha(s_1, s_2) \geq \beta(s_1, s_2)$, and $\alpha(s_2, s_3) \geq \beta(s_2, s_3)$, then $\alpha(s_1, s_3) \geq \beta(s_1, s_3)$.*

Example 1. *Let $A = [-1, 1]$. Define $h_1, h_2 : A \to A$ by $h_1 i = \frac{|i|}{1+|i|}$, $h_2 i = i$. Furthermore, define $\alpha, \beta : A \times A \to [0, \infty)$ via $\alpha(i, j) = e^{i+j}$ and $\beta(i, j) = e^{i+j^2}$. Then, (h_1, h_2) is α, β-triangular admissible.*

Proof. To show that (h_1, h_2) is α, β-admissible, given $i, j \in A$ such that $\alpha(i, j) \geq \beta(i, j)$, then $j \in [0, 1]$. Note that:

$$\alpha(h_1 i, h_2 j) = \alpha\left(\frac{|i|}{1+|i|}, j\right) = e^{\frac{|i|}{1+|i|}+j} \geq e^{\frac{|i|}{1+|i|}+j^2} = \beta(h_1 i, h_2 j)$$

and:

$$\alpha(h_2 i, h_1 j) = \alpha\left(i, \frac{|j|}{1+|j|}\right) = e^{i+\frac{|j|}{1+|j|}} \geq e^{1+(\frac{|j|}{1+|j|})^2} = \beta(h_2 i, h_1 j).$$

Now, given $i, j, k \in A$ such that $\alpha(i, j) \geq \beta(i, j)$ and $\alpha(j, k) \geq \beta(j, k)$, then $j, k \in [0, 1]$. Therefore, $\alpha(i, k) = e^{i+k} \geq e^{i+k^2} = \beta(i, k)$. □

Let $h_1, h_2 : A \to A$ be two mappings. By starting with the initial point $j_0 \in A$, we define the sequence (j_n) in A by $h_1 j_{2n} = j_{2n+1}$ and $h_2 j_{2n+1} = j_{2n+2}$. To facilitate our work, we call the above sequence (j_n) an (h_1, h_2)-sequence with initial point j_0.

Lemma 2. *For the mappings $h_1, h_2 : A \to A$, we assume the following conditions:*

1. *(h_1, h_2) is α, β-triangular admissible and*
2. *there exists $j_0 \in A$ such that $\alpha(h_1 j_0, h_2 h_1 j_0) \geq \beta(h_1 j_0, h_2 h_1 j_0)$ and $\alpha(h_2 h_1 j_0, h_1 j_0) \geq \beta(h_2 h_1 j_0, h_1 j_0)$.*

Then, the (h_1, h_2)-sequence (j_n) with initial point j_0, satisfies $\alpha(j_n, j_m) \geq \beta(j_n, j_m)$ for all $n, m \in \mathbb{N}$.

Proof. For $j_1 = h_1 j_0$ and $j_2 = h_2 j_1$, we get that:

$$\alpha(j_1, j_2) \geq \beta(j_1, j_2) \text{ and } \alpha(j_2, j_1) \geq \beta(j_2, j_1).$$

Since the pair (h_1, h_2) is α, β-triangular admissible, we have:

$$\alpha(h_2 j_1, h_1 j_2) \geq \beta(h_2 j_1, h_1 j_2) \text{ and } \alpha(h_1 j_2, h_2 j_1) \geq \beta(h_1 j_2, h_2 j_1).$$

Now, for $j_3 = h_1 j_2$, we have:

$$\alpha(j_2, j_3) \geq \beta(j_2, j_3) \text{ and } \alpha(j_3, j_2) \geq \beta(j_3, j_2),$$

Repeating the same process, we conclude that (j_n) satisfies:

$$\alpha(j_n, j_{n+1}) \geq \beta(j_n, j_{n+1}) \tag{1}$$

and:

$$\alpha(j_{n+1}, j_n) \geq \beta(j_{n+1}, j_n). \tag{2}$$

By the same process, we can prove that:

$$\alpha(j_{n+1}, j_{n+2}) \geq \beta(j_{n+1}, j_{n+2}). \tag{3}$$

Therefore,

$$\alpha(j_n, j_{n+2}) \geq \beta(j_n, j_{n+2}). \tag{4}$$

Hence, for $m > n$ with $m = n + t$ for some $t \in \mathbb{N}$, we get that:

$$\alpha(j_n, j_m) = \alpha(j_n, j_{n+t}) \geq \beta(j_n, j_{n+t}) = \beta(j_n, j_m).$$

From the definition of α, β-triangular admissible and Equations (1) and (2), we get that:

$$\alpha(j_n, j_m) \geq \beta(j_n, j_m) \text{ for all } n, m \in \mathbb{N}. \tag{5}$$

□

Next, we introduce the concept of the rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction.

Definition 14. *Let p be the modified ω-distance equipped on (A, q), and let $h_1, h_2 : A \to A$ be two self-mappings. Then, we call (h_1, h_2) a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction if there exists a c-comparison function φ such that for all $s_1, s_2 \in A$ with $\alpha(s_1, s_2) \geq \beta(s_1, s_2)$, then we have:*

$$p(h_1 s_1, h_2 s_2) \leq \varphi \max\left\{ p(s_1, h_1 s_1), p(s_2, h_2 s_2), \frac{p(s_1, h_1 s_1) p(s_2, h_2 s_2)}{1 + p(s_1, s_2)} \right\},$$

and:

$$p(h_2 s_1, h_1 s_2) \leq \varphi \max\left\{ p(s_1, h_2 s_1), p(s_2, h_1 s_2), \frac{p(s_1, h_2 s_1) p(s_2, h_1 s_2)}{1 + p(s_2, s_1)} \right\}.$$

For simplicity, we mean by $\mathbb{N}^*$ the set of all non-negative integers.

Lemma 3. *Let p be the modified ω-distance equipped on (A, q). Let $h_1, h_2 : A \to A$ be mappings and φ be a c-comparison. Suppose the following:*

(i) (h_1, h_2) *is α, β-triangular admissible,*
(ii) (h_1, h_2) *is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction, and*
(iii) *there exists $j_0 \in A$ such that:*

$$\alpha(h_1 j_0, h_2 h_1 j_0) \geq \beta(h_1 j_0, h_2 h_1 j_0) \text{ and } \alpha(h_2 h_1 j_0, h_1 j_0) \geq \beta(h_2 h_1 j_0, h_1 j_0).$$

If there exists $k \in \mathbb{N}^*$ such that $p(j_k, j_{k+1}) = 0$ or $p(j_{k+1}, j_k) = 0$, then j_k is a common fixed point of h_1 and h_2, where (j_n) is the (h_1, h_2)-sequence with initial point j_0.

Moreover, if $z \in A$ is a common fixed point of h_1 and h_2, then $p(z, z) = 0$.

Proof. As in Lemma 2, we have $\alpha(j_n, j_m) \geq \beta(j_n, j_m)\ \forall\, n, m \in \mathbb{N}^*$.

Assume that $p(j_k, j_{k+1}) = 0$. If k is even, then $k = 2t$ for some $t \in \mathbb{N}^*$. Therefore, we have $p(j_{2t}, j_{2t+1}) = 0$.

Since (h_1, h_2) is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction, we have:

$$\begin{aligned} p(j_{2t+1}, j_{2t+2}) &= p(h_1 j_{2t}, h_2 j_{2t+1}) \\ &\leq \varphi \max\left\{ p(j_{2t}, h_1 j_{2t}), p(j_{2t+1}, h_1 j_{2t+1}), \frac{p(j_{2t}, h_1 j_{2t}) p(j_{2t+1}, h_2 j_{2t+1})}{1 + p(j_{2t}, j_{2t+1})} \right\} \\ &= \varphi \max\left\{ p(j_{2t}, j_{2t+1}), p(j_{2t+1}, j_{2t+2}), \frac{p(j_{2t}, j_{2t+1}) p(j_{2t+1}, j_{2t+2})}{1 + p(j_{2t}, j_{2t+1})} \right\} \\ &= \varphi \max\{p(j_{2t}, j_{2t+1}), p(j_{2t+1}, j_{2t+2})\}. \end{aligned} \tag{6}$$

Hence, $p(j_{2t+1}, j_{2t+2}) \leq \varphi p(j_{2t+1}, j_{2t+2})$.
Using the properties of the c-comparison function φ, we get:

$$p(j_{2t+1}, j_{2t+2}) = 0. \tag{7}$$

By $(W1)$ of Definition 6, we get:

$$p(j_{2t}, j_{2t+2}) = 0. \tag{8}$$

Furthermore, using $(mW3)$ of Definition 6, we get:

$$q(j_{2t}, j_{2t+2}) = 0. \tag{9}$$

Therefore, $j_{2t} = j_{2t+2}$.
Now,

$$\begin{aligned} p(j_{2t+2}, j_{2t+1}) &= p(h_2 j_{2t+1}, h_1 j_{2t}) \\ &\leq \varphi \max\left\{ p(j_{2t+1}, j_{2t+2}), p(j_{2t}, j_{2t+1}), \frac{p(j_{2t+1}, j_{2t+2}) p(j_{2t}, j_{2t+1})}{1 + p(j_{2t}, j_{2t+1})} \right\} \\ &= \varphi \max\{p(j_{2t+1}, j_{2t+2}), p(j_{2t}, j_{2t+1})\}. \end{aligned} \tag{10}$$

Therefore,

$$p(j_{2t+2}, j_{2t+1}) = 0. \tag{11}$$

Using Equations (8), (9), and (11), we get $q(j_{2t}, j_{2t+1}) = 0$, and so, j_k is a common fixed point of h_1 and h_2.

Using the same process, we can show that j_k is a common fixed point of h_1 and h_2 whenever k is odd.

In a similar manner, we can show that j_k is a common fixed point of h_1 and h_2 if $p(j_{k+1}, j_k) = 0$.
Now, assume that $z \in A$ is a common fixed point of h_1 and h_2.
Since (h_1, h_2) is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction, we have:

$$\begin{aligned} p(z, z) &= p(h_2 z, h_1 z) \\ &\leq \varphi \max\left\{ p(z, z), p(z, z), \frac{p(z, z) p(z, z)}{1 + p(z, z)} \right\} \\ &= \varphi p(z, z). \end{aligned} \tag{12}$$

Hence, $p(z, z) = 0$. □

Now, we introduce our main result:

Theorem 1. *Let p be a modified ω-distance equipped on (A, q). Let $h_1, h_2 : A \to A$ be two mappings and φ be a c-comparison function. Assume the following hypotheses:*

1. *h_1 and h_2 are continuous,*
2. *(h_1, h_2) is α, β-triangular admissible,*
3. *(h_1, h_2) is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction, and*

4. *there exists $j_0 \in A$ such that:*

$$\alpha(h_1 j_0, h_2 h_1 j_0) \geq \beta(h_1 j_0, h_2 h_1 j_0) \text{ and } \alpha(h_2 h_1 j_0, h_1 j_0) \geq \beta(h_2 h_1 j_0, h_1 j_0).$$

Then, the (h_1, h_2)-sequence (j_n) with initial point j_0 converges to a unique common fixed point of h_1 and h_2 in A.

Proof. If there exists $n \in \mathbb{N}^*$ such that $p(j_n, j_{n+1}) = 0$ or $p(j_{n+1}, j_n) = 0$, then by Lemma 3, j_n is a common fixed point of h_1 and h_2. Therefore, we can assume that for all $n \in \mathbb{N}^*$, $p(j_n, j_{n+1}) \neq 0$ and $p(j_{n+1}, j_n) \neq 0$.

If n is odd, then $n = 2s + 1$ for some $s \in \mathbb{N}^*$.

Since the pair (h_1, h_2) is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction, we have:

$$\begin{aligned}
p(j_{2s+1}, j_{2s+2}) &= p(h_1 j_{2s}, h_2 j_{2s+1}) \\
&\leq \varphi \max\left\{ p(j_{2s}, h_1 j_{2s}), p(j_{2s+1}, h_1 j_{2s+1}), \tfrac{p(j_{2s}, h_1 j_{2s}) p(j_{2s+1}, h_2 j_{2s+1})}{1 + p(j_{2s}, j_{2s+1})} \right\} \\
&= \varphi \max\left\{ p(j_{2s}, j_{2s+1}), p(j_{2s+1}, j_{2s+2}), \tfrac{p(j_{2s}, j_{2s+1}) p(j_{2s+1}, j_{2s+2})}{1 + p(j_{2s}, j_{2s+1})} \right\} \\
&= \varphi \max\{ p(j_{2s}, j_{2s+1}), p(j_{2s+1}, j_{2s+2}) \} \\
&= \varphi p(j_{2s}, j_{2s+1}).
\end{aligned} \tag{13}$$

Furthermore, we have:

$$\begin{aligned}
p(j_{2s+2}, j_{2s+1}) &= p(h_2 j_{2s+1}, h_1 j_{2s}) \\
&\leq \varphi \max\left\{ p(j_{2s+1}, h_2 j_{2s+1}), p(j_{2s}, h_1 j_{2s}), \tfrac{p(j_{2s+1}, h_2 j_{2s+1}) p(j_{2s}, h_1 j_{2s})}{1 + p(j_{2s}, j_{2s+1})} \right\} \\
&= \varphi \max\left\{ p(j_{2s+1}, j_{2s+2}), p(j_{2s}, j_{2s+1}), \tfrac{p(j_{2s+1}, j_{2s+2}) p(j_{2s}, j_{2s+1})}{1 + p(j_{2s}, j_{2s+1})} \right\} \\
&= \varphi \max\{ p(j_{2s+1}, j_{2s+2}), p(j_{2s}, j_{2s+1}) \} \\
&= \varphi p(j_{2s}, j_{2s+1}).
\end{aligned} \tag{14}$$

Therefore, we have:

$$\max\{ p(j_{2s+1}, j_{2s+2}), p(j_{2s+2}, j_{2s+1}) \} \leq \varphi p(j_{2s}, j_{2s+1}).$$

Furthermore, if $n \in \mathbb{N}^*$ is even, we have:

$$\max\{ p(j_{2s+1}, j_{2s+2}), p(j_{2s+2}, j_{2s+1}) \} \leq \varphi p(j_{2s}, j_{2s+1}).$$

Therefore, for all $n \in \mathbb{N}^*$, we get:

$$\max\{ p(j_n, j_{n+1}), p(j_{n+1}, j_n) \} \leq \varphi p(j_{n-1}, j_n). \tag{15}$$

Thus,

$$p(j_n, j_{n+1}) \leq \varphi p(j_{n-1}, j_n) \leq \varphi^2 p(j_{n-2}, j_{n-1}) \leq \cdots \leq \varphi^n p(j_0, j_1). \tag{16}$$

Furthermore, we have:

$$p(j_{n+1}, j_n) \leq \varphi^n p(j_0, j_1). \tag{17}$$

Now, we claim that (j_n) is Cauchy.

To show that (j_n) is a left Cauchy sequence, let $n, m \in \mathbb{N}$ with $n > m$.

Using Equation 17 and $(W1)$ of Definition 6, we have:

$$\begin{aligned} p(j_n, j_m) &\leq p(j_n, j_{n-1}) + p(j_{n-1}, j_{n-2}) + \cdots + p(j_{m+1}, j_m) \\ &\leq \varphi p(j_{n-2}, j_{n-1}) + \varphi p(j_{n-3}, j_{n-2}) + \cdots + \varphi p(j_{m-1}, j_m) \\ &= \sum_{k=m}^{n-1} \varphi p(j_{k-1}, j_k) \\ &\leq \sum_{k=m}^{n-1} \varphi^k p(j_0, j_1). \end{aligned} \tag{18}$$

Since φ is a c-comparison function, then $\sum_{k=m}^{\infty} \varphi^k p(j_0, j_1)$ is convergent. Thus, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that:

$$\sum_{k=m}^{\infty} \varphi^k p(j_0, j_1) < \epsilon \text{ for all } m > N. \tag{19}$$

Hence, for $n > m \geq N$, we have:

$$p(j_n, j_m) \leq \sum_{k=m}^{n-1} \varphi^k p(j_0, j_1) \leq \sum_{k=m}^{\infty} \varphi^k p(j_0, j_1) < \epsilon. \tag{20}$$

By Lemma 1, (j_n) is a left Cauchy sequence.

To show that (j_n) is a right Cauchy sequence, let $n, m \in \mathbb{N}$ with $n < m$. Using Equation (16) and (W1) of Definition 6, we get:

$$\begin{aligned} p(j_n, j_m) &\leq p(j_n, j_{n+1}) + p(j_{n+1}, j_{n+2}) + \cdots + p(j_{m-1}, j_m) \\ &= \sum_{k=n}^{m-1} p(j_k, j_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \varphi^k p(j_0, j_1). \end{aligned} \tag{21}$$

Hence, for $m > n \geq N$, we have:

$$p(j_n, j_m) \leq \sum_{k=n}^{m-1} \varphi^k p(j_0, j_1) \leq \sum_{k=n}^{\infty} \varphi^k p(j_0, j_1) < \epsilon. \tag{22}$$

By Lemma 1, (j_n) is a right Cauchy sequence.

Hence, (j_n) is Cauchy. The completeness property of (A, q) implies $j^* \in A$, and so:

$$\lim_{n\to\infty} q(j_{2n}, j^*) = \lim_{n\to\infty} q(j^*, j_{2n}) = 0. \tag{23}$$

Since h_1 is continuous, we have:

$$\begin{aligned} \lim_{n\to\infty} q(h_1 j_{2n}, h_1 j^*) &= \lim_{n\to\infty} q(h_1 j^*, h_1 j_{2n}) \\ \lim_{n\to\infty} q(j_{2n+1}, h_1 j^*) &= \lim_{n\to\infty} q(h_1 j^*, j_{2n+1}) \\ &= 0. \end{aligned}$$

By uniqueness of the limit, we get $j^* = h_1 j^*$. In a similar manner, we can show that $j^* = h_2 j^*$. Consequently, j^* is a common fixed point of h_1 and h_2.

To prove the uniqueness of the common fixed point of h_1 and h_2, if $z \in A$ is a common fixed point of h_1 and h_2, then by Lemma 3, we get that $p(z, z) = 0$.

Assume there exists $v \in A$ such that $v = h_2 v = h_1 v$.

Now,

$$\begin{aligned} p(v,z) &= p(h_1 v, h_2 z) \\ &\leq \varphi \max\left\{ p(v,h_1 v), p(z,h_2 z), \tfrac{p(v,h_1 v)p(z,h_2 z)}{1+p(v,z)} \right\} \\ &= \varphi \max\left\{ p(v,v), p(z,z), \tfrac{p(v,v)p(z,z)}{1+p(v,z)} \right\} \\ &= 0. \end{aligned}$$

Hence, $p(v,z) = 0$. Since $p\,(z,z) = 0$, then by using $(mW3)$ of Definition 6, we get $q\,(v,z) = 0$, and so, $v = z$. □

Next, we introduce an example to show the usability of our work:

Example 2. *Let $A = \mathbb{R}$ be the set of real numbers.*

Define $\alpha, \beta : A \times A \to [0,\infty)$ as follows:

For all $s_1, s_2 \in A$, let $\alpha(s_1,s_2) = e^{|s_1|+|s_2|}$ and $\beta(s_1,s_2) = e^{|s_1+s_2|}$.

Furthermore, define h_1, h_2 on A as follows: $h_1(a) = \frac{1}{2}a$ and $h_2(b) = \frac{1}{3}b$.

Define $q : A \times A \to [0,\infty)$ via:

$$q(s_1,s_2) = \begin{cases} 0 & \text{if } s_1 = s_2, \\ \\ |s_1| + \frac{3}{2}|s_2| & \text{if } s_1 \neq s_2. \end{cases}$$

Define $p : A \times A \to [0,\infty)$ via $p(s_1,s_2) = 2|s_1| + 3|s_2|$. Furthermore, define $\varphi : [0,\infty) \to [0,\infty)$ via $\varphi(\mu) = \frac{6}{7}\mu$.

Then:

(1) (A,q) is a complete quasi metric space,
(2) (A,p) is the $m\omega$-distance on q,
(3) φ is a c-comparison function,
(4) S and T are continuous,
(5) (S,T) is α,β-triangular admissible,
(6) (S,T) is a rational $(\alpha,\beta)_\varphi$-$m\omega$ contraction.

Proof. The proofs of (1)–(4) are obvious.

Now, we prove (5): Given $s_1, s_2, s_3 \in A$, then:

$$\alpha(s_1,s_2) = e^{|s_1|+|s_2|} \geq e^{|s_1+s_2|} = \beta(s_1,s_2).$$

Therefore,

$$\alpha(h_1 s_1, h_2 s_2) = e^{\frac{1}{2}|s_1| + \frac{1}{3}|s_1|} \geq e^{|\frac{1}{2}s_1 + \frac{1}{3}s_2|} = \beta(h_1 s_1, h_2 s_2)$$

and:

$$\alpha(h_2 s_1, h_1 s_2) = e^{\frac{1}{3}|s_1| + \frac{1}{2}|s_2|} \geq e^{|\frac{1}{3}s_1 + \frac{1}{2}s_2|} = \beta(h_2 s_1, h_1 s_2).$$

Furthermore, if $\alpha(s_1,s_2) \geq \beta(s_1,s_2)$ and $\alpha(s_2,s_3) \geq \beta(s_2,s_3)$, then $\alpha(s_1,s_3) \geq \beta(s_1,s_3)$.

To prove (6), given $s_1, s_2 \in A$ with $\alpha(s_1, s_2) \geq \beta(s_1, s_2)$, then, we get that:

$$\begin{aligned}
p(h_1 s_1, h_2 s_2) &= |s_1| + |s_2| \\
&\leq \frac{6}{7}\max\left\{\frac{7}{2}\Big|s_1\Big|, 3\Big|s_2\Big|\right\} \\
&= \frac{6}{7}\max\{p(s_1, h_1 s_1), p(s_2, h_2 s_2)\} \\
&= \frac{6}{7}\max\left\{p(s_1, \frac{1}{2}s_1), p(s_2, \frac{1}{3}s_2)\right\} \\
&\leq \frac{6}{7}\max\left\{p(s_1, h_1 s_1), p(s_2, h_2 s_2), \frac{p(s_1, h_1 s_1)p(s_2, h_2 s_2)}{1 + p(s_1, s_2)}\right\} \\
&= \varphi\max\left\{p(s_1, h_1 s_1), p(s_2, h_2 s_2), \frac{p(s_1, h_1 s_1)p(s_2, h_2 s_2)}{1 + p(s_1, s_2)}\right\}.
\end{aligned}$$

We divide the proof of the second inequality of Definition 14 into two cases.
Case (1): If $|s_1| \leq |s_2|$, then we get that:

$$\begin{aligned}
p(h_2 s_1, h_1 s_2) &= \frac{2}{3}\Big|s_1\Big| + \frac{3}{2}\Big|s_2\Big| \\
&\leq \frac{13}{6}\Big|s_2\Big| \\
&\leq \frac{6}{7}\left(\frac{7}{2}\Big|s_2\Big|\right) \\
&\leq \frac{6}{7}\max\left\{3\Big|s_1\Big|, \frac{7}{2}\Big|s_2\Big|\right\} \\
&= \frac{6}{7}\max\{p(s_1, h_2 s_1), p(s_2, h_1 s_2)\} \\
&\leq \frac{6}{7}\max\left\{p(s_1, h_2 s_1), p(s_2, h_1 s_2), \frac{p(s_1, h_2 s_1)p(s_2, h_1 s_2)}{1 + p(s_2, s_1)}\right\} \\
&= \varphi\max\left\{p(s_1, h_2 s_1), p(s_2, h_1 s_2), \frac{p(s_1, h_2 s_1)p(s_2, h_1 s_2)}{1 + p(s_2, s_1)}\right\}.
\end{aligned}$$

Case (2): If $|s_1| \geq |s_2|$, then we get that:

$$\begin{aligned}
p(h_2 s_1, h_1 s_2) &= \frac{2}{3}\Big|s_1\Big| + \frac{3}{2}\Big|s_2\Big| \\
&\leq \frac{13}{6}\Big|s_1\Big| \\
&\leq \frac{6}{7}(3\Big|s_1\Big|) \\
&\leq \frac{6}{7}\max\left\{3\Big|s_1\Big|, \frac{7}{2}\Big|s_2\Big|\right\} \\
&= \frac{6}{7}\max\{p(s_1, h_2 s_1), p(s_2, h_1 s_2)\} \\
&\leq \frac{6}{7}\max\left\{p(s_1, h_2 s_1), p(s_2, h_1 s_2), \frac{p(s_1, h_2 s_1)p(s_2, h_1 s_2)}{1 + p(s_2, s_1)}\right\} \\
&= \varphi\max\left\{p(s_1, h_2 s_1), p(s_2, h_1 s_2), \frac{p(s_1, h_2 s_1)p(s_2, h_1 s_2)}{1 + p(s_2, s_1)}\right\}.
\end{aligned}$$

Therefore, in each case, we get that:

$$p(h_2 s_1, h_1 s_2) \leq \varphi\max\left\{p(s_1, h_2 s_1), p(s_2, h_1 s_2), \frac{p(s_1, h_1 s_1)p(s_2, h_1 s_2)}{1 + p(s_2, s_1)}\right\}.$$

Hence, the pair (h_1, h_2) is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction. Using Theorem 1, we get that h_1 and h_2 have a unique common fixed point. □

Note that for the (h_1, h_2)-sequence (j_n) with initial point j_0, if h_2 is the identity function, then (j_n) returns to the Picard iteration sequence, i.e., $h_1 j_n = j_{n+1}$ for $n \in \mathbb{N}^*$. Therefore, we can give the following definition:

Definition 15. *We call $\varrho : A \to A$ an α, β-triangular admissible mapping if:*

(i) $\alpha(j,i) \geq \beta(j,i) \Longrightarrow \alpha(\varrho j, i) \geq \beta(\varrho j, i)$ & $\alpha(j, \varrho i) \geq \beta(j, \varrho i)$, *and*
(ii) *if $i, j, k \in A$, $\alpha(i,j) \geq \beta(i,j)$, and $\alpha(j.k) \geq \beta(j,k)$, then $\alpha(i,k) \geq \beta(i,k)$.*

Definition 16. *Let p be modified ω-distance equipped on (A, q). Then, we call $\varrho : A \to A$ a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction if there exists a c-comparison function φ such that if $s_1, s_2 \in A$ and $\alpha(s_1, s_2) \geq \beta(s_1, s_2)$, then:*

$$p(\varrho s_1, s_2) \leq \varphi \max \left\{ p(s_2, s_2), p(s_1, \varrho s_1), \frac{p(s_2, s_2) p(s_1, \varrho s_1)}{1 + p(s_1, s_2)} \right\},$$

and:

$$p(s_1, \varrho s_2) \leq \varphi \max \left\{ p(s_1, s_1), p(s_2, \varrho s_2), \frac{p(s_1, s_1) p(s_2, \varrho s_2)}{1 + p(s_2, s_1)} \right\}.$$

Therefore, according to these definitions, we can derive the following results as consequences of our previous results:

Lemma 4. *For the map $\varrho : A \to A$, we assume the following conditions:*

1. *ϱ is α, β-triangular admissible and*
2. *there exists $j_0 \in A$ such that $\alpha(\varrho j_0, \varrho j_0) \geq \beta(\varrho j_0, \varrho j_0)$ and $\alpha(\varrho j_0, \varrho j_0) \geq \beta(\varrho j_0, \varrho j_0)$.*

Then, for the Picard sequence (j_n) with initial point j_0, we have $\alpha(j_n, j_m) \geq \beta(j_n, j_m)$ for all $n, m \in \mathbb{N}$.

Lemma 5. *Let p be the modified ω-distance equipped on (A, q). Let $\varrho : A \to A$ be a mapping and φ be a c-comparison. Suppose the following:*

(i) *ϱ is α, β-triangular admissible,*
(ii) *ϱ is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction, and*
(iii) *there exists $j_0 \in A$ such that:*

$$\alpha(\varrho j_0, \varrho j_0) \geq \beta(\varrho j_0, \varrho j_0).$$

If there exists $k \in \mathbb{N}^*$ such that $p(j_k, j_{k+1}) = 0$ or $p(j_{k+1}, j_k) = 0$, then j_k is a fixed point of ϱ, where (j_n) is the Picard sequence generated by ϱ with initial point j_0.
Moreover, if $z^* \in A$ is a fixed point of ϱ, then $p(z^*, z^*) = 0$.

Theorem 2. *Let p be the modified ω-distance equipped on (A, q). Let $\varrho : A \to A$ be a mapping and φ be a c-comparison function. Assume the following hypotheses:*

1. *ϱ is continuous,*
2. *ϱ is α, β-triangular admissible,*
3. *ϱ is a rational $(\alpha, \beta)_\varphi$-$m\omega$ contraction, and*
4. *there exists $j_0 \in A$ such that:*

$$\alpha(\varrho j_0, \varrho j_0) \geq \beta(\varrho j_0, \varrho j_0).$$

Then, the Picard sequence (j_n) generated by ϱ with initial point j_0 converges to a unique fixed point of ϱ in A.

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