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Numerical Reckoning Fixed Points of (ρE)-Type Mappings in Modular Vector Spaces

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Abstract: In this paper, we study an iteration process introduced by Thakur et al. for Suzuki mappings in Banach spaces, in the new context of modular vector spaces. We establish existence results for a more recent version of Suzuki generalized non-expansive mappings. The stability and data dependence of the scheme for ρ-contractions is studied as well.

Keywords: modular vector space; iterative process; fixed point; stability; data dependence

1. Introduction

Iterative processes are very important tools for finding numerical solutions of certain classes of problems of nonlinear analysis, which can be formulated in the language of fixed point theory and which cannot be tackled with analytical methods. Notable examples include the problem of finding the roots of polynomials with complex coefficients, the study of variational inequalities and equilibrium problems, algorithms for signal and image processing, etc. Perhaps the best known, due to its key role in the proof of the Banach Contraction Principle, is the Picard iteration process.

Meanwhile, the study of non-expansive mappings stimulated the search of new iteration processes. This was motivated in part by the fact that, unlike the case of contraction mappings, the successive application of a non-expansive mapping does not necessarily lead to a fixed point. The earliest results in this direction were obtained by Krasnosel'skii [1], Mann [2], Halpern [3], Berinde [4], for one-step iterations; Ishikawa [5], etc., for two-step iterations; Noor [6], Agrawal et al. [7], Abbas and Nazir [8], Gürsoy and Karakaya [9], Sintunavarat and Pitea [10], Thakur et al. [11,12], for three-step iterations; and the search for new iteration schemes has remained active ever since.

The iterative processes studied in the above-mentioned works are defined for certain classes of mappings, mainly on Banach spaces with a suitable geometric structure, most often on uniformly convex spaces. While the literature on the subject is becoming quite vast, we believe that it is also important to study iterative processes on modular vector spaces. This is due to the fact that they provide a unified approach to many important spaces which appear in various branches of mathematics, such as Orlicz spaces or Lebesgue spaces. That is why our goal is to study a iterative scheme introduced by Thakur et al. [13] for Suzuki mappings [14] on Banach spaces, in the framework of modular vector spaces. The mappings under consideration are required to satisfy a modular counterpart of the condition (E) from Garcia-Falset et al. [15], which is weaker than Suzuki's condition (C). Recent results in a similar direction have been obtained by Khan [16] and Mitrinović et al. [17].

Modular spaces have been extensively studied by Nakano in his classical monograph [18]. The first examples can be traced back to the early works of Orlicz [19], who introduced what is called now Orlicz spaces. These function spaces are generalizations of L_p spaces where, instead of a p-norm, one works with N-functions (for example $N_1(t) = e^t - t - 1$, $N_2(t) = e^{t^2} - 1$) thus, allowing growth properties more general than power type growth (for details, see for instance [20,21]). These include notable examples such as variable exponent spaces and provide, as well.

The paper is organized as follows. In the first section, we recall the definition of modular vector spaces and their properties needed throughout the paper. In Section 2, we define the mappings satisfying the modular version of the condition (E), providing an example of such a mapping. In Section 3, we study convergence of the iterative scheme introduced by Takur et al. in [13]. The main results of the section are Lemma 3, Theorem 1 and Theorem 2, which give sufficient conditions of convergence and fixed point existence results for (ρE)-type mappings. The fourth section is dedicated to the study of stability and data dependence with respect to ρ-contractive mappings. The main results are Theorem 3 and Theorem 4, respectively.

2. Modular Vector Spaces

Definition 1 ([21])**.** *Let X be a real (or complex) vector space. A function $\rho\colon X \to [0,\infty]$ is called a modular if it satisfies:*

(1) $\rho(x) = 0$ *if and only if* $x = 0$,
(2) $\rho(\alpha x) = \rho(x)$, *for* $|\alpha| = 1$,
(3) $\rho\left(\alpha x + (1-\alpha)\, y\right) \leq \rho\left(x\right) + \rho\left(y\right)$, *where* $\alpha \in [0,1]$,

for any $x, y \in X$. If we replace condition (3) with the following condition

$$\rho\left(\alpha x + (1-\alpha)\, y\right) \leq \alpha\rho\left(x\right) + (1-\alpha)\,\rho\left(y\right),$$

for any $\alpha \in [0,1]$ and any $x, y \in X$, then ρ is called a convex modular.

Unless otherwise specified, throughout this paper, we shall assume that ρ is a convex modular.

Example 1. *Let $\rho\colon \mathbb{R}^2 \to [0,\infty]$, $\rho\left(x_1, x_2\right) = x_1^2 + |x_2|$. It is clear that $\rho\left(x_1, x_2\right) = x_1^2 + |x_2| = 0$ if and only if $(x_1, x_2) = (0,0)$. The last two conditions of the definition are satisfied since both square function and absolute value function are even and convex. Thus, ρ is a convex modular. Notice that this modular does not satisfy the triangle axiom. Take for instance $\rho\left((2,1) + (1,1)\right) = 11 > 7 = \rho\left(2,1\right) + \rho\left(1,1\right)$.*

A convex modular ρ on a vector space X defines naturally a vector subspace as follows.

Definition 2 ([21])**.** *Let ρ be a convex modular function defined on a vector space X. The vector subspace*

$$X_\rho = \{x \in X\colon \lim_{\alpha\to 0} \rho\left(\alpha x\right) = 0\}$$

is called a modular space.

The modular vector space X_ρ can be endowed with a topology associated with the modular ρ by analogy with the metric topology.

Definition 3 ([22])**.** *Let ρ be a modular function defined on a vector space X*

(a) A sequence $\{x_n\} \subset X_\rho$ is called ρ-convergent to some $x \in X_\rho$ if and only if $\lim_{n\to\infty} \rho\left(x_n - x\right) = 0$.
(b) A sequence $\{x_n\} \subset X_\rho$ is called ρ-Cauchy if $\lim_{m,n\to\infty} \rho\left(x_m - x_n\right) = 0$.
(c) We say that X_ρ is ρ-complete if any ρ-Cauchy sequence in X_ρ is ρ-convergent.
(d) A set $C \subset X_\rho$ is called ρ-closed if for any sequence $\{x_n\} \subset C$ which ρ-converges to some point x; it implies that $x \in C$.
(e) A set $C \subset X_\rho$ is called ρ-bounded if $\delta_\rho\left(C\right) = \sup\left\{\rho\left(x - y\right); x, y \in C\right\} < \infty$.
(f) A set $K \subset X_\rho$ is called ρ-compact if any sequence $\{x_n\}$ in K has a subsequence which ρ-converges to a point in K.

(g) *ρ is said to satisfy the Fatou property if $\rho(x-y) \leq \liminf_{n\to\infty} \rho(x-y_n)$ whenever $\{y_n\}$ ρ-converges to y, for any x, y, y_n in X_ρ.*

The property of uniform convexity plays a crucial role while proving results in the framework of normed spaces. The same is true in the context of modular spaces.

Definition 4 (Definition 3.1, [22])**.** *The uniform convexity type properties of the modular ρ are defined for every $r > 0$ and every $\varepsilon > 0$ as follows:*

(a) *Define*

$$D_1(r,\varepsilon) = \left\{(x,y) : x, y \in X_\rho,\ \rho(x) \leq r,\ \rho(y) \leq r,\ \rho(x-y) \geq \varepsilon r\right\}.$$

If $D_1(r,\varepsilon) \neq \emptyset$, let

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) :\ (x,y) \in D_1(r,\varepsilon)\right\}.$$

If $D_1(r,\varepsilon) = \emptyset$, set $\delta_1(r,\varepsilon) = 1$.

We say that ρ satisfies $(UUC1)$ if for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$, depending on s and ε, such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0, \quad \textit{for } r > s.$$

(b) *Define*

$$D_2(r,\varepsilon) = \left\{(x,y) : x, y \in X_\rho,\ \rho(x) \leq r,\ \rho(y) \leq r,\ \rho\left(\frac{x-y}{2}\right) \geq \varepsilon r\right\}.$$

If $D_2(r,\varepsilon) \neq \emptyset$, let

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) :\ (x,y) \in D_2(r,\varepsilon)\right\}.$$

If $D_1(r,\varepsilon) = \emptyset$, set $\delta_1(r,\varepsilon) = 1$.

We say that ρ satisfies $(UUC2)$ if for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta_2(s,\varepsilon) > 0$ depending on s and ε, such that

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0, \quad \textit{for } r > s.$$

The following technical result, whose proof is similar to its modular function spaces counterpart (Lemma 4.2, [23]), will play an important role in the sequel.

Lemma 1. *Let ρ be a convex modular which is $(UUC1)$ and let $\{t_n\} \in (0,1)$ be a sequence bounded away from 0 and 1. If there exists $r > 0$ such that*

$$\limsup_{n\to\infty} \rho(x_n) \leq r,\ \ \limsup_{n\to\infty} \rho(y_n) \leq r,\ \ \lim_{n\to\infty} \rho(t_n x_n + (1-t_n) y_n) = r,$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X_ρ, then $\lim_{n\to\infty} \rho(x_n - y_n) = 0$.

Definition 5. *Let $\{x_n\}$ be a sequence in X_ρ. Let C be a nonempty subset of X_ρ. The function*

$$\tau \colon C \to [0,\infty], \quad \tau(x) = \limsup_{n\to\infty} \rho(x - x_n)$$

is called a ρ-type function. A sequence $\{c_n\}$ in C is called a minimizing sequence of τ if $\lim_{n\to\infty} \tau(c_n) = \inf_{x\in C} \tau(x)$.

For example, take the set of real numbers $\mathbb{R}$ as a modular space with the modular $\rho(x) = |x|$. Consider that C is the subset of the rational numbers $\mathbb{Q} \subset \mathbb{R}$ and the sequence $x_n = \frac{1}{\sqrt{n}}$, $n \geq 1$. The ρ-type function in this case is

$$\tau(x) = \limsup_{n\to\infty} |x - \frac{1}{\sqrt{n}}| = |x|,$$

which is obviously unbounded. As a corresponding minimizing sequence, take for instance the sequence $\{c_n\}$, $c_n = \frac{1}{n}$, $n \geq 1$.

Lemma 2 (Proposition 3.7 [22])**.** *Assume that the modular space X_ρ is ρ-complete and ρ satisfies the Fatou property. Let C be a nonempty convex and ρ-closed subset of X_ρ. Consider the ρ-type function $\tau\colon C \to [0,\infty]$ generated by a sequence $\{x_n\}$ in X_ρ. Assume that $\tau_0 = \inf_{x\in C} \tau(x) < \infty$.*

a) If ρ is $(UUC1)$, then all minimizing sequences of τ are ρ-convergent to the same limit.
b) If ρ is $(UUC2)$ and $\{c_n\}$ is a minimizing sequence of τ, then the sequence $\{c_n/2\}$ ρ-converges to a point which is independent of $\{c_n\}$.

We end this section by recalling a crucial property of the modular.

Definition 6. *Let X_ρ be a modular space. It is said that the modular ρ satisfies the Δ_2-condition if there exists a constant $K \geq 0$ such that*

$$\rho(2x) \leq K\rho(x), \tag{1}$$

for any $x \in X_\rho$. The smallest such constant K will be denoted by ω_2.

For example, the modular $\rho\colon \mathbb{R} \to [0,\infty]$, $\rho(x) = a|x|^\alpha$, $a > 0$, $\alpha > 1$, satisfies the Δ_2-condition with $K = 2^\alpha$. As a counterexample, one may consider the modular $\rho\colon \mathbb{R} \to [0,\infty]$, $\rho(x) = e^{|x|} - |x| - 1$, which does not satisfy the Δ_2-condition (for details, see [20]).

3. Mappings Satisfying the (ρE)-Condition

In 2008, Suzuki [14] introduced a new class of mappings, on normed spaces, which he called *generalized non-expansive mappings*. Soon after, García-Falset et al. [15] provided two kinds of generalizations, one of which is of interest in this paper. Below, we adapt the definition from [15] to the context of modular spaces.

Definition 7. *Let C be a nonempty subset of the modular space X_ρ. A mapping $T\colon C \to X_\rho$ is said to satisfy the (ρE_μ) condition on C, if there exists $\mu \geq 1$ such that*

$$\rho(x - Ty) \leq \mu\rho(x - Tx) + \rho(x - y), \tag{2}$$

for all $x, y \in X_\rho$. One says that T satisfies condition (ρE) whenever T satisfies (ρE_μ) for some $\mu \geq 1$.

Example 2. *The modular ρ introduced in Example 1 endows $\mathbb{R}^2$ with a modular space structure. Take the subset $[-1,1] \times [-1,1] \subset \mathbb{R}^2$ and define a mapping $T\colon\ [-1,1] \times [-1,1] \to [-1,1] \times [-1,1]$ by the rule*

$$T(x_1, x_2) = \begin{cases} \left(x_1, \frac{1}{3}|x_2|\right), & (x_1, x_2) \in [-1,1] \times [-1,1) \\ \left(x_1, -\frac{1}{3}\right), & (x_1, x_2) \in [-1,1] \times \{1\}. \end{cases}$$

Taking $(x_1, x_2) = \left(1, \frac{3}{4}\right)$ *and* $(y_1, y_2) = (1, 1)$, *we see that*

$$\rho\left(T\left(1, \frac{3}{4}\right) - T(1,1)\right) = \frac{7}{12} > \frac{1}{4} = \rho\left(\left(1, \frac{3}{4}\right) - (1,1)\right),$$

meaning that T *is not a* ρ*-non-expansive mapping (see Definition 4.1, [22]).*

Let us now verify that T *satisfies the* (ρE) *condition.*

Case I: *Let* $(x_1, x_2) \in [-1,1] \times [-1,1)$ *and* $(y_1, y_2) \in [-1,1] \times [-1,1]$. *We have*

$$\begin{aligned} \rho\left((x_1, x_2) - T(x_1, x_2)\right) &= |x_2 - \tfrac{1}{3}|x_2| \geq \tfrac{2}{3}|x_2|, \\ \rho\left((x_1, x_2) - (y_1, y_2)\right) &= (x_1 - y_1)^2 + |x_2 - y_2|, \\ \rho\left((x_1, x_2) - T(y_1, y_2)\right) &\leq (x_1 - y_1)^2 + |x_2| + \tfrac{1}{3}|y_2|. \end{aligned}$$

To prove that Condition (ρE) *is satisfied in this case, it is enough to show that the inequality*

$$|x_2| + \frac{1}{3}|y_2| \leq \mu \frac{2}{3}|x_2| + |x_2 - y_2|$$

holds for some $\mu \geq 1$. *Indeed, taking* $\mu = 2$ *and noticing that* $|y_2| \leq |x_2| + |x_2 - y_2|$, *the conclusion follows.*

Case II: *Let now* $(x_1, x_2) \in [-1,1] \times \{1\}$ *and* $(y_1, y_2) \in [-1,1] \times [-1,1]$. *For this case, we see that*

$$\begin{aligned} \rho\left((x_1, x_2) - T(x_1, x_2)\right) &= \tfrac{4}{3}, \\ \rho\left((x_1, x_2) - (y_1, y_2)\right) &= (x_1 - y_1)^2 + |1 - y_2|, \\ \rho\left((x_1, x_2) - T(y_1, y_2)\right) &\leq (x_1 - y_1)^2 + 1 + \tfrac{1}{3}|y_2|. \end{aligned}$$

Similarly as above, it is enough that the inequality

$$1 + \frac{1}{3}|y_2| \leq \mu \frac{4}{3} + |1 - y_2|$$

holds for some $\mu \geq 1$, *which is true since* $|y_2| \leq 1$.

In conclusion, the mapping T *satisfies the* (ρE) *condition for* $\mu = 3$.

4. Convergence Analysis

As before, let C be a subset of a modular space X_ρ. Consider the iterative scheme [13], which we shall call the TTP scheme, defined as follows:

$$\left.\begin{aligned} x_1 &\in C, \\ x_{n+1} &= Ty_n, \\ y_n &= T\left((1 - \alpha_n) x_n + \alpha_n z_n\right), \\ z_n &= (1 - \beta_n) x_n + \beta_n T x_n, \end{aligned}\right\} \tag{3}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

The following results are useful for our purpose.

Lemma 3. *Let* C *be a nonempty* ρ*-closed convex subset of* X_ρ *and let* $T\colon C \to C$ *be a mapping satisfying* (ρE) *with* $F(T) \neq \emptyset$. *For arbitrary chosen* $x_1 \in C$, *let the sequence* $\{x_n\}$ *be generated by the iterative process (3) and suppose* $\rho(x_k - p) < \infty$ *for some* $k \geq 1$. *Then,* $\lim\limits_{n\to\infty} \rho(x_n - p)$ *exists for any* $p \in F(T)$.

Proof. Let $p \in F(T)$. As T satisfies condition (ρE), we have

$$\rho(Tx - p) = \rho(p - Tx) \leq \mu\rho(p - Tp) + \rho(x - p) = \rho(x - p), \text{ for any } x \in C. \tag{4}$$

By (4) it follows that $\rho(Tx - p) \leq \rho(x - p)$, for any $x \in C$, and using this one and the convexity of ρ, one has

$$\begin{aligned} \rho(z_n - p) &= \rho((1-\beta_n)x_n + \beta_n T x_n - p) \\ &\leq (1-\beta_n)\rho(x_n - p) + \beta_n \rho(Tx_n - p) \\ &\leq (1-\beta_n)\rho(x_n - p) + \beta_n \rho(x_n - p) \\ &= \rho(x_n - p). \end{aligned} \tag{5}$$

Similarly, taking into account relation (5), we get

$$\begin{aligned} \rho(y_n - p) &= \rho(T((1-\alpha_n)x_n + \alpha_n z_n) - p) \\ &\leq \rho((1-\alpha_n)x_n + \alpha_n z_n - p) \\ &\leq (1-\alpha_n)\rho(x_n - p) + \alpha_n \rho(z_n - p) \\ &\leq (1-\alpha_n)\rho(x_n - p) + \alpha_n \rho(x_n - p) \\ &= \rho(x_n - p). \end{aligned} \tag{6}$$

Now, using (4) and (6) it follows

$$\begin{aligned} \rho(x_{n+1} - p) &= \rho(Ty_n - p) \\ &\leq \rho(y_n - p) \\ &\leq \rho(x_n - p), \end{aligned} \tag{7}$$

implying that the sequence $\{\rho(x_n - p)\}_{n \geq k}$ is bounded and nonincreasing for any $p \in F(T)$. Thus, the limit $\lim\limits_{n\to\infty} \rho(x_n - p)$ exists. □

Lemma 4. *Let C be a nonempty subset of X_ρ and let $T\colon C \to C$ be a mapping which satisfies condition (ρE). Suppose there exists a bounded sequence $\{x_n\}$ in C such that $\lim\limits_{n\to\infty} \rho(x_n - Tx_n) = 0$ and let τ be the ρ-type generated by $\{x_n\}$. Then, T leaves the minimizing sequences invariant, i.e., if $\{c_n\}$ is a minimizing sequence for τ, then so is $\{Tc_n\}$.*

Proof. Let $\{x_n\}$ be such that $\lim\limits_{n\to\infty} \rho(x_n - Tx_n) = 0$. For arbitrary $x \in C$, we have

$$\rho(x_n - Tx) \leq \mu\rho(x_n - Tx_n) + \rho(x_n - x), \tag{8}$$

which implies that

$$\tau(Tx) = \limsup_{n\to\infty} \rho(x_n - Tx) \leq \limsup_{n\to\infty} \rho(x_n - x) = \tau(x). \tag{9}$$

Let now $\{c_n\}$ be a minimizing sequence. Applying (9), we get

$$\inf_{x\in C} \tau(x) \leq \lim_{n\to\infty} \tau(Tc_n) \leq \lim_{n\to\infty} \tau(c_n) = \inf_{x\in C} \tau(x), \tag{10}$$

which implies that $\lim\limits_{n\to\infty} \tau(Tc_n) = \inf\limits_{x\in C} \tau(x)$, i.e., $\{Tc_n\}$ is a minimizing sequence for τ. □

Proposition 1. *Let C be a nonempty, convex and ρ-closed subset of X_ρ, where X_ρ is ρ-complete and ρ satisfies the Δ_2-condition, is (UUC1), and satisfies the Fatou property. Consider the ρ-type function $\tau\colon C \to [0,\infty]$ generated by a sequence $\{x_n\}$ in X_ρ and suppose $\tau_0 = \inf\limits_{x\in C} \tau(x) < \infty$. Let $\{c_n\}$ and $\{d_n\}$ be two minimizing sequences for τ. Then,*

(i) any convex combination of $\{c_n\}$ and $\{d_n\}$ is a minimizing sequence for τ as well;
(ii) $\lim\limits_{n\to\infty} \rho(c_n - d_n) = 0$.

Proof. (i) Let $e_n = \lambda c_n + (1-\lambda) d_n$, $\lambda \in (0,1)$, $n \geq 1$. For any $x \in C$, we have

$$\rho(e_n - x) \leq \lambda\rho(c_n - x) + (1-\lambda)\rho(d_n - x), n \geq 1,$$

which implies

$$\limsup_{m\to\infty} \rho(e_n - x_m) \leq \lambda \limsup_{m\to\infty} \rho(c_n - x_m) + (1-\lambda)\limsup_{m\to\infty} \rho(d_n - x_m), n \geq 1,$$

i.e.,

$$\tau(e_n) \leq \lambda\tau(c_n) + (1-\lambda)\tau(d_n).$$

Passing to the limit and keeping in mind that $\{c_n\}$ and $\{d_n\}$ are minimizing sequences, we obtain

$$\tau_0 = \inf_{x\in C} \tau(x) \leq \lim_{n\to\infty} \tau(e_n) \leq \lambda\tau_0 + (1-\lambda)\tau_0 = \tau_0, \tag{11}$$

which gives the conclusion.

(ii) Let us notice that, since $e_n = \frac{1}{2}c_n + \frac{1}{2}d_n$, $n \geq 1$, we have $c_n - d_n = 2(e_n - d_n)$, $n \geq 1$. According to (i), $\{e_n\}$ is a minimizing sequence and, according to Lemma 2, all minimizing sequences ρ-converge to the same point, which we denote by z. Thus,

$$\rho(e_n - d_n) = \rho\left(\frac{c_n - d_n}{2}\right) \leq \frac{1}{2}\rho(c_n - z) + \frac{1}{2}\rho(d_n - z), n \geq 1.$$

Thus, on account of (i), we get $\lim_{n\to\infty} \rho(e_n - d_n) = 0$. Similarly, $\lim_{n\to\infty} \rho(e_n - c_n) = 0$. The Δ_2-condition implies the inequality

$$\rho(c_n - d_n) \leq \frac{\omega_2}{2}\left(\rho(e_n - c_n) + \rho(e_n - d_n)\right),$$

which gives the conclusion of (ii) by taking $n \to \infty$. □

Theorem 1. *Let X_ρ be a ρ-complete modular space and C be a nonempty convex ρ-closed and ρ-bounded subset X_ρ. Suppose ρ satisfies the Fatou property, is $(UUC1)$ and satisfies the Δ_2-condition. Let $T\colon C \to C$ be a mapping satisfying condition (ρE) and let the sequence $\{x_n\}$ be generated by the iterative process (3) with $\{\alpha_n\}$ and $\{\beta_n\}$ bounded away from 0 and 1. Then, $F(T) \neq \emptyset$ if and only if $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$*

Proof. Suppose $F(T) \neq \emptyset$ and take $p \in F(T)$. According to Lemma 3, the limit

$$r := \lim_{n\to\infty} \rho(x_n - p)$$

exists. Using the relations (5) and (4) respectively, we have

$$\lim_{n\to\infty} \rho(z_n - p) \leq \lim_{n\to\infty} \rho(x_n - p) = r, \tag{12}$$

$$\limsup_{n\to\infty} \rho(Tx_n - p) \leq \lim_{n\to\infty} \rho(x_n - p) = r. \tag{13}$$

On the other hand, using the inequalities (4) and (7), together with the convexity of ρ, we obtain

$$\begin{aligned}
\rho(x_{n+1} - p) &\leq \rho(y_n - p) \\
&= \rho(T((1-\alpha_n)x_n + \alpha_n z_n) - p) \\
&\leq \rho((1-\alpha_n)x_n + \alpha_n z_n - p) \\
&\leq (1-\alpha_n)\rho(x_n - p) + \alpha_n\rho(z_n - p) \\
&= \rho(x_n - p) - \alpha_n\rho(x_n - p) + \alpha_n\rho(z_n - p),
\end{aligned}$$

which implies

$$\frac{\rho\left(x_{n+1}-p\right)-\rho\left(x_n-p\right)}{\alpha_n} \leq \rho\left(z_n-p\right)-\rho\left(x_n-p\right).$$

Thus,

$$\rho\left(x_{n+1}-p\right)-\rho\left(x_n-p\right) \leq \rho\left(z_n-p\right)-\rho\left(x_n-p\right),$$

i.e.,

$$\rho\left(x_{n+1}-p\right) \leq \rho\left(z_n-p\right).$$

We also have, from condition (4), that $\rho\left(z_n-p\right) \leq \rho\left(x_n-p\right)$, which implies that

$$r=\lim_{n\to\infty}\rho\left(z_n-p\right). \tag{14}$$

It follows

$$\lim_{n\to\infty}\rho\left(\beta_n\left(Tx_n-p\right)+\left(1-\beta_n\right)\left(x_n-p\right)\right)=\lim_{n\to\infty}\rho\left(z_n-p\right)=r. \tag{15}$$

Thus, the conditions of Lemma 1 are satisfied yielding $\lim\limits_{n\to\infty}\rho\left(Tx_n-x_n\right)=0$.

Conversely, assume that $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty}\rho\left(Tx_n-x_n\right)=0$. Let $\tau\colon C\to[0,\infty]$ be the ρ-type function generated by $\{x_n\}$ and let $\{c_n\}$ be a minimizing sequence for τ converging to a point $z\in C$. By Lemma 4, $\{Tc_n\}$ is a minimizing sequence as well and by Proposition 1 $\lim\limits_{n\to\infty}\rho\left(c_n-Tc_n\right)=0$. On the other hand, condition (ρE) gives

$$\rho\left(c_n-Tz\right)\leq\mu\rho\left(c_n-Tc_n\right)+\rho\left(c_n-z\right), n\geq 1.$$

Taking $n\to\infty$, one obtains $\lim\limits_{n\to\infty}\rho\left(c_n-Tz\right)=0$, i.e., c_n ρ-converges to Tz. By the uniqueness of the limit, we have $Tz=z$. $\square$

Theorem 2. *Let C be a nonempty ρ-compact and convex subset of X_ρ and let ρ, T and $\{x_n\}$ be as in Theorem 1. Suppose that $\lim\limits_{n\to\infty}\rho\left(Tx_n-x_n\right)=0$. Then, the sequence $\{x_n\}$ ρ-converges to a fixed point of T.*

Proof. The ρ-compactness of C implies the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which ρ-converges to a point z in C. On the other hand, since T satisfies condition (ρE), we have

$$\rho\left(x_{n_k}-Tz\right)\leq\mu\rho\left(x_{n_k}-Tx_{n_k}\right)+\rho\left(x_{n_k}-z\right),\ \ \mu\geq 1.$$

Noticing that subsequence $\{x_{n_k}\}$ is an a.f.p.s. In addition, we get $\lim\limits_{k\to\infty}\rho\left(x_{n_k}-Tz\right)=0$ and, by the uniqueness of the limit, we have $Tz=z$, i.e., $z\in F(T)$. According to Lemma 3, the limit $\lim\limits_{n\to\infty}\rho\left(x_n-z\right)$ exists and thus $\{x_n\}$ ρ-converges to z. $\square$

5. Stability and Data Dependence

In this section, our goal is to study the stability and data dependence of the TTP scheme (3) for ρ-contractions on modular spaces.

Definition 8. *Let C be a nonempty set of a modular space X_ρ. A mapping $T\colon C\to C$ is called ρ-contraction if there exists a constant $0\leq\theta<1$ such that*

$$\rho\left(Tx-Ty\right)\leq\theta\rho\left(x-y\right), \textit{for all } x,y\in C.$$

The Banach Contraction Principle is valid for ρ-contractions on modular spaces (see [22], Theorem 4.2). Thus, the existence of fixed points for ρ-contractions is guaranteed. It is also straightforward to see that the iteration scheme (3), applied to ρ-contractions, yields the inequality $\rho\left(x_{n+1}-p\right) \leq \theta\rho\left(x_n-p\right)$, where $p \in F(T)$, which implies its convergence to a fixed point.

The following two lemmas will be instrumental in the proofs of the following theorems.

Lemma 5 ([24])**.** *Let $\{\psi_n\}_{n=0}^{\infty}$ and $\{\varphi_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying*

$$\psi_{n+1} \leq (1-\tau_n)\psi_n + \varphi_n,$$

where $\tau_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \tau_n = \infty$ and $\frac{\varphi_n}{\tau_n} \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} \psi_n = 0$.

Lemma 6 ([25])**.** *Let $\{\psi_n\}_{n=0}^{\infty}$ be a nonnegative real sequence for which one supposes there exists $n_0 \in \mathbb{N}$, such that, for all $n \geq n_0$, the following inequality is satisfied:*

$$\psi_{n+1} \leq (1-\tau_n)\psi_n + \tau_n\varphi_n,$$

where $\tau_n \in (0,1)$, $\varphi_n \geq 0$ $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \tau_n = \infty$. Then,

$$0 \leq \limsup_{n\to\infty} \psi_n \leq \limsup_{n\to\infty} \varphi_n. \tag{16}$$

The notion of stability of an iteration process is usually defined for metric spaces (see, for instance, [26,27]). A natural analogue, in the context of modular spaces, is defined as follows.

Definition 9. *Let C be a nonempty set of a modular space X_ρ and let $\{t_n\}_{n=0}^{\infty}$ an arbitrary sequence in C. We say that an iteration process $x_{n+1} = f(T, x_n)$, which converges to a fixed point p, is T-stable if*

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ if and only if } \lim_{n\to\infty} \rho(t_n - p) = 0,$$

where $\varepsilon_n = \rho\left(t_{n+1} - f(T, t_n)\right)$, $n = 0, 1, 2, \ldots$.

Theorem 3. *Let C be a nonempty ρ-closed set of a modular space X_ρ which is ρ-complete and let $T: C \to C$ be a ρ-contraction with a ρ-bounded orbit. Consider the iterative process (3) with $\{\alpha_n\}$ and $\{\beta_n\}$ bounded away from 0 and 1 and satisfying $\delta \leq \alpha_n\beta_n$ for some $\delta > 0$. Suppose the modular ρ satisfies the Δ_2 condition. If $\omega_2\theta^2 \leq 2$, then the iterative process (3) is T-stable.*

Proof. Let $p \in C$ be a fixed point for the mapping T and let $\{t_n\}_{n=0}^{\infty}$ be a sequence in C. Consider the sequence generated by the iterative process (3) $x_{n+1} = f(T, x_n)$, converging to p. Denote $\varepsilon_n = \rho\left(t_{n+1} - f(T, t_n)\right)$ and suppose $\lim_{n\to\infty} \varepsilon_n = 0$. Using the Δ_2 property, the convexity of the modular, as well as the assumption that $\omega_2\theta^2 \leq 2$, we have

$$\begin{aligned}
\rho\left(t_{n+1}-p\right) &= \rho\left(2\left(\frac{1}{2}\left(t_{n+1}-f\left(T,t_n\right)\right)+\frac{1}{2}\left(f\left(T,t_n\right)-p\right)\right)\right) \\
&\leq \frac{\omega_2}{2}\left[\rho\left(t_{n+1}-f\left(T,t_n\right)\right)+\rho\left(f\left(T,t_n\right)-p\right)\right] \\
&\leq \frac{\omega_2}{2}\left[\varepsilon_n+\rho\left(T\left(T\left(\left(1-\alpha_n\right)t_n+\alpha_n\left(\left(1-\beta_n\right)t_n+\beta_n Tt_n\right)\right)\right)-p\right)\right] \\
&\leq \frac{\omega_2}{2}\left[\varepsilon_n+\theta\rho\left(T\left(\left(1-\alpha_n\right)t_n+\alpha_n\left(\left(1-\beta_n\right)t_n+\beta_n Tt_n\right)\right)-p\right)\right] \\
&\leq \frac{\omega_2}{2}\left[\varepsilon_n+\theta^2\rho\left(\left(1-\alpha_n\right)t_n+\alpha_n\left(\left(1-\beta_n\right)t_n+\beta_n Tt_n\right)-p\right)\right] \\
&\leq \frac{\omega_2}{2}\left[\varepsilon_n+\theta^2\left(1-\alpha_n\beta_n+\alpha_n\beta_n\theta\right)\rho\left(t_n-p\right)\right] \\
&= \frac{\omega_2}{2}\varepsilon_n+\frac{\omega_2}{2}\theta^2\left(1-\alpha_n\beta_n\left(1-\theta\right)\right)\rho\left(t_n-p\right) \\
&\leq \left(1-\alpha_n\beta_n\left(1-\theta\right)\right)\rho\left(t_n-p\right)+\frac{\omega_2}{2}\varepsilon_n.
\end{aligned}$$

Applying Lemma 5 for $\psi_n = \rho\left(t_n-p\right)$, $\tau_n = \alpha_n\beta_n\left(1-\theta\right)$ and $\varphi_n = \frac{\omega_2}{2}\varepsilon_n$, we conclude that $\lim_{n\to\infty}\rho\left(t_n-p\right)=0$.

Conversely, suppose $\lim_{n\to\infty} = \rho\left(t_n-p\right)$. We have

$$\begin{aligned}
\varepsilon_n &= \rho\left(t_{n+1}-f\left(T,t_n\right)\right) \\
&\leq \frac{\omega_2}{2}\left(\rho\left(t_{n+1}-p\right)+\rho\left(f\left(T,t_n\right)-p\right)\right) \\
&\leq \frac{\omega_2}{2}\left(\rho\left(t_{n+1}-p\right)+\theta^2\left(1-\alpha_n\beta_n\left(1-\theta\right)\right)\rho\left(t_n-p\right)\right),
\end{aligned}$$

implying that $\lim_{n\to\infty}\varepsilon_n=0$, which completes the proof. □

Definition 10. *Let $T,\widetilde{T}: X_\rho \to X_\rho$ two operators. We say that $\widetilde{T}$ approximates the operator T if, for some $\varepsilon>0$, we have*

$$\|Tx-\widetilde{T}x\|\leq\varepsilon,$$

for all $x\in X_\rho$.

Theorem 4. *Let $\widetilde{T}$ be an approximate operator of a ρ-contraction T such that $\omega_2\theta<2$. Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (3), corresponding to T, and let $\{\widetilde{x}_n\}_{n=1}^{\infty}$ be a iterative sequence generated by the iterative scheme*

$$\left.\begin{aligned}
\widetilde{x}_1 &\in C, \\
\widetilde{x}_{n+1} &= \widetilde{T}\widetilde{y}_n, \\
\widetilde{y}_n &= \widetilde{T}\left(\left(1-\alpha_n\right)\widetilde{x}_n+\alpha_n\widetilde{z}_n\right), \\
\widetilde{z}_n &= \left(1-\beta_n\right)\widetilde{x}_n+\beta_n\widetilde{T}\widetilde{x}_n,
\end{aligned}\right\} \tag{17}$$

for all $n\geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\frac{1}{2}\leq\alpha_n\beta_n\ \forall n\in\mathbb{N}$. If $Tp=p$ and $\widetilde{T}\widetilde{p}=\widetilde{p}$ such that $\lim_{n\to\infty}\widetilde{x}_n=\widetilde{p}$, then

$$\rho\left(p-\widetilde{p}\right)\leq\frac{7\omega_2^2\varepsilon}{2\left(2-\omega_2\theta\right)}.$$

Proof. Using the convexity and the Δ_2 property of the modular, we have

$$\begin{aligned}
\rho\left(z_n-\widetilde{z}_n\right) & =\rho\left(\left(1-\beta_n\right) x_n+\beta_n T x_n-\left(1-\beta_n\right) \widetilde{x}_n-\beta_n \widetilde{T} \widetilde{x}_n\right) \\
& \leq\left(1-\beta_n\right) \rho\left(x_n-\widetilde{x}_n\right)+\beta_n \rho\left(T x_n-\widetilde{T} \widetilde{x}_n\right) \\
& \leq\left(1-\beta_n\right) \rho\left(x_n-\widetilde{x}_n\right)+\beta_n \frac{\omega_2}{2}\left(\rho\left(T x_n-T \widetilde{x}_n\right)+\rho\left(T \widetilde{x}_n-\widetilde{T} \widetilde{x}_n\right)\right) \\
& \leq\left(1-\beta_n\left(1-\frac{\omega_2}{2} \theta\right)\right) \rho\left(x_n-\widetilde{x}_n\right)+\beta_n \frac{\omega_2}{2} \varepsilon .
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
\rho\left(y_n-\widetilde{y}_n\right) & =\rho\left(T\left(\left(1-\alpha_n\right) T x_n+\alpha_n T z_n\right)-\widetilde{T}\left(\left(1-\alpha_n\right) \widetilde{T} \widetilde{x}_n+\alpha_n \widetilde{T} \widetilde{z}_n\right)\right) \\
& \leq \frac{\omega_2}{2}\left(\rho\left(T\left(\left(1-\alpha_n\right) T x_n+\alpha_n T z_n\right)-T\left(\left(1-\alpha_n\right) \widetilde{T} \widetilde{x}_n+\alpha_n \widetilde{T} \widetilde{z}_n\right)\right)\right) \\
& \quad+\frac{\omega_2}{2}\left(\rho\left(T\left(\left(1-\alpha_n\right) \widetilde{T} \widetilde{x}_n+\alpha_n \widetilde{T} \widetilde{z}_n\right)-\widetilde{T}\left(\left(1-\alpha_n\right) \widetilde{T} \widetilde{x}_n+\alpha_n \widetilde{T} \widetilde{z}_n\right)\right)\right) \\
& \leq \frac{\omega_2}{2} \theta\left(\left(1-\alpha_n\right) \rho\left(T x_n-\widetilde{T} \widetilde{x}_n\right)+\alpha_n \rho\left(T z_n-\widetilde{T} \widetilde{z}_n\right)\right)+\frac{\omega_2}{2} \varepsilon \\
& \leq\left[\frac{\omega_2}{2}\right]^2 \theta\left(1-\alpha_n\right)\left(\rho\left(T x_n-T \widetilde{x}_n\right)+\rho\left(T \widetilde{x}_n-\widetilde{T} \widetilde{x}_n\right)\right) \\
& \quad+\left[\frac{\omega_2}{2}\right]^2 \theta \alpha_n\left(\rho\left(T z_n-T \widetilde{z}_n\right)+\rho\left(T \widetilde{z}_n-\widetilde{T} \widetilde{z}_n\right)\right)+\frac{\omega_2}{2} \varepsilon \\
& \leq\left[\frac{\omega_2}{2}\right]^2 \theta\left(\left(1-\alpha_n\right)\left(\theta \rho\left(x_n-\widetilde{x}_n\right)+\varepsilon\right)+\alpha_n\left(\theta \rho\left(z_n-\widetilde{z}_n\right)+\varepsilon\right)\right)+\frac{\omega_2}{2} \varepsilon \\
& \leq\left[\frac{\omega_2}{2}\right]^2 \theta^2\left(\left(1-\alpha_n\right) \rho\left(x_n-\widetilde{x}_n\right)+\alpha_n \rho\left(z_n-\widetilde{z}_n\right)\right)+\left[\frac{\omega_2}{2}\right]^2 \theta \varepsilon+\frac{\omega_2}{2} \varepsilon \\
& \leq\left[\frac{\omega_2}{2}\right]^2 \theta^2\left(1-\alpha_n \beta_n\left(1-\frac{\omega_2}{2} \theta\right)\right) \rho\left(x_n-\widetilde{x}_n\right) \\
& \quad+\varepsilon\left(\alpha_n \beta_n\left[\frac{\omega_2}{2}\right]^3 \theta^2+\left[\frac{\omega_2}{2}\right]^2 \theta+\frac{\omega_2}{2}\right) .
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\rho\left(x_{n+1}-\widetilde{x}_{n+1}\right) & =\rho\left(T y_n-\widetilde{T} \widetilde{y}_n\right) \\
& \leq \frac{\omega_2}{2}\left(\rho\left(T y_n-T \widetilde{y}_n\right)+\rho\left(T \widetilde{y}_n-\widetilde{T} \widetilde{y}_n\right)\right) \\
& \leq \frac{\omega_2}{2}\left(\theta \rho\left(y_n-\widetilde{y}_n\right)+\varepsilon\right) \\
& \leq\left[\frac{\omega_2}{2}\right]^3 \theta^3\left(1-\alpha_n \beta_n\left(1-\frac{\omega_2}{2} \theta\right)\right) \rho\left(x_n-\widetilde{x}_n\right) \\
& \quad+\left(\alpha_n \beta_n\left[\frac{\omega_2}{2}\right]^4 \theta^3+\left[\frac{\omega_2}{2}\right]^3 \theta^2+\left[\frac{\omega_2}{2}\right]^2 \theta+\frac{\omega_2}{2}\right) \varepsilon \\
& \leq\left(1-\alpha_n \beta_n\left(1-\frac{\omega_2}{2} \theta\right)\right) \rho\left(x_n-\widetilde{x}_n\right)+\frac{7 \omega_2}{2} \alpha_n \beta_n \varepsilon .
\end{aligned}$$

Applying now Lemma 6 with $\psi_n=\rho\left(x_n-\widetilde{x}_n\right)$, $\tau_n=\alpha_n \beta_n\left(1-\frac{\omega_2}{2} \theta\right)$ and $\varphi_n=\frac{7 \omega_2 \varepsilon}{2-\omega_2 \theta}$, respectively, we get

$$0 \leq \limsup_{n \rightarrow \infty} \rho\left(x_n-\widetilde{x}_n\right) \leq \limsup_{n \rightarrow \infty} \frac{7 \omega_2 \varepsilon}{2-\omega_2 \theta} . \tag{18}$$

On the other hand, we have the inequality

$$\rho(p-\widetilde{p}) \leq \frac{\omega_2}{2} \rho\left(x_n-\widetilde{x}_n\right)+\left[\frac{\omega_2}{2}\right]^2\left(\rho\left(x_n-p\right)+\rho\left(\widetilde{x}_n-\widetilde{p}\right)\right),$$

in which passing to the limit and using the inequality (18) yields

$$\rho\left(p - \widetilde{p}\right) \leq \frac{7\omega_2^2 \varepsilon}{2\left(2 - \omega_2 \theta\right)},$$

which completes the proof. □

6. Conclusions

In this paper, we have studied the iterative process introduced by Thakur et. al. in [13], in the framework of modular spaces. Sufficient conditions of convergence of the iterative process to fixed points of (ρE)-type mappings were established in Lemma 3, Theorem 1 and Theorem 2, respectively. We have also established conditions for stability and studied the data dependence of the new iterative process with respect to ρ-contractive mappings in Theorem 3 and Theorem 4, respectively.

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