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Linguistic Neutrosophic Numbers Einstein Operator and Its Application in Decision Making

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Abstract: Linguistic neutrosophic numbers (LNNs) include single-value neutrosophic numbers and linguistic variable numbers, which have been proposed by Fang and Ye. In this paper, we define the linguistic neutrosophic number Einstein sum, linguistic neutrosophic number Einstein product, and linguistic neutrosophic number Einstein exponentiation operations based on the Einstein operation. Then, we analyze some of the relationships between these operations. For LNN aggregation problems, we put forward two kinds of LNN aggregation operators, one is the LNN Einstein weighted average operator and the other is the LNN Einstein geometry (LNNEWG) operator. Then we present a method for solving decision-making problems based on LNNEWA and LNNEWG operators in the linguistic neutrosophic environment. Finally, we apply an example to verify the feasibility of these two methods.

Keywords: multiple attribute group decision making (MAGDM); Linguistic neutrosophic; LNN Einstein weighted-average operator; LNN Einstein weighted-geometry (LNNEWG) operator

1. Introduction

Smarandache [1] proposed the neutrosophic set (NS) in 1998. Compared with the intuitionistic fuzzy sets (IFSs), the NS increases the uncertainty measurement, from which decision makers can use the truth, uncertainty and falsity degrees to describe evaluation, respectively. In the NS, the degree of uncertainty is quantified, and these three degrees are completely independent of each other, so, the NS is a generalization set with more capacity to express and deal with the fuzzy data. At present, the study of NS theory has been a part of research that mainly includes the research of the basic theory of NS, the fuzzy decision of NS, and the extension of NS, etc. [2–14]. Recently, Fang and Ye [15] presented the linguistic neutrosophic number (LNN). Soon afterwards, many research topics about LNN were proposed [16–18].

Information aggregation operators have become an important research topic and obtained a wide range of research results. Yager [19] put forward the ordered weighted average (OWA) operator considering the data sorting position. Xu [20] presented the arithmetic aggregation (AA) of IFS. Xu and Yager [21] presented the geometry aggregation (GA) operator of IFS. Zhao [22] proposed generalized aggregation operators based on IFS and proved that AA and GA were special cases of generalized aggregation operator. The operators mentioned above are established based on the algebraic sum and the algebraic product of number sets. They are respectively referred to as a special case of Archimedes t-conorm and t-norm to establish union or intersection operation of the number set. The union and intersection of Einstein operation is a kind of Archimedes t-conorm and t-norm with good smooth characteristics [23]. Wang and Liu [24] built some IF Einstein aggregation operators and proved that the Einstein aggregation operator has better smoothness than the arithmetic aggregation operator. Zhao and Wei [25] put forward the IF Einstein hybrid-average (IFEHA) operator and IF

Einstein hybrid-geometry (IFEHG) operator. Further, Guo etc. [26] applied the Einstein operation to a hesitate fuzzy set. Lihua Yang etc. [27] put forward novel power aggregation operators based on Einstein operations for interval neutrosophic linguistic sets. However, neutrosophic linguistic sets are different from linguistic neutrosophic sets. The former still use two values to describe the evaluation value, while the latter can use a pure language value to describe the evaluation value. As far as we know, this is the first work on Einstein aggregation operators for LNN. It must be noticed that the aggregation operators in References [15–18] are almost based on the most commonly used algebraic product and algebraic sum of LNNs for carrying the combination process, which is not the only operation law that can be chosen to model the intersection and union on LNNs. Thus, we establish the operation rules of LNN based on Einstein operation and put forward the LNN Einstein weighted-average (LNNEWA) operator and LNN Einstein weighted-geometry (LNNEWG) operator. These operators are finally utilized to solve some relevant problems.

The other organizations: in Section 2, concepts of LNN and Einstein are described, operational laws of LNNs based on Einstein operation are defined, and their performance is analyzed. In Section 3, LNNEWA and LNNEWG operators are proposed. In Section 4, multiple attribute group decision making (MAGDM) methods are built based on LNNEWA and LNNEWG operators. In Section 5, an instance is given. In Section 6, conclusions and future research are given.

2. Basic Theories

2.1. LNN and Its Operational Laws

Definition 1. [15] *Set a finite language set* $\Psi = \{\psi_t | t \in [0, k]\}$, *where* ψ_t *is a linguistic variable, k +1 is the cardinality of* Ψ. *Then, we define* $u = \langle \psi_\beta, \psi_\gamma, \psi_\delta \rangle$, *in which* $\psi_\beta, \psi_\gamma, \psi_\delta \in \Psi$ *and* $\beta, \gamma, \delta \in [0, k]$, ψ_β, ψ_δ *and* ψ_γ *expresse truth, falsity and indeterminacy degree, respectively, we call u an LNN.*

Definition 2. [15] *Set three LNNs* $u = \langle \psi_\beta, \psi_\gamma, \psi_\delta \rangle$, $u_1 = \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle$ *and* $u_2 = \langle \psi_{\beta_2}, \psi_{\gamma_2}, \psi_{\delta_2} \rangle$ *in* Ψ *and* $\lambda \geq 0$, *then, the operational rules are as following:*

$$\oplus u_2 = \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle \oplus \langle \psi_{\beta_2}, \psi_{\gamma_2}, \psi_{\delta_2} \rangle = \langle \psi_{\beta_1+\beta_2-\frac{\beta_1\beta_2}{k}}, \psi_{\frac{\gamma_1\gamma_2}{k}}, \psi_{\frac{\delta_1\delta_2}{k}} \rangle; \tag{1}$$

$$u_1 \otimes u_2 = \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle \otimes \langle \psi_{\beta_2}, \psi_{\gamma_2}, \psi_{\delta_2} \rangle = \langle \psi_{\frac{\beta_1\beta_2}{k}}, \psi_{\gamma_1+\gamma_2-\frac{\gamma_1\gamma_2}{k}}, \psi_{\delta_1+\delta_2-\frac{\delta_1\delta_2}{k}} \rangle; \tag{2}$$

$$\lambda u = \lambda \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle = \langle \psi_{k-k(1-\frac{\beta}{k})^\lambda}, \psi_{k(\frac{\gamma}{k})^\lambda}, \psi_{k(\frac{\delta}{k})^\lambda} \rangle; \tag{3}$$

$$u^\lambda = \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle^\lambda = \langle \psi_{k(\frac{\beta}{k})^\lambda}, \psi_{k-k(1-\frac{\gamma}{k})^\lambda}, \psi_{k-k(1-\frac{\delta}{k})^\lambda} \rangle. \tag{4}$$

Definition 3. [15] *Set an LNN* $u = \langle \psi_\beta, \psi_\gamma, \psi_\delta \rangle$ *in* Ψ, *we define* $\zeta(u)$ *as the expectation and* $\eta(u)$ *as the accuracy:*

$$\zeta(u) = (2k + \beta - \gamma - \delta)/3k \tag{5}$$

$$\eta(u) = (\beta - \delta)/k \tag{6}$$

Definition 4. [15]: *Set two LNNs* $u_1 = \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle$ *and* $u_2 = \langle \psi_{\beta_2}, \psi_{\gamma_2}, \psi_{\delta_2} \rangle$ *in* Ψ, *then*

If $\zeta(u_1) > \zeta(u_2)$, *then* $u_1 > u_2$;
If $\zeta(u_1) = \zeta(u_2)$ *then*
If $\eta(u_1) > \eta(u_2)$, *then* $u_1 > u_2$;
If $\eta(u_1) = \eta(u_2)$, *then* $u_1 \sim u_2$.

2.2. Einstein Operation

Definition 5. [28,29] *For any two real Numbers a, b∈ [0, 1], Einstein $\oplus_e$ is an Archimedes t-conorms, Einstein $\otimes_e$ is an Archimedes t-norms, then*

$$\mathrm{a}\oplus_e\mathrm{b} = \frac{a+b}{1+ab}, \quad \mathrm{a}\otimes_e\mathrm{b} = \frac{ab}{1+(1-a)(1-b)}. \tag{7}$$

2.3. Einstein Operation Under the Linguistic Neutrosophic Number

Definition 6. *Set $u = \langle \psi_\beta, \psi_\gamma, \psi_\delta \rangle$, $u_1 = \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle$ and $u_2 = \langle \psi_{\beta_2}, \psi_{\gamma_2}, \psi_{\delta_2} \rangle$ as three LNNs in Ψ, $\lambda \geq 0$, the operation of Einstein $\oplus_e$ and Einstein $\otimes_e$ under the linguistic neutrosophic number are defined as follows:*

$$u_1 \oplus_e u_2 = \langle \psi_{\frac{k^2(\beta_1+\beta_2)}{k^2+\beta_1\beta_2}}, \psi_{\frac{k\gamma_1\gamma_2}{k^2+(k-\gamma_1)(k-\gamma_2)}}, \psi_{\frac{k\delta_1\delta_2}{k^2+(k-\delta_1)(k-\delta_2)}} \rangle; \tag{8}$$

$$u_1 \otimes_e u_2 = \langle \psi_{\frac{k\beta_1\beta_2}{k^2+(k-\beta_1)(k-\beta_2)}}, \psi_{\frac{k^2(\gamma_1+\gamma_2)}{k^2+\gamma_1\gamma_2}}, \psi_{\frac{k^2(\delta_1+\delta_2)}{k^2+\delta_1\delta_2}} \rangle; \tag{9}$$

$$\lambda u = \langle \psi_{k*\frac{(k+\beta)^\lambda-(k-\beta)^\lambda}{(k+\beta)^\lambda+(k-\beta)^\lambda}}, \psi_{k*\frac{2\gamma^\lambda}{(2k-\gamma)^\lambda+\gamma^\lambda}}, \psi_{k*\frac{2\delta^\lambda}{(2k-\delta)^\lambda+\delta^\lambda}} \rangle; \tag{10}$$

$$u^\lambda = \langle \psi_{k*\frac{2\beta^\lambda}{(2k-\beta)^\lambda+\beta^\lambda}}, \psi_{k*\frac{(k+\gamma)^\lambda-(k-\gamma)^\lambda}{(k+\gamma)^\lambda+(k-\gamma)^\lambda}}, \psi_{k*\frac{(k+\delta)^\lambda-(k-\delta)^\lambda}{(k+\delta)^\lambda+(k-\delta)^\lambda}} \rangle. \tag{11}$$

Theorem 1. *Set $u \langle = \psi_\beta, \psi_\gamma, \psi_\delta \rangle$, $u_1 = \langle \psi_{\beta_1}, \psi_{\gamma_1}, \psi_{\delta_1} \rangle$ and $u_2 = \langle \psi_{\beta_2}, \psi_{\gamma_2}, \psi_{\delta_2} \rangle$ as three LNNs in Ψ, $\lambda \geq 0$, then, the operation of Einstein $\oplus_e$ and Einstein $\otimes_e$ have the following performance:*

$$u_1 \oplus_e u_2 = u_2 \oplus_e u_1; \tag{12}$$

$$u_1 \otimes_e u_2 = u_2 \otimes_e u_1; \tag{13}$$

$$\lambda(u_1 \oplus_e u_2) = \lambda u_1 \oplus_e \lambda u_2; \tag{14}$$

$$(u_1 \otimes_e u_2)^\lambda = {u_1}^\lambda \otimes_e {u_2}^\lambda; \tag{15}$$

Proof. Performance (1) and (2) are easy to be obtained, so we omit it; Now we prove the performance (3): According to Definition 6, we can get

$$① \quad u_1 \oplus_e u_2 = \langle \psi_{\frac{k^2(\beta_1+\beta_2)}{k^2+\beta_1\beta_2}}, \psi_{\frac{k\gamma_1\gamma_2}{k^2+(k-\gamma_1)(k-\gamma_2)}}, \psi_{\frac{k\delta_1\delta_2}{k^2+(k-\delta_1)(k-\delta_2)}} \rangle;$$

② $\lambda(u_1 \oplus_e u_2)$

$$= \langle \psi_{k*\frac{(k+\frac{k^2(\beta_1+\beta_2)}{k^2+\beta_1\beta_2})^\lambda-(k-\frac{k^2(\beta_1+\beta_2)}{k^2+\beta_1\beta_2})^\lambda}{(k+\frac{k^2(\beta_1+\beta_2)}{k^2+\beta_1\beta_2})^\lambda+(k-\frac{k^2(\beta_1+\beta_2)}{k^2+\beta_1\beta_2})^\lambda}}, \psi_{k*\frac{2(\frac{k\gamma_1\gamma_2}{k^2+(k-\gamma_1)(k-\gamma_2)})^\lambda}{(2k-\frac{k\gamma_1\gamma_2}{k^2+(k-\gamma_1)(k-\gamma_2)})^\lambda+(\frac{k\gamma_1\gamma_2}{k^2+(k-\gamma_1)(k-\gamma_2)})^\lambda}}, \psi_{k*\frac{2(\frac{k\delta_1\delta_2}{k^2+(k-\delta_1)(k-\delta_2)})^\lambda}{(2k-\frac{k\delta_1\delta_2}{k^2+(k-\delta_1)(k-\delta_2)})^\lambda+(\frac{k\delta_1\delta_2}{k^2+(k-\delta_1)(k-\delta_2)})^\lambda}} \rangle$$

$$= \langle \psi_{k*\frac{(k+\beta_1)^\lambda(k+\beta_2)^\lambda-(k-\beta_1)^\lambda(k-\beta_2)^\lambda}{(k+\beta_1)^\lambda(k+\beta_2)^\lambda+(k-\beta_1)^\lambda(k-\beta_2)^\lambda}}, \psi_{k*\frac{2(\gamma_1\gamma_2)^\lambda}{((2k-\gamma_1)^\lambda(2k-\gamma_2)^\lambda)+(\gamma_1\gamma_2)^\lambda}}, \psi_{k*\frac{2(\delta_1\delta_2)^\lambda}{((2k-\delta_1)^\lambda(2k-\delta_2)^\lambda)+(\delta_1\delta_2)^\lambda}} \rangle;$$

$$③ \quad \lambda u_1 = \langle \psi_{k*\frac{(k+\beta_1)^\lambda-(k-\beta_1)^\lambda}{(k+\beta_1)^\lambda+(k-\beta_1)^\lambda}}, \psi_{k*\frac{2{\gamma_1}^\lambda}{(2k-\gamma_1)^\lambda+{\gamma_1}^\lambda}}, \psi_{k*\frac{2{\delta_1}^\lambda}{(2k-\delta_1)^\lambda+{\delta_1}^\lambda}} \rangle;$$

$$④ \quad \lambda u_2 = \langle \psi_{k*\frac{(k+\beta_2)^\lambda-(k-\beta_2)^\lambda}{(k+\beta_2)^\lambda+(k-\beta_2)^\lambda}}, \psi_{k*\frac{2{\gamma_2}^\lambda}{(2k-\gamma_2)^\lambda+{\gamma_2}^\lambda}}, \psi_{k*\frac{2{\delta_2}^\lambda}{(2k-\delta_2)^\lambda+{\delta_2}^\lambda}} \rangle;$$

⑤ $\lambda u_1 \oplus_e \lambda u_2$

$$= \langle \psi_{\frac{k^2(k*\frac{(k+\beta_1)^\lambda-(k-\beta_1)^\lambda}{(k+\beta_1)^\lambda+(k-\beta_1)^\lambda}+k*\frac{(k+\beta_2)^\lambda-(k-\beta_2)^\lambda}{(k+\beta_2)^\lambda+(k-\beta_2)^\lambda})}{k^2+((k*\frac{(k+\beta_1)^\lambda-(k-\beta_1)^\lambda}{(k+\beta_1)^\lambda+(k-\beta_1)^\lambda})(k*\frac{(k+\beta_2)^\lambda-(k-\beta_2)^\lambda}{(k+\beta_2)^\lambda+(k-\beta_2)^\lambda}))}}, \psi_{\frac{k(k*\frac{2\gamma_1{}^\lambda}{(2k-\gamma_1)^\lambda+\gamma_1{}^\lambda})(k*\frac{2\gamma_2{}^\lambda}{(2k-\gamma_2)^\lambda+\gamma_2{}^\lambda})}{k^2+(k-(k*\frac{2\gamma_1{}^\lambda}{(2k-\gamma_1)^\lambda+\gamma_1{}^\lambda}))(k-(k*\frac{2\gamma_2{}^\lambda}{(2k-\gamma_2)^\lambda+\gamma_2{}^\lambda}))}}, \psi_{\frac{k(k*\frac{2\delta_1{}^\lambda}{(2k-\delta_1)^\lambda+\delta_1{}^\lambda})(k*\frac{2\delta_2{}^\lambda}{(2k-\delta_2)^\lambda+\delta_2{}^\lambda})}{k^2+(k-(k*\frac{2\delta_1{}^\lambda}{(2k-\delta_1)^\lambda+\delta_1{}^\lambda}))(k-(k*\frac{2\delta_2{}^\lambda}{(2k-\delta_2)^\lambda+\delta_2{}^\lambda}))}} \rangle$$

$$= \langle \psi_{k*\frac{(k+\beta_1)^\lambda(k+\beta_2)^\lambda-(k-\beta_1)^\lambda(k-\beta_2)^\lambda}{(k+\beta_1)^\lambda(k+\beta_2)^\lambda+(k-\beta_1)^\lambda(k-\beta_2)^\lambda}}, \psi_{k*\frac{2(\gamma_1\gamma_2)^\lambda}{((2k-\gamma_1)^\lambda(2k-\gamma_2)^\lambda)+(\gamma_1\gamma_2)^\lambda}}, \psi_{k*\frac{2(\gamma_1\delta_2)^\lambda}{((2k-\delta_1)^\lambda(2k-\delta_2)^\lambda)+(\delta_1\delta_2)^\lambda}} \rangle$$

So, we can get $\lambda(u_1 \oplus_e u_2) = \lambda u_1 \oplus_e \lambda u_2$.

Now, we prove the performance (4):

$$① \; u_1{}^\lambda = \langle \psi_{k*\frac{2\beta_1{}^\lambda}{(2k-\beta_1)^\lambda+\beta_1{}^\lambda}}, \psi_{k*\frac{(k+\gamma_1)^\lambda-(k-\gamma_1)^\lambda}{(k+\gamma_1)^\lambda+(k-\gamma_1)^\lambda}}, \psi_{k*\frac{(k+\delta_1)^\lambda-(k-\delta_1)^\lambda}{(k+\delta_1)^\lambda+(k-\delta_1)^\lambda}} \rangle;$$

$$② \; u_2{}^\lambda = \langle \psi_{k*\frac{2\beta_2{}^\lambda}{(2k-\beta_2)^\lambda+\beta_2{}^\lambda}}, \psi_{k*\frac{(k+\gamma_2)^\lambda-(k-\gamma_2)^\lambda}{(k+\gamma_2)^\lambda+(k-\gamma_2)^\lambda}}, \psi_{k*\frac{(k+\delta_2)^\lambda-(k-\delta_2)^\lambda}{(k+\delta_2)^\lambda+(k-\delta_2)^\lambda}} \rangle;$$

$$③ \; u_1{}^\lambda \oplus_e u_2{}^\lambda = \langle \begin{matrix} \psi_{\frac{k(k*\frac{2\beta_1{}^\lambda}{(2k-\beta_1)^\lambda+\beta_1{}^\lambda})(k*\frac{2\beta_2{}^\lambda}{(2k-\beta_2)^\lambda+\beta_2{}^\lambda})}{k^2+(k-(k*\frac{2\beta_1{}^\lambda}{(2k-\beta_1)^\lambda+\beta_1{}^\lambda}))(k-(k*\frac{2\beta_2{}^\lambda}{(2k-\beta_2)^\lambda+\beta_2{}^\lambda}))}}, \\ \psi_{\frac{k^2((k*\frac{(k+\gamma_1)^\lambda-(k-\gamma_1)^\lambda}{(k+\gamma_1)^\lambda+(k-\gamma_1)^\lambda})+(k*\frac{2\beta_2{}^\lambda}{(2k-\beta_2)^\lambda+\beta_2{}^\lambda}))}{k^2+(k*\frac{(k+\gamma_1)^\lambda-(k-\gamma_1)^\lambda}{(k+\gamma_1)^\lambda+(k-\gamma_1)^\lambda})(k*\frac{2\beta_2{}^\lambda}{(2k-\beta_2)^\lambda+\beta_2{}^\lambda})}}, \\ \psi_{\frac{k^2((k*\frac{(k+\delta_1)^\lambda-(k-\delta_1)^\lambda}{(k+\delta_1)^\lambda+(k-\delta_1)^\lambda})+(k*\frac{(k+\delta_2)^\lambda-(k-\delta_2)^\lambda}{(k+\delta_2)^\lambda+(k-\delta_2)^\lambda}))}{k^2+(k*\frac{(k+\delta_1)^\lambda-(k-\delta_1)^\lambda}{(k+\delta_1)^\lambda+(k-\delta_1)^\lambda})(k*\frac{(k+\delta_2)^\lambda-(k-\delta_2)^\lambda}{(k+\delta_2)^\lambda+(k-\delta_2)^\lambda})}} \end{matrix} \rangle$$

$$= \langle \psi_{k*\frac{2(\beta_1\beta_2)^\lambda}{((2k-\beta_1)^\lambda(2k-\beta_2)^\lambda)+(\beta_1\beta_2)^\lambda}}, \psi_{k*\frac{(k+\gamma_1)^\lambda(k+\gamma_2)^\lambda-(k-\gamma_1)^\lambda(k-\gamma_2)^\lambda}{(k+\gamma_1)^\lambda(k+\gamma_2)^\lambda+(k-\gamma_1)^\lambda(k-\gamma_2)^\lambda}}, \psi_{k*\frac{(k+\delta_1)^\lambda(k+\delta_2)^\lambda-(k-\delta_1)^\lambda(k-\delta_2)^\lambda}{(k+\delta_1)^\lambda(k+\delta_2)^\lambda+(k-\delta_1)^\lambda(k-\delta_2)^\lambda}} \rangle;$$

$$④ \; u_1 \otimes_e u_2 = \langle \psi_{\frac{k\beta_1\beta_2}{k^2+(k-\beta_1)(k-\beta_2)}}, \psi_{\frac{k^2(\gamma_1+\gamma_2)}{k^2+\gamma_1\gamma_2}}, \psi_{\frac{k^2(\delta_1+\delta_2)}{k^2+\delta_1\delta_2}} \rangle;$$

$$⑤ \; (u_1 \otimes_e u_2)^\lambda = \langle \psi_{k*\frac{2(\frac{k\beta_1\beta_2}{k^2+(k-\beta_1)(k-\beta_2)})^\lambda}{(2k-\frac{k\beta_1\beta_2}{k^2+(k-\beta_1)(k-\beta_2)})^\lambda+(\frac{k\beta_1\beta_2}{k^2+(k-\beta_1)(k-\beta_2)})^\lambda}}, \psi_{k*\frac{(k+\frac{k^2(\gamma_1+\gamma_2)}{k^2+\gamma_1\gamma_2})^\lambda-(k-\frac{k^2(\gamma_1+\gamma_2)}{k^2+\gamma_1\gamma_2})^\lambda}{(k+\frac{k^2(\gamma_1+\gamma_2)}{k^2+\gamma_1\gamma_2})^\lambda+(k-\frac{k^2(\gamma_1+\gamma_2)}{k+\gamma_1\gamma_2})^\lambda}}, \psi_{k*\frac{(k+\frac{k^2(\delta_1+\delta_2)}{k^2+\delta_1\delta_2})^\lambda-(k-\frac{k^2(\delta_1+\delta_2)}{k^2+\delta_1\delta_2})^\lambda}{(k+\frac{k^2(\delta_1+\delta_2)}{k^2+\delta_1\delta_2})^\lambda+(k-\frac{k^2(\delta_1+\delta_2)}{k^2+\delta_1\delta_2})^\lambda}} \rangle$$

$$= \langle \psi_{k*\frac{2(\beta_1\beta_2)^\lambda}{((2k-\beta_1)^\lambda(2k-\beta_2)^\lambda)+(\beta_1\beta_2)^\lambda}}, \psi_{k*\frac{(k+\gamma_1)^\lambda(k+\gamma_2)^\lambda-(k-\gamma_1)^\lambda(k-\gamma_2)^\lambda}{(k+\gamma_1)^\lambda(k+\gamma_2)^\lambda+(k-\gamma_1)^\lambda(k-\gamma_2)^\lambda}}, \psi_{k*\frac{(k+\delta_1)^\lambda(k+\delta_2)^\lambda-(k-\delta_1)^\lambda(k-\delta_2)^\lambda}{(k+\delta_1)^\lambda(k+\delta_2)^\lambda+(k-\delta_1)^\lambda(k-\delta_2)^\lambda}} \rangle;$$

So, we can get $(u_1 \oplus_e u_2)^\lambda = u_1{}^\lambda \oplus_e u_2{}^\lambda$. □

3. Einstein Aggregation Operators

3.1. LNNEWA Operator

Definition 7. *Set a LNN* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *in* Ψ*, for i = 1,2, ..., z, we define the LNNEWA operator:*

$$LNNEWA(u_1, u_2, \ldots u_z) = \overset{z}{\underset{i=1}{\oplus_e}} \epsilon_i u_i, \tag{16}$$

with the relative weight vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_z)^T$, $\sum_{i=1}^{z} \epsilon_i = 1$ *and* $\epsilon_i \in [0, 1]$.

Theorem 2. *Set a collection* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *in* Ψ*, for i = 1,2, ... ,z, then according to the LNNEWA aggregation operator, we can get the following result:*

$$\begin{aligned} LNNEWA(u_1, u_2, \ldots u_z) &= \overset{z}{\underset{i=1}{\oplus_e}} \epsilon_i u_i \\ &= \langle \psi_{k*\frac{\prod_{i=1}^{z}(k+\beta_i)^{\epsilon_i}-\prod_{i=1}^{z}(k-\beta_i)^{\epsilon_i}}{\prod_{i=1}^{z}(k+\beta_i)^{\epsilon_i}+\prod_{i=1}^{z}(k-\beta_i)^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{z}\gamma_i{}^{\epsilon_i}}{\prod_{i=1}^{z}(2k-\gamma_i)^{\epsilon_i}+\prod_{i=1}^{z}\gamma_i{}^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{z}\delta_i{}^{\epsilon_i}}{\prod_{i=1}^{z}(2k-\delta_i)^{\epsilon_i}+\prod_{i=1}^{z}\delta_i{}^{\epsilon_i}}} \rangle \end{aligned} \tag{17}$$

with the relative weight vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_z)^T$, $\sum_{i=1}^{z} \epsilon_i = 1$ *and* $\epsilon_i \in [0, 1]$.

Proof.

$$① \ \epsilon_i u_i = \langle \psi_{k*\frac{(k+\beta_i)^{\epsilon_i}-(k-\beta_i)^{\epsilon_i}}{(k+\beta_i)^{\epsilon_i}+(k-\beta_i)^{\epsilon_i}}}, \psi_{k*\frac{2\gamma_i^{\epsilon_i}}{(2k-\gamma_i)^{\epsilon_i}+\gamma_i^{\epsilon_i}}}, \psi_{k*\frac{2\delta_i^{\epsilon_i}}{(2k-\delta_i)^{\epsilon_i}+\delta_i^{\epsilon_i}}} \rangle;$$

$$② \ z = 2\,, LNNEWA(u_1, u_2) = \mathop{\oplus_e}_{i=1}^{2} \epsilon_i u_i$$
$$= \langle \psi_{\frac{k^2(k*\frac{(k+\beta_1)^{\epsilon_1}-(k-\beta_1)^{\epsilon_1}}{(k+\beta_1)^{\epsilon_1}+(k-\beta_1)^{\epsilon_1}}+k*\frac{(k+\beta_2)^{\epsilon_2}-(k-\beta_2)^{\epsilon_2}}{(k+\beta_2)^{\epsilon_2}+(k-\beta_2)^{\epsilon_2}})}{k^2+((k*\frac{(k+\beta_1)^{\epsilon_1}-(k-\beta_1)^{\epsilon_1}}{(k+\beta_1)^{\epsilon_1}+(k-\beta_1)^{\epsilon_1}})(k*\frac{(k+\beta_2)^{\epsilon_2}-(k-\beta_2)^{\epsilon_2}}{(k+\beta_2)^{\epsilon_2}+(k-\beta_2)^{\epsilon_2}})}}, \psi_{\frac{k(k*\frac{2\gamma_1^{\epsilon_1}}{(2k-\gamma_1)^{\epsilon_1}+\gamma_1^{\epsilon_1}})(k*\frac{2\gamma_2^{\epsilon_2}}{(2k-\gamma_2)^{\epsilon_2}+\gamma_2^{\epsilon_2}})}{k^2+(k-(k*\frac{2\gamma_1^{\epsilon_1}}{(2k-\gamma_1)^{\epsilon_1}+\gamma_1^{\epsilon_1}}))(k-(k*\frac{2\gamma_2^{\epsilon_2}}{(2k-\gamma_2)^{\epsilon_2}+\gamma_2^{\epsilon_2}}))}}, \psi_{\frac{k(k*\frac{2\delta_1^{\epsilon_1}}{(2k-\delta_1)^{\epsilon_1}+\delta_1^{\epsilon_1}})(k*\frac{2\delta_2^{\epsilon_2}}{(2k-\delta_2)^{\epsilon_2}+\delta_2^{\epsilon_2}})}{k^2+(k-(k*\frac{2\delta_1^{\epsilon_1}}{(2k-\delta_1)^{\epsilon_1}+\delta_1^{\epsilon_1}}))(k-(k*\frac{2\delta_2^{\epsilon_2}}{(2k-\delta_2)^{\epsilon_2}+\delta_2^{\epsilon_2}}))}} \rangle$$
$$= \langle \psi_{k*\frac{(k+\beta_1)^{\epsilon_1}(k+\beta_2)^{\epsilon_2}-(k-\beta_1)^{\epsilon_1}(k-\beta_2)^{\epsilon_2}}{(k+\beta_1)^{\epsilon_1}(k+\beta_2)^{\epsilon_2}+(k-\beta_1)^{\epsilon_1}(k-\beta_2)^{\epsilon_2}}}, \psi_{k*\frac{2\gamma_1^{\epsilon_1}\gamma_2^{\epsilon_2}}{(2k-\gamma_1)^{\epsilon_1}(2k-\gamma_2)^{\epsilon_2}+\gamma_1^{\epsilon_1}\gamma_2^{\epsilon_2}}}, \psi_{k*\frac{2\delta_1^{\epsilon_1}\delta_2^{\epsilon_2}}{(2k-\delta_1)^{\epsilon_1}(2k-\delta_1)^{\epsilon_2}+\delta_1^{\epsilon_1}\delta_2^{\epsilon_2}}} \rangle$$
$$= \langle \psi_{k*\frac{\prod_{i=1}^{2}(k+\beta_i)^{\epsilon_i}-\prod_{i=1}^{2}(k-\beta_i)^{\epsilon_i}}{\prod_{i=1}^{2}(k+\beta_i)^{\epsilon_i}+\prod_{i=1}^{2}(k-\beta_i)^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{2}\gamma_i^{\epsilon_i}}{\prod_{i=1}^{2}(2k-\gamma_i)^{\epsilon_i}+\prod_{i=1}^{2}\gamma_i^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{2}\delta_i^{\epsilon_i}}{\prod_{i=1}^{2}(2k-\delta_i)^{\epsilon_i}+\prod_{i=1}^{2}\delta_i^{\epsilon_i}}} \rangle.$$

Suppose $z = m$, according t formula (17), we can get

$$\begin{aligned} LNNEWA(u_1, u_2, \ldots u_m) &= \mathop{\oplus_e}_{i=1}^{m} \epsilon_i u_i \\ &= \langle \psi_{k*\frac{\prod_{i=1}^{m}(k+\beta_i)^{\epsilon_i}-\prod_{i=1}^{m}(k-\beta_i)^{\epsilon_i}}{\prod_{i=1}^{m}(k+\beta_i)^{\epsilon_i}+\prod_{i=1}^{m}(k-\beta_i)^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{m}\gamma_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\gamma_i)^{\epsilon_i}+\prod_{i=1}^{m}\gamma_i^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{m}\delta_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\delta_i)^{\epsilon_i}+\prod_{i=1}^{m}\delta_i^{\epsilon_i}}} \rangle, \end{aligned} \tag{18}$$

Then z = m + 1, the following can be found:

$$LNNEWA(u_1, u_2, \ldots \ u_m, u_{m+1}) = (\mathop{\oplus_e}_{i=1}^{m} \epsilon_i u_i) \oplus_e \epsilon_{m+1} u_{m+1}$$
$$= \left\langle \begin{array}{l} \psi_{k*\frac{\prod_{i=1}^{m}(k+\beta_i)^{\epsilon_i}-\prod_{i=1}^{m}(k-\beta_i)^{\epsilon_i}}{\prod_{i=1}^{m}(k+\beta_i)^{\epsilon_i}+\prod_{i=1}^{m}(k-\beta_i)^{\epsilon_i}}} \\ \psi_{k*\frac{2\prod_{i=1}^{m}\gamma_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\gamma_i)^{\epsilon_i}+\prod_{i=1}^{m}\gamma_i^{\epsilon_i}}}, \\ \psi_{k*\frac{2\prod_{i=1}^{m}\delta_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\delta_i)^{\epsilon_i}+\prod_{i=1}^{m}\delta_i^{\epsilon_i}}} \end{array} \right\rangle \oplus_e \left\langle \begin{array}{l} \psi_{k*\frac{(k+\beta_{m+1})^{\epsilon_{m+1}}-(k-\beta_{m+1})^{\epsilon_{m+1}}}{(k+\beta_{m+1})^{\epsilon_{m+1}}+(k-\beta_{m+1})^{\epsilon_{m+1}}}}, \\ \psi_{k*\frac{2\gamma_{m+1}^{\epsilon_{m+1}}}{(2k-\gamma_{m+1})^{\epsilon_{m+1}}+\gamma_{m+1}^{\epsilon_{m+1}}}}, \\ \psi_{k*\frac{2\delta_{m+1}^{\epsilon_{m+1}}}{(2k-\delta_{m+1})^{\epsilon_{m+1}}+\delta_{m+1}^{\epsilon_{m+1}}}} \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} \psi_{\frac{k^2((k*\frac{\prod_{i=1}^{m}(k+\beta_i)^{\epsilon_i}-\prod_{i=1}^{m}(k-\beta_i)^{\epsilon_i}}{\prod_{i=1}^{k}(k+\beta_i)^{\epsilon_i}+\prod_{i=1}^{k}(k-\beta_i)^{\epsilon_i}})+(k*\frac{(k+\beta_{m+1})^{\epsilon_{m+1}}-(k-\beta_{m+1})^{\epsilon_{m+1}}}{(k+\beta_{m+1})^{\epsilon_{m+1}}+(k-\beta_{m+1})^{\epsilon_{m+1}}}))}{k^2+(k*\frac{\prod_{i=1}^{m}(k+\beta_i)^{\epsilon_i}-\prod_{i=1}^{m}(k-\beta_i)^{\epsilon_i}}{\prod_{i=1}^{m}(k+\beta_i)^{\epsilon_i}+\prod_{i=1}^{m}(k-\beta_i)^{\epsilon_i}})(k*\frac{(k+\beta_{m+1})^{\epsilon_{m+1}}-(k-\beta_{m+1})^{\epsilon_{m+1}}}{(k+\beta_{m+1})^{\epsilon_{m+1}}+(k-\beta_{m+1})^{\epsilon_{m+1}}})}}, \\ \psi_{\frac{k(k*\frac{2\prod_{i=1}^{m}\gamma_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\gamma_i)^{\epsilon_i}+\prod_{i=1}^{m}\gamma_i^{\epsilon_i}})(k*\frac{2\gamma_{m+1}^{\epsilon_{m+1}}}{(2k-\gamma_{m+1})^{\epsilon_{m+1}}+\gamma_{m+1}^{\epsilon_{m+1}}})}{k^2+(k-(k*\frac{2\prod_{i=1}^{m}\gamma_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\gamma_i)^{\epsilon_i}+\prod_{i=1}^{m}\gamma_i^{\epsilon_i}}))(k-(k*\frac{2\gamma_{m+1}^{\epsilon_{m+1}}}{(2k-\gamma_{m+1})^{\epsilon_{m+1}}+\gamma_{m+1}^{\epsilon_{m+1}}}))}}, \\ \psi_{\frac{k(k*\frac{2\prod_{i=1}^{m}\delta_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\delta_i)^{\epsilon_i}+\prod_{i=1}^{m}\delta_i^{\epsilon_i}})(k*\frac{2\delta_{m+1}^{\epsilon_{m+1}}}{(2k-\delta_{m+1})^{\epsilon_{m+1}}+\delta_{m+1}^{\epsilon_{m+1}}})}{k^2+(k-(k*\frac{2\prod_{i=1}^{m}\delta_i^{\epsilon_i}}{\prod_{i=1}^{m}(2k-\delta_i)^{\epsilon_i}+\prod_{i=1}^{m}\delta_i^{\epsilon_i}}))(k-(k*\frac{2\delta_{m+1}^{\epsilon_{m+1}}}{(2k-\delta_{m+1})^{\epsilon_{m+1}}+\delta_{m+1}^{\epsilon_{m+1}}}))}} \end{array} \right\rangle$$
$$= \langle \psi_{k*\frac{\prod_{i=1}^{m+1}(k+\beta_i)^{\epsilon_i}-\prod_{i=1}^{m+1}(k-\beta_i)^{\epsilon_i}}{\prod_{i=1}^{m+1}(k+\beta_i)^{\epsilon_i}+\prod_{i=1}^{m+1}(k-\beta_i)^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{m+1}\gamma_i^{\epsilon_i}}{\prod_{i=1}^{m+1}(2k-\gamma_i)^{\epsilon_i}+\prod_{i=1}^{m+1}\gamma_i^{\epsilon_i}}}, \psi_{k*\frac{2\prod_{i=1}^{m+1}\delta_i^{\epsilon_i}}{\prod_{i=1}^{m+1}(2k-\delta_i)^{\epsilon_i}+\prod_{i=1}^{m+1}\delta_i^{\epsilon_i}}} \rangle.$$

So, Equation (17) is satisfied for any z according to the above results.
This proves Theorem 1. □

Theorem 3. *(Idempotency). Set an LNN* $u = \langle \psi_\beta, \psi_\gamma, \psi_\delta \rangle$ *in* Ψ*, for every* u_i *in* u *is equal to* u*, we can get:*

$$LNNEWA(u_1, u_2, \ldots u_z) = LNNEWA(u, u \ldots u) = u.$$

Proof. For $u_i = u$, *then* $\beta_i = \beta$; $\gamma_i = \gamma$; $\delta_i = \delta$ = (*i* = 1, 2, …,z), the following result can be found:

$$
\begin{aligned}
&LNNEWA(u_1, u_2, \ldots u_z) = LNNEWA\ (u, u \ldots u) = (\overset{z}{\underset{i=1}{\oplus_e}} \epsilon_i u) \\
&= \langle \psi_{k * \frac{\prod_{i=1}^z (k+\beta)^{\epsilon_i} - \prod_{i=1}^z (k-\beta)^{\epsilon_i}}{\prod_{i=1}^z (k+\beta)^{\epsilon_i} + \prod_{i=1}^z (k-\beta)^{\epsilon_i}}}, \psi_{k * \frac{2 \prod_{i=1}^z \gamma^{\epsilon_i}}{\prod_{i=1}^z (2k-\gamma)^{\epsilon_i} + \prod_{i=1}^z \gamma^{\epsilon_i}}}, \psi_{k * \frac{2 \prod_{i=1}^z \delta^{\epsilon_i}}{\prod_{i=1}^z (2k-\delta)^{\epsilon_i} + \prod_{i=1}^z \delta^{\epsilon_i}}} \rangle \\
&= \langle \psi_{k * \frac{(k+\beta)-(k-\beta)}{(k+\beta)+(k-\beta)}}, \psi_{k * \frac{2\gamma}{(2k-\gamma)+\gamma}}, \psi_{k * \frac{2\delta}{(2k-\delta)+\delta}} \rangle \\
&= \langle \psi_\beta, \psi_\gamma, \psi_\delta \rangle = u
\end{aligned}
$$

Theorem 4. *(Monotonicity) set two collections of LNNs* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *and* $u_i{}' = \langle \psi_{\beta_i{}'}, \psi_{\gamma_i{}'}, \psi_{\delta_i{}'} \rangle$ *(i = 1, 2, … , z) in* Ψ, *if* $u_i \le u_i{}'$ *then*

$$LNNEWA(u_1, u_2, \ldots u_z) \le LNNEWA(u_1{}', u_2{}', \ldots u_z{}').$$

Proof. For $u_i \le u_i{}'$, then $\epsilon_i u_i \le \epsilon_i u_i{}'$

So, we can easily obtain:

$$\overset{z}{\underset{i=1}{\oplus_e}} \epsilon_i u_i \le \overset{z}{\underset{i=1}{\oplus_e}} \epsilon_i u_i{}'$$

For $LNNEWA(u_1, u_2, \ldots u_z) = \overset{z}{\underset{i=1}{\oplus_e}} \epsilon_i u_i$ and $LNNEWA(u_1{}', u_2{}', \ldots u_z{}') = \overset{z}{\underset{i=1}{\oplus_e}} \epsilon_i u_i{}'$, then we can get: $LNNEWA(u_1, u_2, \ldots u_z) \le LNNEWA(u_1{}', u_2{}', \ldots u_z{}')$. □

Theorem 5. *(Boundedness) Let a collection* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *in* Ψ, $u^- = \langle min(\psi_{\beta_i}), max(\psi_{\gamma_i}), max(\psi_{\delta_i}) \rangle$ *and* $u^+ = \langle max(\psi_{\beta_i}), min(\psi_{\gamma_i}), min(\psi_{\delta_i}) \rangle$, *we can get:*

$$u^- \le LNNEWA(u_1, u_2, \ldots u_z) \le u^+.$$

Proof. The following can be obtained by using Theorem 3:

$$\mathrm{u}^- = \mathrm{LNNEWA}(\mathrm{u}^-, \mathrm{u}^- \ldots \mathrm{u}^-),\ \mathrm{u}^+ = \mathrm{LNNEWA}(\mathrm{u}^+, \mathrm{u}^+ \ldots \mathrm{u}^+).$$

The following can be obtained by using Theorem 4:

$$\mathrm{LNNEWA}\ (\mathrm{u}^-, \mathrm{u}^- \ldots \mathrm{u}^-) \le \mathrm{LNNEWA}(\mathrm{u}_1, \mathrm{u}_2, \ldots \mathrm{u}_z) \le \mathrm{LNNEWA}(\mathrm{u}^+, \mathrm{u}^+ \ldots \mathrm{u}^+).$$

Above all, we can get:

$$\mathrm{u}^- \le \mathrm{LNNEWA}(\mathrm{u}_1, \mathrm{u}_2, \ldots \mathrm{u}_z) \le \mathrm{u}^+.$$

□

3.2. LNNEWG Operators

Definition 8. *Set a collection* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *in* Ψ, *for i = 1, 2, … , z, we define the LNNEWG operator:*

$$LNNEWG(u_1, u_2, \ldots u_z) = \overset{z}{\underset{i=1}{\otimes_e}} (u_i)^{\epsilon_i}, \tag{19}$$

with the relative weight vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_z)^T$, $\sum_{i=1}^z \epsilon_i = 1$ *and* $\epsilon_i \in [0, 1]$.

Theorem 6. *Set a collection* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *in* Ψ*, for i = 1,2, … ,z, then according to the LNNEWG aggregation operator, we can get the following result:*

$$\begin{aligned} LNNEWG(u_1, u_2, \ldots \ u_z) &= \overset{z}{\underset{i=1}{\otimes_e}} (u_i)^{\epsilon_i} \\ &= \langle \psi_{k * \frac{2\prod_{i=1}^{z} \beta_i^{\epsilon_i}}{\prod_{i=1}^{z} (2k-\beta_i)^{\epsilon_i} + \prod_{i=1}^{z} \beta_i^{\epsilon_i}}}, \psi_{k * \frac{\prod_{i=1}^{z} (k+\gamma_i)^{\epsilon_i} - \prod_{i=1}^{z} (k-\gamma_i)^{\epsilon_i}}{\prod_{i=1}^{z} (k+\gamma_i)^{\epsilon_i} + \prod_{i=1}^{z} (k-\gamma_i)^{\epsilon_i}}}, \psi_{k * \frac{\prod_{i=1}^{z} (k+\delta_i)^{\epsilon_i} - \prod_{i=1}^{z} (k-\delta_i)^{\epsilon_i}}{\prod_{i=1}^{z} (k+\delta_i)^{\epsilon_i} + \prod_{i=1}^{z} (k-\delta_i)^{\epsilon_i}}} \rangle \end{aligned} \tag{20}$$

with the relative weight vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_z)^T$, $\sum_{i=1}^{z} \epsilon_i = 1$ *and* $\epsilon_i \in [0, 1]$.

Theorem 7. *(Idempotency) Set a collection* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *in* Ψ*, for i = 1,2, … ,z, for every* u_i *in u is equal to u, we can get*

$$LNNEWG(u_1, u_2, \ldots u_z) = LNNEWG(u, u \ldots u) = u.$$

Theorem 8. *(Monotonicity). Set two collections of LNNs* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *and* $u_i{}' = \langle \psi_{\beta_i{}'}, \psi_{\gamma_i{}'}, \psi_{\delta_i{}'} \rangle$ *(i = 1, 2, … , z) in* Ψ*, if* $u_i \le u_i{}'$ *then*

$$LNNEWG(u_1, u_2, \ldots u_z) \le LNNEWG(u_1{}', u_2{}', \ldots u_z{}').$$

Theorem 9. *(Boundedness) Let a collection* $u_i = \langle \psi_{\beta_i}, \psi_{\gamma_i}, \psi_{\delta_i} \rangle$ *in* Ψ*,* $u^- = \langle min(\psi_{\beta_i}), max(\psi_{\gamma_i}), max(\psi_{\delta_i}) \rangle$ *and* $u^+ = \langle max(\psi_{\beta_i}), min(\psi_{\gamma_i}), min(\psi_{\delta_i}) \rangle$*, we can get:*

$$u^- \le LNNEWG(u_1, u_2, \ldots u_z) \le u^+$$

We omit the proof here because it is similar to Theorems 2–5.

4. Methods with LNNEWA or LNNEWG Operator

We introduce two MAGDM methods with the LNNEWA or LNNEWG operator in LNN information.

Now, we suppose that a collection of alternatives is expressed $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_m\}$ and a collection of attributes is expressed $E = \{E_1, E_2, \ldots, E_n\}$. Then, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{\mathrm{n}})^{\mathrm{T}}$ with $\sum_{\mathrm{i}=1}^{\mathrm{n}} \epsilon_{\mathrm{i}} = 1$ *and* $\epsilon_{\mathrm{i}} \in [0, 1]$ is the weight vector of $E_i (i = 1, 2, \ldots, n)$. Establishing a set of experts $D = \{D_1, D_2, \ldots, D_t\}$, $\mu = (\mu_1, \mu_2, \ldots, \mu_t)^T$ with $1 \ge \mu_j \ge 0$ and $\sum_{j=1}^{t} \mu_j = 1$ is the weight vector of $D_i (i = 1, 2, \ldots, t)$. Assuming that the expert $D_y (y = 1, 2, \ldots, t)$ uses the LNNs to give out the assessed value $\theta_{ij}^{(y)}$ for alternative Θ_i with the attribute E_j, the value $\theta_{ij}^{(y)}$ can be written as $\theta_{ij}^{(y)} = \langle \psi^y_{\beta_{ij}}, \psi^y_{\gamma_{ij}}, \psi^y_{\delta_{ij}} \rangle (y = 1, 2, \ldots, t\,; i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, $\psi^y_{\beta_{ij}}, \psi^y_{\gamma_{ij}}, \psi^y_{\delta_{ij}} \in \Psi$. Then, the decision evaluation matrix can be found. Table 1 is the decision matrix.

Table 1. The decision matrix using linguistic neutrosophic numbers (LNN).

	E_1	…	E_n
Θ_1	$\langle \psi^y_{\beta_{11}}, \psi^y_{\gamma_{11}}, \psi^y_{\delta_{11}} \rangle$	…	$\langle \psi^y_{\beta_{1n}}, \psi^y_{\gamma_{1n}}, \psi^y_{\delta_{1n}} \rangle$
Θ_2	$\langle \psi^y_{\beta_{21}}, \psi^y_{\gamma_{21}}, \psi^y_{\delta_{21}} \rangle$	…	$\langle \psi^y_{\beta_{2n}}, \psi^y_{\gamma_{2n}}, \psi^y_{\delta_{2n}} \rangle$
…	…	…	…
Θ_m	$\langle \psi^y_{\beta_{m1}}, \psi^y_{\gamma_{m1}}, \psi^y_{\delta_{m1}} \rangle$	…	$\langle \psi^y_{\beta_{mn}}, \psi^y_{\gamma_{mn}}, \psi^y_{\delta_{mn}} \rangle$

The decision steps are described as follows:

Step 1: the integrated matrix can be obtained by the *LNNEWA* operator:

$$\begin{aligned}\theta_{ij} &= \langle \psi_{\beta_{ij}}, \psi_{\gamma_{ij}}, \psi_{\delta_{ij}} \rangle = LNNEWA(\theta^1_{ij}, \theta^2_{ij}, \ldots, \theta^t_{ij}) = \overset{t}{\underset{l=1}{\oplus_e}} \theta_l \theta^l_{ij} \\ &= \langle \psi_{k*\frac{\prod_{l=1}^t (k+\beta^l_{ij})^{\mu_l} - \prod_{l=1}^t (k-\beta^l_{ij})^{\mu_l}}{\prod_{l=1}^t (k+\beta^l_{ij})^{\mu_l} + \prod_{l=1}^t (k-\beta^l_{ij})^{\mu_l}}}, \psi_{k*\frac{2\prod_{l=1}^t {\gamma^l_{ij}}^{\mu_l}}{\prod_{l=1}^t (2k-\gamma^l_{ij})^{\mu_l} + \prod_{l=1}^t {\gamma^l_{ij}}^{\mu_l}}}, \psi_{k*\frac{2\prod_{l=1}^t {\delta^l_{ij}}^{\mu_l}}{\prod_{l=1}^t (2k-\delta^l_{ij})^{\mu_l} + \prod_{l=1}^t {\delta^l_{ij}}^{\mu_l}}} \end{aligned} \tag{21}$$

Step 2: the total collective LNN θ_i $(i = 1, 2, \ldots, m)$ can be obtained by the *LNNWEA* or *LNNEWG* operator.

$$\begin{aligned}\theta_i &= LNNEWA(\theta_{i1}, \theta_{i2}, \ldots, \theta_{in}) = \overset{n}{\underset{j=1}{\oplus_e}} \epsilon_{ij} \theta_{ij} \\ &= \langle \psi_{k*\frac{\prod_{j=1}^n (k+\beta_{ij})^{\epsilon_{ij}} - \prod_{j=1}^n (k-\beta_{ij})^{\epsilon_{ij}}}{\prod_{j=1}^n (k+\beta_{ij})^{\epsilon_{ij}} + \prod_{j=1}^n (k-\beta_{ij})^{\epsilon_{ij}}}}, \psi_{k*\frac{2\prod_{j=1}^n {\gamma_{ij}}^{\epsilon_{ij}}}{\prod_{j=1}^n (2k-\gamma_{ij})^{\epsilon_{ij}} + \prod_{j=1}^n {\gamma_{ij}}^{\epsilon_{ij}}}}, \psi_{k*\frac{2\prod_{j=1}^n {\delta_{ij}}^{\epsilon_{ij}}}{\prod_{j=1}^n (2k-\delta_{ij})^{\epsilon_{ij}} + \prod_{j=1}^n {\delta_{ij}}^{\epsilon_{ij}}}} \rangle \end{aligned} \tag{22}$$

Or

$$\begin{aligned}\theta_i &= LNNEWG(\theta_{i1}, \theta_{i2}, \ldots, \theta_{in}) = \overset{n}{\underset{j=1}{\otimes_e}} (\theta_{ij})^{\epsilon_{ij}} \\ &= \langle \psi_{k*\frac{2\prod_{j=1}^n {\beta_{ij}}^{\epsilon_{ij}}}{\prod_{j=1}^n (2k-\beta_{ij})^{\epsilon_{ij}} + \prod_{j=1}^n {\beta_{ij}}^{\epsilon_{ij}}}}, \psi_{k*\frac{\prod_{j=1}^n (k+\gamma_{ij})^{\epsilon_{ij}} - \prod_{j=1}^n (k-\gamma_{ij})^{\epsilon_{ij}}}{\prod_{j=1}^n (k+\gamma_{ij})^{\epsilon_{ij}} + \prod_{j=1}^n (k-\gamma_{ij})^{\epsilon_{ij}}}}, \psi_{k*\frac{\prod_{j=1}^n (k+\delta_{ij})^{\epsilon_{ij}} - \prod_{j=1}^n (k-\delta_{ij})^{\epsilon_{ij}}}{\prod_{j=1}^n (k+\delta_{ij})^{\epsilon_{ij}} + \prod_{j=1}^n (k-\delta_{ij})^{\epsilon_{ij}}}} \rangle \end{aligned} \tag{23}$$

Step 3: according to Definition 3, we can calculate $\zeta(\theta_i)$ and $\eta(\theta_i)$ of every LNN $\Theta_i (i = 1, 2, \ldots, m)$.
Step 4: According to $\zeta(\theta_i)$, then we can rank the alternatives and the best one can be chosen out.
Step 5: End.

5. Illustrative Examples

5.1. Numerical Example

Now, we adopt illustrative examples of the MAGDM problems to verify the proposed decision methods. An investment company wants to find a company to invest. Now, there are four companies $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}$ to be considered as candidates, the first is for selling cars (Θ_1), the second is for selling food (Θ_2), the third is for selling computers (Θ_3), and the last is for selling arms (Θ_4). Next, three experts $D = \{D_1, D_2, D_3\}$ are invited to evaluate these companies, their weight vector is $\mu = (0.37, 0.33, 0.3)^T$. The experts make evaluations of the alternatives according to three attributes $E = \{E_1, E_2, E_3\}$, E_1 is the ability of risk, E_2 is the ability of growth, and E_3 is the ability of environmental impact, the weight vector of them is $\epsilon = (0.35, 0.25, 0.4)^T$. Then, the experts use LNNs to make the evaluation values with a linguistic set $\Psi = \{\psi_0 =$ extremely poor, $\psi_1 =$ very poor, $\psi_2 =$ poor, $\psi_3 =$ slightly poor, $\psi_4 =$ medium , $\psi_5 =$ slightlygood, $\psi_6 =$ good, $\psi_7 =$ very good, $\psi_8 =$ extremely good$\}$.

Then, the decision evaluation matrix can be established, Tables 2–4 show them.

Table 2. The decision matrix based on the data of D_1.

	E_1	E_2	E_3
Θ_1	$\langle \psi^1_6, \psi^1_1, \psi^1_2 \rangle$	$\langle \psi^1_7, \psi^1_2, \psi^1_1 \rangle$	$\langle \psi^1_6, \psi^1_2, \psi^1_2 \rangle$
Θ_2	$\langle \psi^1_7, \psi^1_1, \psi^1_1 \rangle$	$\langle \psi^1_7, \psi^1_3, \psi^1_2 \rangle$	$\langle \psi^1_7, \psi^1_2, \psi^1_1 \rangle$
Θ_3	$\langle \psi^1_6, \psi^1_2, \psi^1_2 \rangle$	$\langle \psi^1_7, \psi^1_1, \psi^1_1 \rangle$	$\langle \psi^1_6, \psi^1_2, \psi^1_2 \rangle$
Θ_4	$\langle \psi^1_7, \psi^1_1, \psi^1_2 \rangle$	$\langle \psi^1_7, \psi^1_2, \psi^1_3 \rangle$	$\langle \psi^1_7, \psi^1_2, \psi^1_1 \rangle$

Table 3. The decision matrix based on the data of D_2.

	$\boldsymbol{E_1}$	$\boldsymbol{E_2}$	$\boldsymbol{E_3}$
Θ_1	$\langle \psi_6^2, \psi_1^2, \psi_2^2 \rangle$	$\langle \psi_6^2, \psi_1^2, \psi_1^2 \rangle$	$\langle \psi_4^2, \psi_2^2, \psi_3^2 \rangle$
Θ_2	$\langle \psi_7^2, \psi_2^2, \psi_3^2 \rangle$	$\langle \psi_6^2, \psi_1^2, \psi_1^2 \rangle$	$\langle \psi_4^2, \psi_2^2, \psi_3^2 \rangle$
Θ_3	$\langle \psi_5^2, \psi_1^2, \psi_2^2 \rangle$	$\langle \psi_5^2, \psi_1^2, \psi_2^2 \rangle$	$\langle \psi_5^2, \psi_4^2, \psi_2^2 \rangle$
Θ_4	$\langle \psi_6^2, \psi_1^2, \psi_1^2 \rangle$	$\langle \psi_5^2, \psi_1^2, \psi_1^2 \rangle$	$\langle \psi_5^2, \psi_2^2, \psi_3^2 \rangle$

Table 4. The decision matrix based on the data of D_3.

.	$\boldsymbol{E_1}$	$\boldsymbol{E_2}$	$\boldsymbol{E_3}$
Θ_1	$\langle \psi_7^3, \psi_3^3, \psi_4^3 \rangle$	$\langle \psi_7^3, \psi_3^3, \psi_3^3 \rangle$	$\langle \psi_5^3, \psi_2^3, \psi_5^3 \rangle$
Θ_2	$\langle \psi_6^3, \psi_3^3, \psi_4^3 \rangle$	$\langle \psi_5^3, \psi_1^3, \psi_2^3 \rangle$	$\langle \psi_6^3, \psi_2^3, \psi_3^3 \rangle$
Θ_3	$\langle \psi_7^3, \psi_2^3, \psi_4^3 \rangle$	$\langle \psi_6^3, \psi_1^3, \psi_2^3 \rangle$	$\langle \psi_7^3, \psi_2^3, \psi_4^3 \rangle$
Θ_4	$\langle \psi_7^3, \psi_2^3, \psi_3^3 \rangle$	$\langle \psi_5^3, \psi_2^3, \psi_1^3 \rangle$	$\langle \psi_6^3, \psi_1^3, \psi_1^3 \rangle$

Now, the proposed method is applied to manage this MAGDM problem and the computational procedures are as follows:

Step 1: the overall decision matrix can be obtained by the *LNNEWA* operator in Table 5.

Table 5. The overall decision matrix.

	$\boldsymbol{E_1}$	$\boldsymbol{E_2}$	$\boldsymbol{E_3}$
Θ_1	$\langle \psi_{6.3671}, \psi_{1.4116}, \psi_{2.4888} \rangle$	$\langle \psi_{6.7366}, \psi_{1.8191}, \psi_{1.4116} \rangle$	$\langle \psi_{5.1343}, \psi_{2.000}, \psi_{3.0637} \rangle$
Θ_2	$\langle \psi_{6.7630}, \psi_{1.7705}, \psi_{2.2397} \rangle$	$\langle \psi_{6.2295}, \psi_{1.5275}, \psi_{1.5997} \rangle$	$\langle \psi_{6.0042}, \psi_{2.000}, \psi_{2.0355} \rangle$
Θ_3	$\langle \psi_{6.1200}, \psi_{1.5997}, \psi_{2.4888} \rangle$	$\langle \psi_{6.2067}, \psi_{1.000}, \psi_{1.5564} \rangle$	$\langle \psi_{6.1200}, \psi_{2.5427}, \psi_{2.4888} \rangle$
Θ_4	$\langle \psi_{6.7366}, \psi_{1.2370}, \psi_{1.8191} \rangle$	$\langle \psi_{5.9645}, \psi_{1.5997}, \psi_{1.5275} \rangle$	$\langle \psi_{6.2067}, \psi_{1.6329}, \psi_{1.4602} \rangle$

Step 2: the total collective LNN $\theta_i (i = 1, 2, \ldots, m)$ can be obtained by the *LNNWEA* operator:

$$\theta_1 = \langle \psi_{6.0661}, \psi_{1.7313}, \psi_{2.3644} \rangle, \theta_2 = \langle \psi_{6.0961}, \psi_{1.7929}, \psi_{1.9840} \rangle,$$
$$\theta_3 = \langle \psi_{5.7523}, \psi_{1.7260}, \psi_{2.2199} \rangle, \text{ and } \theta_4 = \langle \psi_{6.4198}, \psi_{1.4753}, \psi_{1.5957} \rangle.$$

Step 3: according to Definition 3, the expected values of $\zeta(\theta_i)$ for $\theta_i (i = 1, 2, 3, 4)$ can be calculated:

$$\zeta(\theta_1) = 0.7488, \zeta(\theta_2) = 0.7633, \zeta(\theta_3) = 0.7419, \text{ and } \zeta(\theta_4) = 0.8062.$$

Based on the expected values, four alternatives can be ranked $\Theta_4 \succ \Theta_2 \succ \Theta_1 \succ \Theta_3$, thus, company Θ_4 is the optimal choice.

Now, the *LNNEWG* operator was used to manage this MAGDM problem:

Step 1′: the overall decision matrix can be obtained by the *LNNEWA* operator;

Step 2′: the total collective LNN θ_i $(i = 1, 2, \ldots, m)$ can be obtained by the *LNNEWG* operator, which are as following:

$$\theta_1 = \langle \psi_{5.9491}, \psi_{1.7507}, \psi_{2.4660} \rangle, \theta_2 = \langle \psi_{6.5864}, \psi_{1.8026}, \psi_{2.0000} \rangle, \theta_3 = \langle \psi_{6.8354}, \psi_{1.8390}, \psi_{2.2614} \rangle,$$
$$\text{and } \theta_4 = \langle \psi_{6.3950}, \psi_{1.4868}, \psi_{1.6033} \rangle.$$

Step 3′: according to Definition 3, the expected values of $\zeta(\theta_i)$ for $\theta_i (i = 1, 2, 3, 4)$ can be calculated:

$$\zeta(\theta_1) = 0.7389, \zeta(\theta_2) = 0.7827, (\theta_3) = 0.7806, \text{ and } \zeta(\theta_4) = 0.8043.$$

Based on the expected values, four alternatives can be ranked $\Theta_4 \succ \Theta_2 \succ \Theta_3 \succ \Theta_1$, thus, company Θ_4 is still the optimal choice.

Clearly, there exists a small difference in sorting between these two kinds of methods. However, we can get the same optimal choice by using the LNNEWA and LNNEWG operators. The proposed methods are effective ranking methods for the MCDM problem.

5.2. Comparative Analysis

Now, we do some comparisons with other related methods for LNN, all the results are shown in Table 6.

Table 6. The ranking orders by utilizing three different methods.

Method	Result	Ranking Order	The Best Alternative
Method 1 based on arithmetic averaging in [15]	$\zeta(\theta_1) = 0.7528$, $\zeta(\theta_2) = 0.7777$, $\zeta(\theta_3) = 0.7613$, $\zeta(\theta_4) = 0.8060$.	$\theta_4 > \theta_2 > \theta_3 > \theta_1$	θ_4
Method 2 based on geometric averaging in [15]	$\zeta(\theta_1) = 0.7397$, $\zeta(\theta_2) = 0.7747$, $\zeta(\theta_3) = 0.7531$, $\zeta(\theta_4) = 0.8035$.	$\theta_4 > \theta_2 > \theta_3 > \theta_1$	θ_4
Method 3 based on Bonferroni Mean in [16] (p = q = 1)	$\zeta(\theta_1) = 0.7298$, $\zeta(\theta_2) = 0.7508$, $\zeta(\theta_3) = 0.7424$ $\zeta(\theta_4) = 0.7864$.	$\theta_4 > \theta_2 > \theta_3 > \theta_1$	θ_4
The proposed method	$\zeta(\theta_1) = 0.7488$, $\zeta(\theta_2) = 0.7633$, $\zeta(\theta_3) = 0.7419$ $\zeta(\theta_4) = 0.8062$.	$\theta_4 > \theta_2 > \theta_1 > \theta_3$	θ_4

As shown in Table 6, we can see that company θ_4 is the best choice for investing by using four methods. Many methods such as arithmetic averaging, geometric averaging, and Bonferroni mean can all be used in LNN to handle the multiple attribute decision-making problems and can get similar results. Additionally, The Einstein aggregation operator is smoother than the algebra aggregation operator, which is used in the literature [15,16]. Compared to the existing literature [2–14], LNNs can express and manage pure linguistic evaluation values, while other literature [2–14] cannot do that. In this paper, a new MAGDM method was presented by using the LNNEWA or LNNEWG operator based on LNN environment.

6. Conclusions

A new approach for solving MAGDM problems was proposed in this paper. First, we applied the Einstein operation to a linguistic neutrosophic set and established the new operation rules of this linguistic neutrosophic set based on the Einstein operator. Second, we combined some aggregation operators with the linguistic neutrosophic set and defined the linguistic neutrosophic number Einstein weight average operator and the linguistic neutrosophic number Einstein weight geometric (LNNEWG) operator according the new operation rules. Finally, by using the LNNEWA and LNNEWG operator, two methods for handling MADGM problem were presented. In addition, these two methods were introduced into a concrete example to show the practicality and advantages of the proposed approach. In future, we will further study the Einstein operation in other neutrosophic environment just like the refined neutrosophic set [30]. At the same time, we will use these aggregation operators in many actual fields, such as campaign management, decision making and clustering analysis and so on [31–33].

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