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Neutrosophic Quadruple BCI-Positive Implicative Ideals

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Abstract: By considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, Smarandache introduced the concept of neutrosophic quadruple numbers. Using the concept of neutrosophic quadruple numbers based on a set, Jun et al. constructed neutrosophic quadruple BCK/BCI-algebras and implicative neutrosophic quadruple BCK-algebras. The notion of a neutrosophic quadruple BCI-positive implicative ideal is introduced, and several properties are dealt with in this article. We establish the relationship between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal. Given nonempty subsets I and J of a BCI-algebra, conditions for the neutrosophic quadruple (I, J) -set to be a neutrosophic quadruple BCI-positive implicative ideal are provided.

Keywords: neutrosophic quadruple BCK/BCI-number; neutrosophic quadruple BCK/BCI-algebra; neutrosophic quadruple (BCI-positive implicative) ideal

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1. Introduction

A BCK/BCI-algebra is a class of logical algebras introduced by K. Iséki (see [1,2]) and was extensively investigated by several researchers. Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [3–10]. Smarandache introduced the notion of neutrosophic sets with wide applications in sciences (see [11–13]), which is a more general stage to extend the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Smarandache [14] introduced the concept of neutrosophic quadruple numbers by considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF), where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers. Using the notion of neutrosophic quadruple numbers based on a set, Jun et al. [15] constructed neutrosophic quadruple BCK/BCI-algebras and implicative neutrosophic quadruple BCK-algebras (see also [16]).

In this paper, we introduce the notion of a neutrosophic quadruple BCI-positive implicative ideal, and investigate several properties. We consider relations between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal. Given nonempty subsets I and J of a BCI-algebra U , we provide conditions for the neutrosophic quadruple (I, J) -set to be a neutrosophic quadruple BCI-positive implicative ideal.

2. Preliminaries

A *BCI-algebra* is a set U with a special element 0 and a binary operation $*$ that satisfies the following conditions:

- (I) $(\forall p, q, r \in U) (((p * q) * (p * r)) * (r * q) = 0),$
- (II) $(\forall p, q \in U) ((p * (p * q)) * q = 0),$
- (III) $(\forall p \in U) (p * p = 0),$
- (IV) $(\forall p, q \in U) (p * q = 0, q * p = 0 \Rightarrow p = q).$

If a BCI-algebra U satisfies the following identity:

- (V) $(\forall p \in U) (0 * p = 0),$

then U is called a *BCK-algebra*. In a BCK/BCI-algebra U , the following conditions are valid.

$$(\forall p \in U) (p * 0 = p), \quad (1)$$

$$(\forall p, q, r \in U) (p \leq q \Rightarrow p * r \leq q * r, r * q \leq r * p), \quad (2)$$

$$(\forall p, q, r \in U) ((p * q) * r = (p * r) * q), \quad (3)$$

$$(\forall p, q, r \in U) ((p * r) * (q * r) \leq p * q) \quad (4)$$

where $p \leq q$ if and only if $p * q = 0$.

Any BCI-algebra U satisfies the following conditions (see [17]):

$$(\forall p, q \in U) (p * (p * (p * q))) = p * q, \quad (5)$$

$$(\forall p, q \in U) (0 * (p * q) = (0 * p) * (0 * q)), \quad (6)$$

$$(\forall p, q \in U) (0 * (0 * (p * q))) = (0 * q) * (0 * p)). \quad (7)$$

By a *subalgebra* of a BCK/BCI-algebra U , we mean a nonempty subset S of U such that $p * q \in S$ for all $p, q \in S$. We say that a subset G of a BCK/BCI-algebra U is an *ideal* of U if it satisfies:

$$0 \in G, \quad (8)$$

$$(\forall p \in U) (\forall q \in G) (p * q \in G \Rightarrow p \in G). \quad (9)$$

A subset G of a BCI-algebra U is called a *BCI-positive implicative ideal* of U (see [18,19]) if it satisfies (8) and

$$(\forall p, q, r \in U) (((p * r) * r) * (q * r) \in G, q \in G \Rightarrow p * r \in G), \quad (10)$$

For further information regarding BCK/BCI-algebras and neutrosophic set theory, we refer the reader to the books [17,20] and the site [21] respectively. We will use neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let U be a set. A *neutrosophic quadruple U-number* is an ordered quadruple (a, pT, qI, rF) , where $a, p, q, r \in U$ and T, I, F have their usual neutrosophic logic meanings (see [15]).

The set of all neutrosophic quadruple U -numbers which is denoted by $\mathcal{N}(U)$, that is,

$$\mathcal{N}(U) := \{(a, pT, qI, rF) \mid a, p, q, r \in U\},$$

is called the *neutrosophic quadruple set* based on U . In particular, if U is a BCK/BCI-algebra, then a neutrosophic quadruple U -number is called a *neutrosophic quadruple BCK/BCI-number* and $\mathcal{N}(U)$ is called the *neutrosophic quadruple BCK/BCI-set*.

We define a binary operation $\square$ on the neutrosophic quadruple BCK/BCI-set $\mathcal{N}(U)$ by

$$(a, pT, qI, rF) \square (b, uT, vI, wF) = (a * b, (p * u)T, (q * v)I, (z * w)F)$$

for all $(a, pT, qI, rF), (b, uT, vI, wF) \in \mathcal{N}(U)$. Given $a_1, a_2, a_3, a_4 \in U$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F),$$

which is called the *zero neutrosophic quadruple BCK/BCI-number*. Then $(\mathcal{N}(U); \square, \tilde{0})$ is a BCK/BCI-algebra (see [15]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $\mathcal{N}(U)$.

We define an order relation “ $\ll$ ” and the equality “ $=$ ” on the neutrosophic quadruple BCK/BCI-algebra $\mathcal{N}(U)$ as follows:

$$\begin{aligned}\tilde{p} \ll \tilde{q} &\Leftrightarrow p_i \leq q_i \text{ for } i = 1, 2, 3, 4, \\ \tilde{p} = \tilde{q} &\Leftrightarrow p_i = q_i \text{ for } i = 1, 2, 3, 4\end{aligned}$$

for all $\tilde{p} = (p_1, p_2T, p_3I, p_4F), \tilde{q} = (q_1, q_2T, q_3I, q_4F) \in \mathcal{N}(U)$. It is easy to verify that “ $\ll$ ” is an equivalence relation on $\mathcal{N}(U)$.

Let U be a BCK/BCI-algebra. Given nonempty subsets I and J of U , consider the set

$$\mathcal{N}(I, J) := \{(a, pT, qI, rF) \in \mathcal{N}(U) \mid a, p \in I \text{ \& } q, r \in J\},$$

which is called the *neutrosophic quadruple (I, J) -set*.

The neutrosophic quadruple (I, J) -set $\mathcal{N}(I, J)$ with $I = J$ is denoted by $\mathcal{N}(I)$, and it is called the *neutrosophic quadruple I -set*.

3. Neutrosophic Quadruple BCI-Positive Implicative Ideals

In what follows, let U and $\mathcal{N}(U)$ be a BCI-algebra and a neutrosophic quadruple BCI-algebra, respectively, unless otherwise specified.

Definition 1. Given nonempty subsets I and J of U , if $\mathcal{N}(I, J)$ is a BCI-positive implicative ideal of $\mathcal{N}(U)$, we say $\mathcal{N}(I, J)$ is a *neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$* .

Example 1. Consider a BCI-algebra $U = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 1.

Table 1. Cayley table for the binary operation “ $*$ ”.

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Then the neutrosophic quadruple BCI-algebra $\mathcal{N}(U)$ has 81 elements. If we take $I = \{0, a\}$ and $J = \{0, a\}$, then

$$\mathcal{N}(I, J) = \{\tilde{0}, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{\beta}_4, \tilde{\beta}_5, \tilde{\beta}_6, \tilde{\beta}_7, \tilde{\beta}_8, \tilde{\beta}_9, \tilde{\beta}_{10}, \tilde{\beta}_{11}, \tilde{\beta}_{12}, \tilde{\beta}_{13}, \tilde{\beta}_{14}, \tilde{\beta}_{15}\}$$

and it is routine to check that $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ where

$$\begin{aligned}\tilde{0} &= (0, 0T, 0I, 0F), \tilde{\beta}_1 = (0, 0T, 0I, aF), \tilde{\beta}_2 = (0, 0T, aI, 0F), \tilde{\beta}_3 = (0, 0T, aI, aF), \\ \tilde{\beta}_4 &= (0, aT, 0I, 0F), \tilde{\beta}_5 = (0, aT, 0I, aF), \tilde{\beta}_6 = (0, aT, aI, 0F), \tilde{\beta}_7 = (0, aT, aI, aF),\end{aligned}$$

$$\begin{aligned}\tilde{\beta}_8 &= (a, 0T, 0I, 0F), \tilde{\beta}_9 = (a, 0T, 0I, aF), \tilde{\beta}_{10} = (a, 0T, aI, 0F), \tilde{\beta}_{11} = (a, 0T, aI, aF), \\ \tilde{\beta}_{12} &= (a, aT, 0I, 0F), \tilde{\beta}_{13} = (a, aT, 0I, aF), \tilde{\beta}_{14} = (a, aT, aI, 0F), \tilde{\beta}_{15} = (a, aT, aI, aF).\end{aligned}$$

Proposition 1. Given nonempty subsets I and J of U , the neutrosophic quadruple BCI-positive implicative ideal $\mathcal{N}(I, J)$ of $\mathcal{N}(U)$ satisfies the following assertions.

$$(\forall \tilde{p}, \tilde{q}, \tilde{r} \in \mathcal{N}(U))(((\tilde{p} \sqcup \tilde{r}) \sqcup \tilde{r}) \sqcup (\tilde{q} \sqcup \tilde{r}) \in \mathcal{N}(I, J) \Rightarrow (\tilde{p} \sqcup \tilde{q}) \sqcup \tilde{r} \in \mathcal{N}(I, J)), \quad (11)$$

$$(\forall \tilde{p}, \tilde{q} \in \mathcal{N}(U))(((\tilde{p} \sqcup \tilde{q}) \sqcup \tilde{q}) \sqcup (\tilde{0} \sqcup \tilde{q}) \in \mathcal{N}(I, J) \Rightarrow \tilde{p} \sqcup \tilde{q} \in \mathcal{N}(I, J)). \quad (12)$$

Proof. Let $\mathcal{N}(I, J)$ be a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ for any nonempty subsets I and J of U . Assume that $((\tilde{p} \sqcup \tilde{r}) \sqcup \tilde{r}) \sqcup (\tilde{q} \sqcup \tilde{r}) \in \mathcal{N}(I, J)$ for all $\tilde{p}, \tilde{q}, \tilde{r} \in \mathcal{N}(U)$. Since

$$\begin{aligned}(((\tilde{p} \sqcup \tilde{q}) \sqcup \tilde{r}) \sqcup \tilde{r}) \sqcup (\tilde{0} \sqcup \tilde{r}) &= (((\tilde{p} \sqcup \tilde{r}) \sqcup \tilde{r}) \sqcup \tilde{q}) \sqcup ((\tilde{q} \sqcup \tilde{q}) \sqcup \tilde{r}) \\ &= (((\tilde{p} \sqcup \tilde{r}) \sqcup \tilde{r}) \sqcup \tilde{q}) \sqcup ((\tilde{q} \sqcup \tilde{r}) \sqcup \tilde{q}) \\ &\leq ((\tilde{p} \sqcup \tilde{r}) \sqcup \tilde{r}) \sqcup (\tilde{q} \sqcup \tilde{r}),\end{aligned}$$

we have $((\tilde{p} \sqcup \tilde{q}) \sqcup \tilde{r}) \sqcup \tilde{r} \sqcup (\tilde{0} \sqcup \tilde{r}) \in \mathcal{N}(I, J)$. Since $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal, it follows that $(\tilde{p} \sqcup \tilde{q}) \sqcup \tilde{r} \in \mathcal{N}(I, J)$. Hence (11) is valid. If we take $\tilde{q} = \tilde{0}$ and $\tilde{r} = \tilde{q}$ in (11), then we get (12). $\square$

We consider relations between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal.

Theorem 1. For any nonempty subsets I and J of U , if $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$, then it is a neutrosophic quadruple ideal of $\mathcal{N}(U)$.

Proof. Assume that $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$. Let $\tilde{p} = (p_1, p_2T, p_3I, p_4F)$ and $\tilde{q} = (q_1, q_2T, q_3I, q_4F)$ be elements of $\mathcal{N}(U)$ such that $\tilde{q} \in \mathcal{N}(I, J)$ and $\tilde{p} \sqcup \tilde{q} \in \mathcal{N}(I, J)$. Then

$$((\tilde{p} \sqcup \tilde{0}) \sqcup \tilde{0}) \sqcup (\tilde{q} \sqcup \tilde{0}) = \tilde{p} \sqcup \tilde{q} \in \mathcal{N}(I, J),$$

which implies that $\tilde{p} = \tilde{p} \sqcup \tilde{0} \in \mathcal{N}(I, J)$. Therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple ideal of $\mathcal{N}(U)$. $\square$

The converse of Theorem 1 is not true as seen in the following example.

Example 2. Consider a BCI-algebra $U = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 2.

Table 2. Cayley table for the binary operation “ $*$ ”.

$*$	0	1	a
0	0	0	0
1	1	0	0
a	a	1	0

Then the neutrosophic quadruple BCI-algebra $\mathcal{N}(U)$ has 81 elements. If we take $I = \{0\}$ and $J = \{0\}$, then $\mathcal{N}(I, J) = \{\tilde{0}\}$ is a neutrosophic quadruple ideal of $\mathcal{N}(U)$. But it is not a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ since

$$(((a, aT, aI, aF) \sqcup (1, 1T, 1I, 1F)) \sqcup (1, 1T, 1I, 1F)) \sqcup (\tilde{0} \sqcup (1, 1T, 1I, 1F)) = \tilde{0} \in \mathcal{N}(I, J)$$

and $(a, aT, aI, aF) \sqsubseteq (1, 1T, 1I, 1F) = (1, 1T, 1I, 1F) \notin \mathcal{N}(I, J)$.

Given nonempty subsets I and J of U , we provide conditions for the set $\mathcal{N}(I, J)$ to be a neutrosophic quadruple BCI-positive implicative ideal.

Theorem 2. If I and J are BCI-positive implicative ideal of U , then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Assume that I and J are BCI-positive implicative ideal of U . Obviously $\tilde{0} \in \mathcal{N}(I, J)$. Let $\tilde{p} = (p_1, p_2T, p_3I, p_4F)$, $\tilde{q} = (q_1, q_2T, q_3I, q_4F)$ and $\tilde{r} = (r_1, r_2T, r_3I, r_4F)$ be elements of $\mathcal{N}(U)$ such that $\tilde{q} \in \mathcal{N}(I, J)$ and $((\tilde{p} \sqsubseteq \tilde{r}) \sqsubseteq \tilde{r}) \sqsubseteq (\tilde{q} \sqsubseteq \tilde{r}) \in \mathcal{N}(I, J)$. Then $q_i \in I$ and $q_j \in J$ for $i = 1, 2$ and $j = 3, 4$. Also

$$\begin{aligned} ((\tilde{p} \sqsubseteq \tilde{r}) \sqsubseteq \tilde{r}) \sqsubseteq (\tilde{q} \sqsubseteq \tilde{r}) &= (((p_1, p_2T, p_3I, p_4F) \sqsubseteq (r_1, r_2T, r_3I, r_4F)) \sqsubseteq (r_1, r_2T, r_3I, r_4F)) \\ &\quad \sqsubseteq ((q_1, q_2T, q_3I, q_4F) \sqsubseteq (r_1, r_2T, r_3I, r_4F)) \\ &= ((p_1 * r_1, (p_2 * r_2)T, (p_3 * r_3)I, (p_4 * r_4)F) \sqsubseteq (r_1, r_2T, r_3I, r_4F)) \\ &\quad \sqsubseteq (q_1 * r_1, (q_2 * r_2)T, (q_3 * r_3)I, (q_4 * r_4)F) \\ &= (((p_1 * r_1) * r_1, ((p_2 * r_2) * r_2)T, ((p_3 * r_3) * r_3)I, ((p_4 * r_4) * r_4)F)) \\ &\quad \sqsubseteq (q_1 * r_1, (q_2 * r_2)T, (q_3 * r_3)I, (q_4 * r_4)F) \\ &= (((p_1 * r_1) * r_1) * (q_1 * r_1), (((p_2 * r_2) * r_2) * (q_2 * r_2))T, \\ &\quad (((p_3 * r_3) * r_3) * (q_3 * r_3))I, (((p_4 * r_4) * r_4) * (q_4 * r_4))F) \\ &\in \mathcal{N}(I, J), \end{aligned}$$

and so $((p_i * r_i) * r_i) * (q_i * r_i) \in I$ and $((p_j * r_j) * r_j) * (q_j * r_j) \in J$ for $i = 1, 2$ and $j = 3, 4$. it follows from (10) that $p_i * r_i \in I$ and $p_j * r_j \in J$ for $i = 1, 2$ and $j = 3, 4$. Hence

$$\begin{aligned} \tilde{p} \sqsubseteq \tilde{r} &= (p_1, p_2T, p_3I, p_4F) \sqsubseteq (r_1, r_2T, r_3I, r_4F) \\ &= (p_1 * r_1, (p_2 * r_2)T, (p_3 * r_3)I, (p_4 * r_4)F) \in \mathcal{N}(I, J). \end{aligned}$$

Therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$. $\square$

Corollary 1. If I is a BCI-positive implicative ideal of U , then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 3. Let I and J be ideals of U which satisfies the following condition.

$$(\forall p, q \in U)((p * q) * q) * (0 * q) \in I \cap J \Rightarrow p * q \in I \cap J. \quad (13)$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Obviously $\tilde{0} \in \mathcal{N}(I, J)$. Let $\tilde{p} = (p_1, p_2T, p_3I, p_4F)$, $\tilde{q} = (q_1, q_2T, q_3I, q_4F)$ and $\tilde{r} = (r_1, r_2T, r_3I, r_4F)$ be elements of $\mathcal{N}(U)$ such that $\tilde{r} \in \mathcal{N}(I, J)$ and $((\tilde{p} \sqsubseteq \tilde{q}) \sqsubseteq \tilde{q}) \sqsubseteq (\tilde{r} \sqsubseteq \tilde{q}) \in \mathcal{N}(I, J)$. Then $r_1, r_2 \in I$, $r_3, r_4 \in J$ and

$$\begin{aligned} ((\tilde{p} \sqsubseteq \tilde{q}) \sqsubseteq \tilde{q}) \sqsubseteq (\tilde{r} \sqsubseteq \tilde{q}) &= (((p_1 * q_1) * q_1) * (r_1 * q_1), (((p_2 * q_2) * q_2) * (r_2 * q_2))T, \\ &\quad (((p_3 * q_3) * q_3) * (r_3 * q_3))I, (((p_4 * q_4) * q_4) * (r_4 * q_4))F) \\ &\in \mathcal{N}(I, J), \end{aligned}$$

that is,

$$((p_i * q_i) * q_i) * (r_i * q_i) \in I \text{ and } ((p_j * q_j) * q_j) * (r_j * q_j) \in J \quad (14)$$

for $i = 1, 2$ and $j = 3, 4$. Note that

$$\begin{aligned} &(((p_k * q_k) * q_k) * (0 * q_k)) * (((p_k * q_k) * q_k) * (r_k * q_k)) \\ &\leq (r_k * q_k) * (0 * q_k) \leq r_k * 0 = r_k \end{aligned}$$

for $k = 1, 2, 3, 4$ by (I), (1) and (4). Since I and J are ideals of U , it follows that

$$\begin{aligned} &(((p_i * q_i) * q_i) * (0 * q_i)) * (((p_i * q_i) * q_i) * (r_i * q_i)) \in I, \\ &(((p_j * q_j) * q_j) * (0 * q_j)) * (((p_j * q_j) * q_j) * (r_j * q_j)) \in J \end{aligned} \quad (15)$$

for $i = 1, 2$ and $j = 3, 4$. Combining (14) and (15), we get

$$((p_i * q_i) * q_i) * (0 * q_i) \in I \text{ and } ((p_j * q_j) * q_j) * (0 * q_j) \in J$$

for $i = 1, 2$ and $j = 3, 4$. Using (13) implies that $p_i * q_i \in I$ and $p_j * q_j \in J$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\begin{aligned} \tilde{p} \sqcap \tilde{q} &= (p_1, p_2 T, p_3 I, p_4 F) \sqcap (q_1, q_2 T, q_3 I, q_4 F) \\ &= (p_1 * q_1, (p_2 * q_2) T, (p_3 * q_3) I, (p_4 * q_4) F) \in \mathcal{N}(I, J). \end{aligned}$$

Therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$. $\square$

Corollary 2. Let I be an ideal of U which satisfies the following condition.

$$(\forall p, q \in U)((p * q) * q) * (0 * q) \in I \Rightarrow p * q \in I. \quad (16)$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 4. Let I and J be ideals of U which satisfies the following condition.

$$(\forall p, q, r \in U)((p * r) * (q * r)) * r \in I \cap J \Rightarrow (p * q) * r \in I \cap J. \quad (17)$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Suppose that $((p * q) * q) * (0 * q) \in I \cap J$ for all $p, q \in U$. Then $((p * q) * (0 * q)) * q = ((p * q) * q) * (0 * q) \in I \cap J$, which implies from (17) and (1) that $p * q = (p * 0) * q \in I \cap J$. Therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 3. $\square$

Corollary 3. Let I be an ideal of U which satisfies the following condition.

$$(\forall p, q, r \in U)((p * r) * (q * r)) * r \in I \Rightarrow (p * q) * r \in I. \quad (18)$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 5. Let I and J be subsets of U such that

$$0 \in I \cap J, \quad (19)$$

$$(\forall p, q, r \in U)((p * q) * q) * (0 * q) * r \in I \cap J, r \in I \cap J \Rightarrow p * q \in I \cap J. \quad (20)$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. If we take $q = 0$ in (20) and use (1) and (III), then

$$(\forall p, r \in U)(p * r \in I \cap J, r \in I \cap J \Rightarrow p \in I \cap J).$$

Hence I and J are ideals of U . Assume that $((p * q) * q) * (0 * q) \in I \cap J$ for all $p, q \in U$. Then

$$(((p * q) * q) * (0 * q)) * 0 = ((p * q) * q) * (0 * q) \in I \cap J,$$

It follows from (19) and (20) that $p * q \in I \cap J$. Consequently, $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 3. $\square$

Corollary 4. Let I be a subset of U such that

$$0 \in I, \quad (21)$$

$$(\forall p, q, r \in U)((((p * q) * q) * (0 * q)) * r \in I, r \in I \Rightarrow p * q \in I). \quad (22)$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 6. Let I, J, G and H be ideals of U such that $G \subseteq I$ and $H \subseteq J$. If G and H are BCI-positive implicative ideals of U , then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Let $p, q, r \in U$ be such that $((p * q) * q) * (0 * q) \in I \cap J$. Then

$$\begin{aligned} & (((p * ((p * q) * q) * (0 * q))) * q) * q) * (0 * q) \\ &= (((p * q) * q) * (0 * q)) * (((p * q) * q) * (0 * q)) \\ &= 0 \in G \cap H, \end{aligned}$$

and so $(p * q) * (((p * q) * q) * (0 * q)) = (p * (((p * q) * q) * (0 * q))) * q \in G \cap H \subseteq I \cap J$ since G and H are BCI-positive implicative ideals of U . Thus $p * q \in I \cap J$, and therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 3. $\square$

Corollary 5. Let I and G be ideals of U such that $G \subseteq I$. If G is a BCI-positive implicative ideal of U , then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 7. Let I, J, G and H be ideals of U such that $G \subseteq I, H \subseteq J$ and

$$(\forall p, q \in U)((p * q) * q) * (0 * q) \in G \cap H \Rightarrow p * q \in G \cap H. \quad (23)$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Let $p, q, r \in U$ be such that $r \in G \cap H$ and $((p * q) * q) * (r * q) \in G \cap H$. Since

$$(((p * q) * q) * (0 * q)) * (((p * q) * q) * (r * q)) \leq (r * q) * (0 * q) \leq r * 0 = r \in G \cap H,$$

we have $((p * q) * q) * (0 * q) \in G \cap H$. It follows from (23) that $p * q \in G \cap H$. Hence G and H are BCI-positive implicative ideals of U , and therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 6. $\square$

Corollary 6. Let I and G be ideals of U such that $G \subseteq I$ and

$$(\forall p, q \in U)((p * q) * q) * (0 * q) \in G \Rightarrow p * q \in G. \quad (24)$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 8. Let I, J, G and H be ideals of U such that $G \subseteq I, H \subseteq J$ and

$$(\forall p, q \in U)((p * r) * (q * r)) * r \in G \cap H \Rightarrow (p * q) * r \in G \cap H). \quad (25)$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Let $p, q \in U$ be such that $((p * q) * q) * (0 * q) \in G \cap H$. Then

$$((p * q) * (0 * q)) * q = ((p * q) * q) * (0 * q) \in G \cap H.$$

It follows from (25) and (1) that $p * q = (p * 0) * q \in G \cap H$. Hence $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 7. $\square$

Proof. If we put $q = 0$ and $r = q$ in (25), then we have the condition (23). Hence $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 7. $\square$

Corollary 7. Let I and G be ideals of U such that $G \subseteq I$ and

$$(\forall p, q \in U)((p * r) * (q * r)) * r \in G \Rightarrow (p * q) * r \in G). \quad (26)$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

4. Conclusions

By considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, Smarandache have introduced the concept of neutrosophic quadruple numbers. Using the notion of neutrosophic quadruple numbers based on a set (instead of real or complex numbers), Jun et al. have constructed neutrosophic quadruple BCK/BCI-algebras and implicative neutrosophic quadruple BCK-algebras. In this manuscript, we have introduced the concept of a neutrosophic quadruple BCI-positive implicative ideal, and investigated several properties. We have discussed relations between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal. Given nonempty subsets I and J of a BCI-algebra U , we have provided conditions for the neutrosophic quadruple (I, J) -set to be a neutrosophic quadruple BCI-positive implicative ideal. In the forthcoming research and papers, we will continue these ideas and will define new notions in several algebraic structures.

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References

1. Iséki, K. On BCI-algebras. *Math. Semin. Notes* **1980**, *8*, 125–130.
2. Iséki, K.; Tanaka, S. An introduction to the theory of BCK-algebras. *Math. Jpn.* **1978**, *23*, 1–26.
3. Borumand Saeid, A.; Jun, Y.B. Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points. *Ann. Fuzzy Math. Inform.* **2017**, *14*, 87–97. [[CrossRef](#)]
4. Jun, Y.B. Neutrosophic subalgebras of several types in BCK/BCI-algebras. *Ann. Fuzzy Math. Inform.* **2017**, *14*, 75–86.

5. Jun, Y.B.; Kim, S.J.; Smarandache, F. Interval neutrosophic sets with applications in BCK/BCI-algebra. *Axioms* **2018**, *7*, 23. [CrossRef]
6. Jun, Y.B.; Smarandache, F.; Bordbar, H. Neutrosophic $\mathcal{N}$ -structures applied to BCK/BCI-algebras. *Information* **2017**, *8*, 128. [CrossRef]
7. Jun, Y.B.; Smarandache, F.; Song, S.Z.; Khan, M. Neutrosophic positive implicative $\mathcal{N}$ -ideals in BCK/BCI-algebras. *Axioms* **2018**, *7*, 3. [CrossRef]
8. Khan, M.; Anis, S.; Smarandache, F.; Jun, Y.B. Neutrosophic $\mathcal{N}$ -structures and their applications in semigroups. *Ann. Fuzzy Math. Inform.* **2017**, *14*, 583–598.
9. Öztürk, M.A.; Jun, Y.B. Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points. *J. Int. Math. Virtual Inst.* **2018**, *8*, 1–17.
10. Song, S.Z.; Smarandache, F.; Jun, Y.B. Neutrosophic commutative $\mathcal{N}$ -ideals in BCK-algebras. *Information* **2017**, *8*, 130. [CrossRef]
11. Smarandache, F. *Neutrosophy, Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning; American Research Press: Rehoboth, DE, USA, 1998; 105p. Available online: <http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf> (accessed on 27 March 2019).
12. Smarandache, F. *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability*; American Reserch Press: Rehoboth, NM, USA, 1999.
13. Smarandache, F. Neutrosophic set—A generalization of the intuitionistic fuzzy set. *Int. J. Pure Appl. Math.* **2005**, *24*, 287–297.
14. Smarandache, F. Neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, and the multiplication of neutrosophic quadruple numbers. *Neutrosophic Sets Syst.* **2015**, *10*, 96–98.
15. Jun, Y.B.; Song, S.Z.; Smarandache, F.; Bordbar, H. Neutrosophic quadruple BCK/BCI-algebras. *Axioms* **2018**, *7*, 41. [CrossRef]
16. Muhiuddin, G.; Al-Kenani, A.N.; Roh, E.H.; Jun, Y.B. Implicative neutrosophic quadruple BCK-algebras and ideals. *Symmetry* **2019**, *11*, 277. [CrossRef]
17. Huang, Y. *BCI-Algebra*; Science Press: Beijing, China, 2006.
18. Liu, Y.L.; Xu, Y.; Meng, J. BCI-implicative ideals of BCI-algebras. *Inform. Sci.* **2007**, *177*, 4987–4996. [CrossRef]
19. Liu, Y.L.; Zhang, X.H. Characterization of weakly positive implicative BCI-algebras. *J. Hanzhong Teach. Coll.* **1994**, *1*, 4–8.
20. Meng, J.; Jun, Y.B. *BCK-Algebras*; Kyungmoonsa Co.: Seoul, Korea, 1994.
21. Neutrosophy, Neutrosophic and Applications. Available online: <http://fs.gallup.unm.edu/neutrosophy.htm> (accessed on 27 March 2019)

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