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p-Regularity and p-Regular Modification in $\top$-Convergence Spaces

Qiu Jin [1,*], Lingqiang Li [1,2,3,*] and Guangming Lang [2,3]

[1] Department of Mathematics, Liaocheng University, Liaocheng 252059, China
[2] School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, China; langguangming1984@126.com
[3] Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science and Technology, Changsha 410114, China
* Correspondence: jinqiu79@126.com (Q.J.); jinqiu@lcu.edu.cn (L.L.); Tel.: +86-150-6355-2700 (Q.J.); +86-152-0650-6635 (L.L.)

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Abstract: Fuzzy convergence spaces are extensions of convergence spaces. $\top$-convergence spaces are important fuzzy convergence spaces. In this paper, p-regularity (a relative regularity) in $\top$-convergence spaces is discussed by two equivalent approaches. In addition, lower and upper p-regular modifications in $\top$-convergence spaces are further investigated and studied. Particularly, it is shown that lower (resp., upper) p-regular modification and final (resp., initial) structures have good compatibility.

Keywords: fuzzy topology; fuzzy convergence; $\top$-convergence space; regularity

1. Introduction

Convergence spaces [1] are generalizations of topological spaces. Regularity is an important property in convergence spaces. In general, there are two equivalent approaches to characterize regularity. One approach is stated through a diagonal condition of filters [2,3], the other approach is represented through a closure condition of filters [4]. In [5,6], for a pair of convergence structures p, q on the same underlying set, Wilde-Kent-Richardson considered a relative regularity (called p-regularity) both from two equivalent approaches. When $p = q$, p-regularity is nothing but regularity. Wilde-Kent [6] further presented a theory of lower and upper p-regular modifications in convergence spaces. Said precisely, for convergence structures p, q on a set X, the lower (resp., upper) p-regular modification of q is defined as the finest (resp., coarsest) p-regular convergence structure coarser (resp., finer) than q.

Fuzzy convergence spaces are natural extensions of convergence spaces. Quite recently, two types of fuzzy convergence spaces received wide attention: (1) stratified L-generalized convergence spaces (resp., stratified L-convergence spaces) initiated by Jäger [7] (resp., Flores [8]) and then developed by many scholars [8–30]; and (2) $\top$-convergence spaces introduced by Fang [31] and then discussed by many researchers [32–36]. Regularity in stratified L-generalized convergence spaces (resp., stratified L-convergence spaces) was studied by Jäger [37] (resp., Boustique-Richardson [38,39]), p-regularity and p-regular modifications in stratified L-generalized convergence spaces and that in stratified L-convergence spaces were discussed by Li [40,41]. Regularity in $\top$-convergence spaces by different diagonal conditions of $\top$-filters were researched by Fang [31] and Li [42], respectively. Regularity in $\top$-convergence spaces by closure condition of $\top$-filters were studied by Reid and Richardson [36]. In this paper, we shall discuss p-regularity and p-regular modifications in $\top$-convergence spaces.

The contents are arranged as follows. Section 2 recalls some notions and notations for later use. Section 3 presents p-regularity in $\top$-convergence spaces by a diagonal condition of $\top$-filters and a closure condition of $\top$-filters, respectively. Section 4 mainly discusses p-regular modifications in $\top$-convergence spaces. The lower and upper p-regular modifications in $\top$-convergence spaces are investigated and researched. Especially, it is shown that lower (resp., upper) p-regular modification and final (resp., initial) structures have good compatibility.

2. Preliminaries

In this paper, if not otherwise stated, $L = (L, \leq)$ is always a complete lattice with a top element $\top$ and a bottom element $\perp$, which satisfies the distributive law $\alpha \wedge (\bigvee_{i \in I} \beta_i) = \bigvee_{i \in I}(\alpha \wedge \beta_i)$. A lattice with these conditions is called a complete Heyting algebra. The operation $\rightarrow: L \times L \longrightarrow L$ given by

$$\alpha \rightarrow \beta = \bigvee \{\gamma \in L : \alpha \wedge \gamma \leq \beta\}.$$

is called the residuation with respect to $\wedge$. We collect here some basic properties of the binary operations $\wedge$ and $\rightarrow$ [43].

(1) $a \rightarrow b = \top \Leftrightarrow a \leq b$;
(2) $a \wedge b \leq c \Leftrightarrow b \leq a \rightarrow c$;
(3) $a \wedge (a \rightarrow b) \leq b$;
(4) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$;
(5) $(\bigvee_{j \in J} a_j) \rightarrow b = \bigwedge_{j \in J}(a_j \rightarrow b)$; (6) $a \rightarrow (\bigwedge_{j \in J} b_j) = \bigwedge_{j \in J}(a \rightarrow b_j)$.

A function $\mu : X \rightarrow L$ is said to be an L-fuzzy set in X, and all L-fuzzy sets in X are denoted as L^X. The operators $\vee, \wedge, \rightarrow$ on L can be translated onto L^X pointwisely. Precisely, for any $\mu, \nu, \mu_t (t \in T) \in L^X$,

$$(\bigvee_{t \in T} \mu_t)(x) = \bigvee_{t \in T} \mu_t(x), \quad (\bigwedge_{t \in T} \mu_t)(x) = \bigwedge_{t \in T} \mu_t(x), (\mu \rightarrow \nu)(x) = \mu(x) \rightarrow \nu(x).$$

Let $f : X \longrightarrow Y$ be a function. We define $f^{\rightarrow} : L^X \longrightarrow L^Y$ by $f^{\rightarrow}(\mu)(y) = \bigvee_{f(x)=y} \mu(x)$ for $\mu \in L^X$ and $y \in Y$, and define $f^{\leftarrow} : L^Y \longrightarrow L^X$ by $f^{\leftarrow}(\nu)(x) = \nu(f(x))$ for $\nu \in L^Y$ and $x \in X$ [43].

Let μ, ν be L-fuzzy sets in X. The subsethood degree of μ, ν, denoted as $S_X(\mu, \nu)$, is defined by $S_X(\mu, \nu) = \bigwedge_{x \in X}(\mu(x) \rightarrow \nu(x))$ [44–46]

Lemma 1. [31,42,47] *Let $f : X \longrightarrow Y$ be a function and $\mu_1, \mu_2 \in L^X$, $\lambda_1, \lambda_2 \in L^Y$. Then*

(1) $S_X(\mu_1, \mu_2) \leq S_Y(f^{\rightarrow}(\mu_1), f^{\rightarrow}(\mu_2))$,
(2) $S_Y(\lambda_1, \lambda_2) \leq S_X(f^{\leftarrow}(\lambda_1), f^{\leftarrow}(\lambda_2))$,
(3) $S_Y(f^{\rightarrow}(\mu_1), \lambda_1) = S_X(\mu_1, f^{\leftarrow}(\lambda_1))$.

2.1. $\top$-Filters and $\top$-Convergence Spaces

Definition 1. *[43,48] A nonempty subset $\mathbb{F} \subseteq L^X$ is said to be a $\top$-filter on the set X if it satisfies the following three conditions:*

(TF1) $\forall \lambda \in \mathbb{F}$, $\bigvee_{x \in X} \lambda(x) = \top$,
(TF2) $\forall \lambda, \mu \in \mathbb{F}$, $\lambda \wedge \mu \in \mathbb{F}$,
(TF3) *if* $\bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda) = \top$, *then* $\lambda \in \mathbb{F}$.

The set of all $\top$-filters on X is denoted by $\mathbb{F}_L^{\top}(X)$.

Definition 2. *[43] A nonempty subset $\mathbb{B} \subseteq L^X$ is referred to be a $\top$-filter base on the set X if it holds that:*

(TB1) $\forall \lambda \in \mathbb{B}$, $\bigvee_{x \in X} \lambda(x) = \top$,
(TB2) *if* $\lambda, \mu \in \mathbb{B}$, *then* $\bigvee_{\nu \in \mathbb{B}} S_X(\nu, \lambda \wedge \mu) = \top$.

Each $\top$-filter base generates a $\top$-filter $\mathbb{F}_{\mathbb{B}}$ by

$$\mathbb{F}_{\mathbb{B}} := \{\lambda \in L^X | \bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \top\}.$$

Example 1. *[31,43] Let $f : X \longrightarrow Y$ be a function.*

(1) *For any $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, the family $\{f^{\rightarrow}(\lambda) | \lambda \in \mathbb{F}\}$ forms a $\top$-filter base on Y. The generated $\top$-filter is denoted as $f^{\Rightarrow}(\mathbb{F})$, called the image of $\mathbb{F}$ under f. It is known that $\mu \in f^{\Rightarrow}(\mathbb{F}) \Longleftrightarrow f^{\leftarrow}(\mu) \in \mathbb{F}$.*

(2) *For any $\mathbb{G} \in \mathbb{F}_L^{\top}(Y)$, the family $\{f^{\leftarrow}(\mu) | \mu \in \mathbb{G}\}$ forms a $\top$-filter base on Y iff $\bigvee_{y \in f(X)} \mu(y) = \top$ holds for all $\mu \in \mathbb{G}$. The generated $\top$-filter (if exists) is denoted as $f^{\Leftarrow}(\mathbb{G})$, called the inverse image of $\mathbb{G}$ under f. It is known that $\mathbb{G} \subseteq f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G}))$ holds whenever $f^{\Leftarrow}(\mathbb{G})$ exists. Furthermore, $f^{\Leftarrow}(\mathbb{G})$ always exists and $\mathbb{G} = f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G}))$ whenever f is surjective.*

(3) *For any $x \in X$, the family $[x]_{\top} =: \{\lambda \in L^X | \lambda(x) = \top\}$ is a $\top$-filter on X, and $f^{\Rightarrow}([x]_{\top}) = [f(x)]_{\top}$.*

Lemma 2. *Let $f : X \longrightarrow Y$ be a function.*

(1) *If $\mathbb{B}$ is a $\top$-filter base of $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, then $\{f^{\rightarrow}(\lambda) | \lambda \in \mathbb{B}\}$ is a $\top$-filter base of $f^{\Rightarrow}(\mathbb{F})$, see Example 2.9 (1) in [31].*

(2) *If $\mathbb{B}$ is a $\top$-filter base of $\mathbb{G} \in \mathbb{F}_L^{\top}(Y)$ and $f^{\Leftarrow}(\mathbb{G})$ exists, then $\{f^{\leftarrow}(\mu) | \mu \in \mathbb{B}\}$ is a $\top$-filter base of $f^{\Leftarrow}(\mathbb{G})$, see Example 2.9 (2) in [31].*

(3) *Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}_L^{\top}(X)$ and $\mathbb{B}$ be a $\top$-filter base of $\mathbb{F}$. Then $\mathbb{B} \subseteq \mathbb{G}$ implies that $\mathbb{F} \subseteq \mathbb{G}$, see Lemma 2.5 (1) in [42].*

(4) *Let $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $\mathbb{B}$ be a $\top$-filter base of $\mathbb{F}$. Then $\bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda)$, see Lemma 3.1 in [36].*

For each $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, we define $\Lambda(\mathbb{F}) : L^X \longrightarrow L$ as

$$\forall \lambda \in L^X, \Lambda(\mathbb{F})(\lambda) = \bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda),$$

then $\Lambda(\mathbb{F})$ is a tightly stratified L-filter on X [47].

In the following, we recall some notions and notations collected in [29].

Definition 3. *[31] Let X be a nonempty set. Then a function $q : \mathbb{F}_L^{\top}(X) \longrightarrow 2^X$ is said to be a $\top$-convergence structure on X if it satisfies the following two conditions:*

(TC1) *$\forall x \in X$, $[x]_{\top} \xrightarrow{q} x$;*

(TC1) *if $\mathbb{F} \xrightarrow{q} x$ and $\mathbb{F} \subseteq \mathbb{G}$, then $\mathbb{G} \xrightarrow{q} x$.*

where $\mathbb{F} \xrightarrow{q} x$ is short for $x \in q(\mathbb{F})$. The pair (X, q) is said to be a $\top$-convergence space.

A function $f : X \longrightarrow X'$ between $\top$-convergence spaces (X, q), (X', q') is said to be continuous if $f^{\Rightarrow}(\mathbb{F}) \xrightarrow{q'} f(x)$ for any $\mathbb{F} \xrightarrow{q} x$.

We denote the category consisting of $\top$-convergence spaces and continuous functions as $\top$-**CS** . It has been known that $\top$-**CS** is topological over **SET** [31].

For a source $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$, the initial structure, q on X is defined by

$$\mathbb{F} \xrightarrow{q} x \Longleftrightarrow \forall i \in I, f_i^{\Rightarrow}(\mathbb{F}) \xrightarrow{q_i} f_i(x) \text{ [35,49]}.$$

For a sink $((X_i, q_i) \xrightarrow{f_i} X)_{i \in I}$, the final structure, q on X is defined as

$$\mathbb{F} \xrightarrow{q} x \Longleftrightarrow \begin{cases} \mathbb{F} \supseteq [x]_{\top}, & x \notin \cup_{i \in I} f_i(X_i); \\ \mathbb{F} \supseteq f_i^{\Rightarrow}(\mathbb{G}_i), & \exists i \in I, x_i \in X_i, \mathbb{G}_i \in \mathbb{F}_L^{\top}(X_i) \text{ s.t } f(x_i) = x, \mathbb{G}_i \xrightarrow{q_i} x_i. \end{cases}$$

When $X = \cup_{i\in I} f_i(X_i)$, the final structure q can be characterized as

$$\mathbb{F} \xrightarrow{q} x \Longleftrightarrow \mathbb{F} \supseteq f_i^{\Rightarrow}(\mathbb{G}_i) \text{ for some } \mathbb{G}_i \xrightarrow{q_i} x_i \text{ with } f(x_i) = x.$$

Let $\top(X)$ denote the set of all $\top$-convergence structures on a set X. For $p, q \in \top(X)$, we say that q is finer than p (or p is coarser than q), denoted as $p \leq q$ for short, if the identity $\mathrm{id}_X : (X, q) \longrightarrow (X, p)$ is continuous. It has been known that $(\top(X), \leq)$ forms a completed lattice. The discrete (resp., indiscrete) structure δ (resp., ι) is the top (resp., bottom) element of $(T(X), \leq)$, where δ is defined as $\mathbb{F} \xrightarrow{\delta} x$ iff $\mathbb{F} \supseteq [x]_\top$; and ι is defined as $\mathbb{F} \xrightarrow{\iota} x$ for all $\mathbb{F} \in \mathbb{F}_L^\top(X)$, $x \in X$ [42].

Remark 1. *When $L = \{\perp, \top\}$, $\top$-convergence spaces degenerate into convergence spaces. Therefore, $\top$-convergence spaces are natural generalizations of convergence spaces.*

3. p-Regularity in $\top$-Convergence Spaces

In this section, we shall discuss the p-regularity in $\top$-convergence spaces. Two equivalent approaches are considered, one approach using diagonal $\top$-filter and the other approach using closure of $\top$-filter. Moreover, it will be proved that p-regularity is preserved under the initial and final structures in the category $\top$-**CS**.

At first, we recall the notions of diagonal $\top$-filter and closure of $\top$-filter to define p-regularity.

Let J, X be any sets and $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ be any function. Then we define a function $\hat{\phi} : L^X \to L^J$ as

$$\forall \lambda \in L^X, \forall j \in J, \hat{\phi}(\lambda)(j) = \Lambda(\phi(j))(\lambda) = \bigvee_{\mu \in \phi(j)} S_X(\mu, \lambda).$$

For any $\mathbb{F} \in \mathbb{F}_L^\top(J)$, it is known that the subset of L^X defined by

$$k\phi\mathbb{F} := \{\lambda \in L^X | \hat{\phi}(\lambda) \in \mathbb{F}\}$$

forms a $\top$-filter on X, called diagonal $\top$-filter of $\mathbb{F}$ under ϕ [31]. It was shown that $S_X(\lambda, \mu) \leq S_J(\hat{\phi}(\lambda), \hat{\phi}(\mu))$ for any $\lambda, \mu \in L^X$.

Lemma 3. *Let $f : X \longrightarrow Y$ and $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ be functions. Then for any $\mathbb{F} \in \mathbb{F}_L^\top(J)$ we have $f^{\Rightarrow}(k\phi\mathbb{F}) = k(f^{\Rightarrow} \circ \phi)\mathbb{F}$.*

Proof. $f^{\Rightarrow}(k\phi\mathbb{F}) \subseteq k(f^{\Rightarrow} \circ \phi)\mathbb{F}$. By Lemma 2 (3) we need only check that $f^{\rightarrow}(\lambda) \in k(f^{\Rightarrow} \circ \phi)\mathbb{F}$ for any $\lambda \in k\phi\mathbb{F}$. Take $\lambda \in k\phi\mathbb{F}$ then $\hat{\phi}(\lambda) \in \mathbb{F}$. Please note that $\forall j \in J$,

$$\hat{\phi}(\lambda)(j) = \bigvee_{\mu \in \phi(j)} S_X(\mu, \lambda) \leq \bigvee_{\mu \in \phi(j)} S_Y(f^{\rightarrow}(\mu), f^{\rightarrow}(\lambda)) \leq \bigvee_{\nu \in f^{\Rightarrow}(\phi(j))} S_Y(\nu, f^{\rightarrow}(\lambda)) = \widehat{f^{\Rightarrow} \circ \phi}(f^{\rightarrow}(\lambda))(j),$$

i.e., $\hat{\phi}(\lambda) \leq \widehat{f^{\Rightarrow} \circ \phi}(f^{\rightarrow}(\lambda))$, and so $\widehat{f^{\Rightarrow} \circ \phi}(f^{\rightarrow}(\lambda)) \in \mathbb{F}$, i.e., $f^{\rightarrow}(\lambda) \in k(f^{\Rightarrow} \circ \phi)\mathbb{F}$.

$k(f^{\Rightarrow} \circ \phi)\mathbb{F} \subseteq f^{\Rightarrow}(k\phi\mathbb{F})$. For any $\lambda \in k(f^{\Rightarrow} \circ \phi)\mathbb{F}$ we have $\widehat{f^{\Rightarrow} \circ \phi}(\lambda) \in \mathbb{F}$. By Lemma 2 (4), $\forall j \in J$,

$$\widehat{f^{\Rightarrow} \circ \phi}(\lambda)(j) = \bigvee_{\nu \in f^{\Rightarrow}(\phi(j))} S_Y(\nu, \lambda) = \bigvee_{\mu \in \phi(j)} S_Y(f^{\rightarrow}(\mu), \lambda) \leq \bigvee_{\mu \in \phi(j)} S_X(\mu, f^{\leftarrow}(\lambda)) = \hat{\phi}(f^{\leftarrow}(\lambda))(j),$$

i.e., $\widehat{f^{\Rightarrow} \circ \phi}(\lambda) \leq \hat{\phi}(f^{\leftarrow}(\lambda))$, and so $\hat{\phi}(f^{\leftarrow}(\lambda)) \in \mathbb{F}$, i.e., $f^{\leftarrow}(\lambda) \in k\phi\mathbb{F}$ then $\lambda \in f^{\Rightarrow}(k\phi\mathbb{F})$. $\square$

Definition 4. *[36] Let (X, p) be a $\top$-convergence space. For each $\lambda \in L^X$, the L-set $\overline{\lambda}_p \in L^X$ defined by*

$$\forall x \in X, \overline{\lambda}_p(x) = \bigvee_{\mathbb{F} \xrightarrow{p} x} \Lambda(\mathbb{F})(\lambda) = \bigvee_{\mathbb{F} \xrightarrow{p} x} \bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda)$$

is called closure of λ w.r.t p.

For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$, the closure of $\mathbb{F}$ regarding p, denoted as $cl_p(\mathbb{F})$, is defined to be the $\top$-filter generated by $\{\overline{\lambda}_p | \lambda \in \mathbb{F}\}$ as a $\top$-filter base.

Lemma 4. *[36] Let (X, p) be a $\top$-convergence space. Then for all $\lambda, \mu \in L^X$ we get*

(1) $\lambda \leq \overline{\lambda}_p$;
(2) $\lambda \leq \mu$ *implies* $\overline{\lambda}_p \leq \overline{\mu}_p$;
(3) $S_X(\lambda, \mu) \leq S_X(\overline{\lambda}, \overline{\mu})$.

Let $\mathbb{N}$ be the set of natural numbers containing 0 and let (X, p) be a $\top$-convergence space. For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$, we define $cl_p^0(\mathbb{F}) = \mathbb{F}$. Furthermore, for any $n \in \mathbb{N}$, we define the $n+1$ th iteration of the closure $\top$-filter of $\mathbb{F}$ as $cl_p^{n+1}(\mathbb{F}) = cl_p(cl_p^n(\mathbb{F}))$ if $cl_p^n(\mathbb{F})$ has been defined.

The next proposition collects the properties of closure of $\top$-filters. We omit the obvious proofs.

Proposition 1. *Let (X, p) be a $\top$-convergence space and $\mathbb{F}, \mathbb{G} \in \mathbb{F}_L^\top(X)$. Then for any $n \in \mathbb{N}$,*

(1) $cl_p^n(\mathbb{F}) \subseteq \mathbb{F}$,
(2) *if* $\mathbb{F} \subseteq \mathbb{G}$, *then* $cl_p^n(\mathbb{F}) \subseteq cl_p^n(\mathbb{G})$,
(3) *if* $p' \in T(X)$ *and* $p \leq p'$, *then* $cl_p^n(\mathbb{F}) \subseteq cl_{p'}^n(\mathbb{F})$.

Definition 5. *A function $f : (X, q) \longrightarrow (Y, p)$ between $\top$-convergence spaces is said to be a closure function if $\overline{f^{\rightarrow}(\lambda)}_p \leq f^{\rightarrow}(\overline{\lambda}_q)$ for any $\lambda \in L^X$.*

Proposition 2. *Suppose that $f : (X, q) \longrightarrow (Y, p)$ is a function between $\top$-convergence spaces and $\mathbb{F} \in \mathbb{F}_L^\top(X)$, $n \in \mathbb{N}$.*

(1) *If f is a continuous function, then $f^{\Rightarrow}(cl_q^n(\mathbb{F})) \supseteq cl_p^n(f^{\Rightarrow}(\mathbb{F}))$.*
(2) *If f is a closure function, then $f^{\Rightarrow}(cl_q^n(\mathbb{F})) \subseteq cl_p^n(f^{\Rightarrow}(\mathbb{F}))$.*

Proof. (1) Let's prove it by mathematical induction.

Firstly, we check $\overline{f^{\leftarrow}(\lambda)}_q \leq f^{\leftarrow}(\overline{\lambda}_p)$ for any $\lambda \in L^Y$. In fact, for any $x \in X$, by continuity of f we obtain

$$\begin{aligned} \overline{f^{\leftarrow}(\lambda)}_q(x) &= \bigvee_{\mathbb{G} \xrightarrow{q} x} \bigvee_{\mu \in \mathbb{G}} S_X(\mu, f^{\leftarrow}(\lambda)) = \bigvee_{\mathbb{G} \xrightarrow{q} x} \bigvee_{\mu \in \mathbb{G}} S_Y(f^{\rightarrow}(\mu), \lambda) \\ &\leq \bigvee_{f^{\Rightarrow}(\mathbb{G}) \xrightarrow{p} f(x)} \bigvee_{f^{\rightarrow}(\mu) \in f^{\Rightarrow}(\mathbb{G})} S_Y(f^{\rightarrow}(\mu), \lambda) \leq \bigvee_{\mathbb{H} \xrightarrow{p} f(x)} \bigvee_{\nu \in \mathbb{H}} S_Y(\nu, \lambda) = f^{\leftarrow}(\overline{\lambda}_p)(x). \end{aligned}$$

Secondly, we prove $f^{\Rightarrow}(cl_q^n(\mathbb{F})) \supseteq cl_p^n(f^{\Rightarrow}(\mathbb{F}))$ when $n = 1$. Let $\lambda \in f^{\Rightarrow}(\mathbb{F})$, i.e., $f^{\leftarrow}(\lambda) \in \mathbb{F}$. Then by $\overline{f^{\leftarrow}(\lambda)}_q \leq f^{\leftarrow}(\overline{\lambda}_p)$ we have $f^{\leftarrow}(\overline{\lambda}_p) \in cl_q(\mathbb{F})$, i.e., $\overline{\lambda}_p \in f^{\Rightarrow}(cl_q(\mathbb{F}))$. It follows by Lemma 2 (3) that $f^{\Rightarrow}(cl_q(\mathbb{F})) \supseteq cl_p(f^{\Rightarrow}(\mathbb{F}))$.

Thirdly, we assume that $f^{\Rightarrow}(cl_q^n(\mathbb{F})) \supseteq cl_p^n(f^{\Rightarrow}(\mathbb{F}))$ when $n = k$. Then we prove $f^{\Rightarrow}(cl_q^n(\mathbb{F})) \supseteq cl_p^n(f^{\Rightarrow}(\mathbb{F}))$ when $n = k+1$. In fact,

$$f^{\Rightarrow}(cl_q^{k+1}(\mathbb{F})) = f^{\Rightarrow}(cl_q(cl_q^k(\mathbb{F}))) \supseteq cl_p(f^{\Rightarrow}(cl_q^k(\mathbb{F}))) \supseteq cl_p(cl_p^k(f^{\Rightarrow}(\mathbb{F}))) = cl_p^{k+1}(f^{\Rightarrow}(\mathbb{F})).$$

(2) We prove only that the inequality holds for $n = 1$, and the rest of the proof is similar to (1).

For any $\lambda \in \mathbb{F}$, we have $\overline{\lambda}_q \in cl_q(\mathbb{F})$ and then $f^{\rightarrow}(\lambda) \in f^{\Rightarrow}(\mathbb{F})$. From f is a closure function, we conclude that $f^{\rightarrow}(\overline{\lambda}_q) \geq \overline{f^{\rightarrow}(\lambda)}_p \in cl_p(f^{\Rightarrow}(\mathbb{F}))$ and so $f^{\rightarrow}(\overline{\lambda}_q) \in cl_p(f^{\Rightarrow}(\mathbb{F}))$. By Lemma 2 (1), (3) we obtain $f^{\Rightarrow}(cl_q(\mathbb{F})) \subseteq cl_p(f^{\Rightarrow}(\mathbb{F}))$. □

Now, we tend our attention to p-regularity and its equivalent characterization. In the following, we shorten a pair of $\top$-convergence spaces (X, p) and (X, q) as (X, p, q).

Definition 6. *Let (X, p, q) be a pair of $\top$-convergence spaces. Then q is said to be p-regular if the following condition p-**(TC)** is fulfilled.*

p-**(TC)**: $\forall \mathbb{F} \in \mathbb{F}_L^{\top}(X), \forall x \in X, \mathbb{F} \xrightarrow{q} x \Longrightarrow cl_p(\mathbb{F}) \xrightarrow{q} x.$

Remark 2. *When $L = \{\perp, \top\}$, a $\top$-convergence space degenerates into a convergence space, and the condition p-**(TC)** degenerates into the crisp p-regularity condition in [5]. When $p = q$, the condition p-**(TC)** is precisely the regular characterization in [36].*

We say a pair of $\top$-convergence spaces (X, p, q) fulfill the Fischer $\top$-diagonal condition whenever

p-**(TR)**: Let J, X be any sets, $\psi : J \longrightarrow X$, and $\phi : J \longrightarrow \mathbb{F}_L^{\top}(X)$ such that $\phi(j) \xrightarrow{p} \psi(j)$, for each $j \in J$. Then for each $\mathbb{F} \in \mathbb{F}_L^{\top}(J)$ and each $x \in X$, $k\phi\mathbb{F} \xrightarrow{q} x$ implies $\psi^{\Rightarrow}(\mathbb{F}) \xrightarrow{q} x$.

Remark 3. *When $L = \{\perp, \top\}$, a $\top$-convergence space degenerates into a convergence space, and the condition p-**(TC)** degenerates into the Fischer diagonal condition $R_{p,q}$ in [6]. When $p = q$, the condition p-**(TR)** is precisely the diagonal condition **(TR)** in [31].*

In the following, we shall show that p-regularity can be described by Fischer $\top$-diagonal condition p-**(TR)**.

Lemma 5. *Let (X, p, q) be a pair of $\top$-convergence spaces and let J, X, ϕ, ψ be defined as in p-**(TR)**. Then $S_X(\overline{\mu}_p, \lambda) \leq S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda))$ for all $\lambda, \mu \in L^X$.*

Proof. Let $\lambda, \mu \in L^X$.

$$\begin{aligned} S_X(\overline{\mu}_p, \lambda) &= \bigwedge_{x \in X} ([\bigvee_{\mathbb{G} \xrightarrow{p} x} \Lambda(\mathbb{G})(\mu)] \to \lambda(x)), \text{ by } \psi(j) \in X, \phi(j) \xrightarrow{p} \psi(j) \\ &\leq \bigwedge_{j \in J} (\Lambda(\phi(j))(\mu) \to \lambda(\psi(j))) = \bigwedge_{j \in J} (\hat{\phi}(\mu)(j) \to \psi^{\leftarrow}(\lambda)(j)) \\ &= S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)). \quad \square \end{aligned}$$

Theorem 1. *(Theorem 4.8 in [36] for $p = q$) Let (X, p, q) be a pair of $\top$-convergence spaces. Then p-**(TC)**$\Longleftrightarrow$$p$-**(TR)**.*

Proof. p-**(TC)**$\Longrightarrow$$p$-**(TR)**. Let J, X, ϕ, ψ be defined as in p-**(TR)**. Assume that $\mathbb{F} \in \mathbb{F}_L^{\top}(J)$ and $k\phi\mathbb{F} \xrightarrow{q} x$. Then it follows by p-**(TC)** that $cl_p(k\phi\mathbb{F}) \xrightarrow{q} x$.

Next we prove that $cl_p(k\phi\mathbb{F}) \subseteq \psi^{\Rightarrow}(\mathbb{F})$. Indeed, for any $\lambda \in cl_p(k\phi\mathbb{F})$, we have

$$\top = \bigvee_{\mu \in k\phi\mathbb{F}} S_X(\overline{\mu}_p, \lambda) \overset{\text{Lemma 5}}{\leq} \bigvee_{\mu \in k\phi\mathbb{F}} S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)) = \bigvee_{\hat{\phi}(\mu) \in \mathbb{F}} S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)) \leq \bigvee_{\nu \in \mathbb{F}} S_J(\nu, \psi^{\leftarrow}(\lambda)),$$

which means $\psi^{\leftarrow}(\lambda) \in \mathbb{F}$, i.e., $\lambda \in \psi^{\Rightarrow}(\mathbb{F})$.

Now we have known that $cl_p(k\phi\mathbb{F}) \xrightarrow{q} x$ and $cl_p(k\phi\mathbb{F}) \subseteq \psi^{\Rightarrow}(\mathbb{F})$. Therefore, $\psi^{\Rightarrow}(\mathbb{F}) \xrightarrow{q} x$, as desired.

p-**(TR)**$\Longrightarrow$$p$-**(TC)**. Let

$$J = \{(\mathbb{G}, y) \in \mathbb{F}_L^{\top}(X) \times X | \mathbb{G} \xrightarrow{p} y\}; \psi : J \longrightarrow X, (\mathbb{G}, y) \mapsto y; \phi : J \longrightarrow \mathbb{F}_L^{\top}(X), (\mathbb{G}, y) \mapsto \mathbb{G}.$$

Then $\forall j \in J, \phi(j) \xrightarrow{p} \psi(j)$. Please note that $j = (\mathbb{G}, y) \in J \Longleftrightarrow \mathbb{G} = \phi(j), y = \psi(j)$.

(1) For any $\lambda, \mu \in L^X$, $S_X(\overline{\mu}_p, \lambda) = S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda))$. Indeed,

$$\begin{aligned} S_X(\overline{\mu}_p, \lambda) &= \bigwedge_{y \in X}([\bigvee_{\mathbb{G} \xrightarrow{p} y} \Lambda(\mathbb{G})(\mu)] \to \lambda(y)) = \bigwedge_{y \in X} \bigwedge_{(\mathbb{G}, y) \in J} (\Lambda(\mathbb{G})(\mu) \to \lambda(y)) \\ &= \bigwedge_{j=(\mathbb{G},y) \in J} (\Lambda(\phi(j))(\mu) \to \lambda(\psi(j))) = \bigwedge_{j \in J} (\hat{\phi}(\mu)(j) \to \psi^{\leftarrow}(\lambda)(j)) \\ &= S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)). \end{aligned}$$

(2) For each $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, the family $\{\hat{\phi}(\lambda) | \lambda \in \mathbb{F}\}$ forms a $\top$-filter base on J. Indeed,
(TB1): For any $\lambda \in \mathbb{F}$, by $[y]_{\top} \xrightarrow{p} y$ for any $y \in X$, we have

$$\bigvee_{j \in J} \hat{\phi}(\lambda)(j) = \bigvee_{j \in J} \Lambda(\phi(j))(\lambda) = \bigvee_{y \in X} \bigvee_{\mathbb{G} \xrightarrow{p} y} \Lambda(\mathbb{G})(\lambda) \geq \bigvee_{y \in X} \Lambda([y]_{\top})(\lambda) = \bigvee_{y \in X} \lambda(y) = \top.$$

(TB2): For any $\lambda, \mu \in \mathbb{F}$, note that for any $j \in J$,

$$\begin{aligned} \hat{\phi}(\lambda)(j) \wedge \hat{\phi}(\mu)(j) &= \bigvee_{\lambda_1 \in \phi(j)} S_X(\lambda_1, \lambda) \wedge \bigvee_{\mu_1 \in \phi(j)} S_X(\mu_1, \mu) \\ &\leq \bigvee_{\lambda_1, \mu_1 \in \phi(j)} S_X(\lambda_1 \wedge \mu_1, \lambda \wedge \mu) \\ &\leq \bigvee_{\nu \in \phi(j)} S_X(\nu, \lambda \wedge \mu) = \hat{\phi}(\lambda \wedge \mu)(j), \end{aligned}$$

i.e., $\hat{\phi}(\lambda) \wedge \hat{\phi}(\mu) \leq \hat{\phi}(\lambda \wedge \mu)$. It follows easily that (TB2) is satisfied. We denote the $\top$-filter generated by $\{\hat{\phi}(\lambda) | \lambda \in \mathbb{F}\}$ as $\mathbb{F}^{\phi}$.

(3) For each $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, $k\phi\mathbb{F}^{\phi} \supseteq \mathbb{F}$. Indeed, for any $\lambda \in \mathbb{F}$, we have $\hat{\phi}(\lambda) \in \mathbb{F}^{\phi}$, i.e., $\lambda \in k\phi\mathbb{F}^{\phi}$.

(4) For each $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, $\psi^{\Rightarrow}(\mathbb{F}^{\phi}) = cl_p(\mathbb{F})$. Indeed,

$$\lambda \in \psi^{\Rightarrow}(\mathbb{F}^{\phi}) \iff \psi^{\leftarrow}(\lambda) \in \mathbb{F}^{\phi} \iff \bigvee_{\mu \in \mathbb{F}} S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)) = \top \overset{(1)}{\iff} \bigvee_{\mu \in \mathbb{F}} S_X(\overline{\mu}_p, \lambda) = \top \iff \lambda \in cl_p(\mathbb{F}).$$

Assume that $\mathbb{F} \xrightarrow{q} x$, then by (3), we have $k\phi\mathbb{F}^{\phi} \supseteq \mathbb{F}$, and so $k\phi\mathbb{F}^{\phi} \xrightarrow{q} x$. From p-**(TR)** and (4), we get that $cl_p(\mathbb{F}) = \psi^{\Rightarrow}(\mathbb{F}^{\phi}) \xrightarrow{q} x$. Therefore, the condition p-**(TC)** is satisfied. □

The next theorem shows that p-regularity is preserved under initial structures.

Theorem 2. *Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of $\top$-convergence spaces such that each q_i is p_i-regular. If q (resp., p) is the initial structure on X regarding the source $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$ (resp., $(X \xrightarrow{f_i} (X_i, p_i))_{i \in I}$), then q is also p-regular.*

Proof. Let $\psi : J \longrightarrow X$ and $\phi : J \longrightarrow \mathbb{F}_L^{\top}(X)$ be any function such that $\phi(j) \xrightarrow{p} \psi(j)$ for any $j \in J$. Then

$$\forall i \in I, \forall j \in J, (f_i^{\Rightarrow} \circ \phi)(j) = f_i^{\Rightarrow}(\phi(j)) \xrightarrow{p_i} f_i(\psi(j)) = (f_i \circ \psi)(j).$$

Let $\mathbb{F} \in \mathbb{F}_L^{\top}(J)$ satisfy $k\phi\mathbb{F} \xrightarrow{q} x$. Then by definition of q and Lemma 3 we have

$$\forall i \in I, k(f_i^{\Rightarrow} \circ \phi)\mathbb{F} = f_i^{\Rightarrow}(k\phi\mathbb{F}) \xrightarrow{q_i} f_i(x).$$

Since q_i is p_i-regular we have $f_i^{\Rightarrow}\psi^{\Rightarrow}(\mathbb{F}) = (f_i \circ \psi)^{\Rightarrow}(\mathbb{F}) \xrightarrow{q_i} f_i(x)$. By definition of q we have $\psi^{\Rightarrow}(\mathbb{F}) \xrightarrow{q} x$. Thus q is p-regular. □

The next theorem shows that p-regularity is preserved under final structures with some additional assumptions.

Theorem 3. *Let $\{(X_i, q_i, p_i)\}_{i\in I}$ be pairs of $\top$-convergence spaces such that each q_i is p_i-regular. Let q (resp., p) be the final structure on X relative to the sink $((X_i, q_i) \xrightarrow{f_i} X)_{i\in I}$ (resp., $((X_i, p_i) \xrightarrow{f_i} X)_{i\in I}$). If $X = \cup_{i\in I} f_i(X_i)$ and each $f_i : (X_i, p_i) \longrightarrow (X, p)$ is a closure function, then q is also p-regular.*

Proof. Let $\mathbb{F} \in \mathbb{F}_L^{\top}(X) \xrightarrow{q} x$. Then by definition of q, there exists $i \in I, x_i \in X_i, \mathbb{G}_i \in \mathbb{F}_L^{\top}(X_i), f_i(x_i) = x$ such that $\mathbb{G}_i \xrightarrow{q_i} x_i$ and $f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F}$. Because q_i is p_i-regular we get $cl_{p_i}(\mathbb{G}_i) \xrightarrow{q_i} x_i$ and then $f_i^{\Rightarrow}(cl_{p_i}(\mathbb{G}_i)) \xrightarrow{q} x$. By f_i is a closure function and Proposition 2 (2) it follows that $cl_p(f_i^{\Rightarrow}(\mathbb{G}_i)) \xrightarrow{q} x$. Hence $cl_p(\mathbb{F}) \xrightarrow{q} x$ from $cl_p(f_i^{\Rightarrow}(\mathbb{G}_i)) \subseteq cl_p(\mathbb{F})$. Thus q is p-regular. □

For any $\{q_i\}_{i\in I} \subseteq \top(X)$, note that the supremum (resp., infimum) of $\{q_i\}_{i\in I}$ in the lattice $\top(X)$, denoted as $\sup\{q_i|i \in I\}$ (resp., $\inf\{q_i|i \in I\}$), is precisely the initial structure (resp., final structure) regarding the source $(X \xrightarrow{id_X} (X, q_i))_{i\in I}$ (resp., the sink $((X, q_i) \xrightarrow{id_X} X)_{i\in I}$). By Theorems 2 and 3, we obtain easily the following corollary. It will show us that p-regularity is preserved under supremum and infimum in the lattice $\top(X)$.

Corollary 1. *Let $\{q_i|i \in I\} \subseteq \top(X)$ and $p \in \top(X)$ with each (X, q_i) being p-regular. Then both $\inf\{q_i\}_{i\in I}$ and $\sup\{q_i\}_{i\in I}$ are all p-regular.*

4. Lower (Upper) p-Regular Modifications in $\top$-Convergence Spaces

In this section, we shall consider the p-regular modifications in $\top$-convergence spaces.

Lemma 6. *Let p, q be $\top$-convergence structures on X.*

(1) If q is p-regular, then $\mathbb{F} \xrightarrow{q} x$ implies $cl_p^n(\mathbb{F}) \xrightarrow{q} x$ for any $n \in \mathbb{N}$.
(2) If q is p-regular, then q is p'-regular for any $p \leq p'$.
(3) The indiscrete structure ι is p-regular for any $p \in \top(X)$.

Proof. It is obvious. □

4.1. Lower p-Regular Modification

It has been known that p-regularity is preserved under supremum in the lattice $\top(X)$ (see Corollary 1), and the indiscrete structure ι is p-regular for any $p \in \top(X)$ (see Lemma 6 (3)). So, it follows easily that for a pair of $\top$-convergence spaces (X, p, q), there is a finest p-regular $\top$-convergence structure $\gamma_p q$ on X which is coarser than q.

Definition 7. *Let (X, p, q) be a pair of $\top$-convergence spaces. Then the $\top$-convergence structure $\gamma_p q$ on X is said to be the lower p-regular modification of q.*

The following theorem gives a characterization on lower p-regular modification.

Theorem 4. *For any $p, q \in \top(X)$, $\mathbb{F} \xrightarrow{\gamma_p q} x \Longleftrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ s.t. $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$.*

Proof. We define q' as $\mathbb{F} \xrightarrow{q'} x \Longleftrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ s.t. $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$, then we prove $\gamma_p q = q'$.

Obviously, $q' \in \top(X)$ and $q' \leq q$. We check that q' is p-regular. In fact, let $\mathbb{F} \xrightarrow{q'} x$. Then there exists $n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ such that $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$. It follows that $cl_p(\mathbb{F}) \supseteq cl_p(cl_p^n(\mathbb{G})) = cl_p^{n+1}(\mathbb{G})$, so $cl_p(\mathbb{F}) \xrightarrow{q'} x$. Now, we have proved that q' is p-regular.

Let r be p-regular with $r \leq q$. We prove below $r \leq q'$. In fact, let $\mathbb{F} \xrightarrow{q'} x$. Then there exists $n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ such that $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$, so $\mathbb{G} \xrightarrow{r} x$ by $q \leq r$. Because r is p-regular it follows by Lemma 6 (1) that $\mathbb{F} \supseteq cl_p^n(\mathbb{G}) \xrightarrow{r} x$. Therefore, $r \leq q'$. □

Theorem 5. *If $f : (X, q) \longrightarrow (X', q')$ and $f : (X, p) \longrightarrow (X', p')$ are both continuous function between $\top$-convergence spaces then so is $f : (X, \gamma_p q) \longrightarrow (X', \gamma_{p'} q')$.*

Proof. For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and $x \in X$.

$$
\begin{aligned}
\mathbb{F} \xrightarrow{\gamma_p q} x &\Longrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x \text{ s.t. } \mathbb{F} \supseteq cl_p^n(\mathbb{G}) \\
&\Longrightarrow \exists n \in \mathbb{N}, f^{\Rightarrow}(\mathbb{G}) \xrightarrow{q'} f(x) \text{ s.t. } f^{\Rightarrow}(\mathbb{F}) \supseteq f^{\Rightarrow}(cl_p^n(\mathbb{G})) \\
&\Longrightarrow \exists n \in \mathbb{N}, f^{\Rightarrow}(\mathbb{G}) \xrightarrow{q'} f(x) \text{ s.t. } f^{\Rightarrow}(\mathbb{F}) \supseteq cl_{p'}^n(f^{\Rightarrow}(\mathbb{G})) \\
&\Longrightarrow f^{\Rightarrow}(\mathbb{F}) \xrightarrow{\gamma_{p'} q'} (f(x)),
\end{aligned}
$$

where the second implication uses the continuity of $f : (X, q) \longrightarrow (X', q')$, and the third implication uses the continuity of $f : (X, p) \longrightarrow (X', p')$ and Proposition 2(1). □

The following theorem exhibits us that lower p-regular modification and final structures have good compatibility.

Theorem 6. *Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of spaces in $\top$-**CS** and let q be the final structure relative to the sink $((X_i, q_i) \xrightarrow{f_i} X)_{i \in I}$ with $X = \cup_{i \in I} f_i(X_i)$. If $p \in \top(X)$ such that each $f_i : (X_i, p_i) \longrightarrow (X, p)$ is a continuous closure function, then $\gamma_p q$ is the final structure relative to the sink $((X_i, \gamma_{p_i} q_i) \xrightarrow{f_i} X)_{i \in I}$.*

Proof. Let s denote the final structure relative to the sink $((X_i, \gamma_{p_i} q_i) \xrightarrow{f_i} X)_{i \in I}$. Then for any $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and $x \in X$

$$
\begin{aligned}
\mathbb{F} \xrightarrow{s} x &\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, \mathbb{G}_i \xrightarrow{\gamma_{p_i} q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F}, \text{by Theorem 4} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i(x_i) = x, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } cl_{p_i}^n(\mathbb{H}_i) \subseteq \mathbb{G}_i, f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F}, \text{by Proposition 2 (1)} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i^{\Rightarrow}(\mathbb{H}_i) \xrightarrow{q} x \text{ s.t. } cl_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq f_i^{\Rightarrow}(cl_{p_i}^n(\mathbb{H}_i)) \subseteq f_i^{\Rightarrow}(\mathbb{G}_i), f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i^{\Rightarrow}(\mathbb{H}_i) \xrightarrow{q} x \text{ s.t. } cl_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq \mathbb{F} \\
&\Longrightarrow \mathbb{F} \xrightarrow{\gamma_p q} x.
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
\mathbb{F} \xrightarrow{\gamma_p q} x &\Longrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(\mathbb{H}_i) \subseteq \mathbb{G}, cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } cl_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq cl_p^n(\mathbb{G}), cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(cl_{p_i}^n(\mathbb{H}_i)) \subseteq cl_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, cl_{p_i}^n(\mathbb{H}_i) \xrightarrow{\gamma_{p_i} q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(cl_{p_i}^n(\mathbb{H}_i)) \subseteq \mathbb{F} \\
&\Longrightarrow \mathbb{F} \xrightarrow{s} x,
\end{aligned}
$$

where the fourth implication follows by Proposition 2(2). □

The following corollary tells us that lower p-regular modification has good compatibility with infimum in the lattice $\top(X)$.

Corollary 2. *Assume that* $\{q_i | i \in I\} \subseteq \top(X)$, $p \in \top(X)$ *and* $q = \inf\{q_i | i \in I\}$. *Then* $\gamma_p q = \inf\{\gamma_p q_i | i \in I\}$.

4.2. Upper p-Regular Modification

Similar to the crisp case, the discrete structure δ is not always p-regular for any $p \in \top(X)$. This shows that for a given $q \in \top(X)$, there may not exist p-regular $\top$-convergence structure on X finer than q.

Definition 8. *Let* (X, p, q) *be a pair of* $\top$*-convergence spaces. If there exists a coarsest p-regular* $\top$*-convergence structure* $\gamma^p q$ *on* X *finer than* q*, then it is said to be the upper p-regular modification of* q.

It has been known that the existence of $\gamma^p q$ depends on the existence of a p-regular $\top$-convergence structure finer than q (see Corollary 1), and $\gamma_p \delta$ is the finest p-regular $\top$-convergence structure. So, it follows easily that $\gamma^p q$ exists if and only if $q \leq \gamma_p \delta$. By Theorem 4, we obtain the following result.

Theorem 7. *Let* (X, p, q) *be a pair of* $\top$*-convergence spaces. Then*

$$\gamma^p q \text{ exists} \Longleftrightarrow \forall x \in X, \forall n \in \mathbb{N}, cl_p^n([x]_\top) \xrightarrow{q} x.$$

Proof. For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and any $x \in X$, from Theorem 4 we obtain

$$\mathbb{F} \xrightarrow{\gamma_p \delta} x \Longleftrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{\delta} x \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F}.$$

Necessity. Let $\gamma^p q$ exist. Then $q \leq \gamma_p \delta$. So, for any $x \in X$, $n \in \mathbb{N}$

$$[x]_\top \xrightarrow{\delta} x \Longrightarrow cl_p^n([x]_\top) \xrightarrow{\gamma_p \delta} x \Longrightarrow cl_p^n([x]_\top) \xrightarrow{q} x.$$

Sufficiency. Let $cl_p^n([x]_\top) \xrightarrow{q} x$ for any $x \in X$, $n \in \mathbb{N}$. Then

$$\begin{array}{rcl} \mathbb{F} \xrightarrow{\gamma_p \delta} x & \Longrightarrow & \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{\delta} x \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ & \Longrightarrow & \exists n \in \mathbb{N}, [x]_\top \subseteq \mathbb{G} \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ & \overset{\text{Proposition 1 (2)}}{\Longrightarrow} & \exists n \in \mathbb{N}, cl_p^n([x]_\top) \subseteq cl_p^n(\mathbb{G}) \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ & \Longrightarrow & \exists n \in \mathbb{N} \text{ s.t. } cl_p^n([x]_\top) \subseteq \mathbb{F} \\ & \Longrightarrow & \mathbb{F} \xrightarrow{q} x. \end{array}$$

It follows that $q \leq \gamma_p \delta$, so $\gamma^p q$ exists. □

The following theorem gives a characterization on upper p-regular modification if it exists.

Theorem 8. *Let* (X, p, q) *be a pair of* $\top$*-convergence spaces and* $\gamma^p q$ *exists. Then*

$$\mathbb{F} \xrightarrow{\gamma^p q} x \Longleftrightarrow \forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x.$$

Proof. We define q' as $\mathbb{F} \xrightarrow{q'} x \Longleftrightarrow \forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x$.

(1) $q' \in \top(X)$. It is obvious.

(2) $q \leq q'$. In fact, let $\mathbb{F} \xrightarrow{q'} x$ then $\mathbb{F} = cl_p^0(\mathbb{F}) \xrightarrow{q} x$.

(3) q' is p-regular. In fact, let $\mathbb{F} \xrightarrow{q'} x$. Then for any $n \in \mathbb{N}$ it holds that $cl_p^n(cl_p(\mathbb{F})) = cl_p^{n+1}(\mathbb{F}) \xrightarrow{q} x$, which means $cl_p(\mathbb{F}) \xrightarrow{q'} x$. So, q' is p-regular.

(4) Let r be p-regular with $q \leq r$. Then $q' \leq r$. In fact, let $\mathbb{F} \xrightarrow{r} x$ then for any $n \in \mathbb{N}$, by Proposition 6 (1) it holds that $cl_p^n(\mathbb{F}) \xrightarrow{r} x$ and so $cl_p^n(\mathbb{F}) \xrightarrow{q} x$ by $q \leq r$. That means $\mathbb{F} \xrightarrow{q'} x$.

By (1)–(4) we get that q' is the coarsest p-regular $\top$-convergence structure finer than q. Therefore, $\gamma^p q = q'$. □

Theorem 9. *Let $f : (X, q) \longrightarrow (X', q')$ be a continuous function, and $f : (X, p) \longrightarrow (X', p')$ be a closure function between $\top$-convergence spaces. If $\gamma^p q$ and $\gamma^{p'} q'$ exist then $f : (X, \gamma^p q) \longrightarrow (X', \gamma^{p'} q')$ is continuous.*

Proof. Let $\mathbb{F} \xrightarrow{\gamma^p q} x$. Then $\forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x$. Since $f : (X, q) \longrightarrow (X', q')$ is a continuous function and $f : (X, p) \longrightarrow (X', p')$ is a closure function it holds that

$$\forall n \in \mathbb{N}, cl_{p'}^n(f^{\Rightarrow}(\mathbb{F})) \supseteq f^{\Rightarrow}(cl_p^n(\mathbb{F})) \xrightarrow{q'} f(x),$$

so $f^{\Rightarrow}(\mathbb{F}) \xrightarrow{\gamma^{p'} q'} f(x)$, as desired. □

The following theorem exhibits us that the upper p-regular modification has good compatibility with initial structures.

Theorem 10. *Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of spaces in $\top$-**CS** and q be the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$. Let $p \in \top(X)$ such that each $f_i : (X, p) \longrightarrow (X_i, p_i)$ is continuous closure function. If $\gamma^{p_i} q_i$ exists for all $i \in I$ then so does $\gamma^p q$, and $\gamma^p q$ is precisely the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, \gamma^{p_i} q_i))_{i \in I}$.*

Proof. At first, we show the existence of $\gamma^p q$. By Theorem 7, it suffices to check that $cl_p^n([x]_\top) \xrightarrow{q} x$ for any $x \in X, n \in \mathbb{N}$. In fact, by the existence of $\gamma^{p_i} q_i$ we have $cl_{p_i}^n([f_i(x)]) \xrightarrow{q_i} f_i(x)$ for any $i \in I, x \in X, n \in \mathbb{N}$. Then by each $f_i : (X, p) \longrightarrow (X_i, p_i)$ being a continuous closure function it holds that

$$f_i^{\Rightarrow}(cl_p^n([x]_\top) = cl_{p_i}^n(f_i^{\Rightarrow}([x]_\top)) = cl_{p_i}^n([f_i(x)]_\top) \xrightarrow{q_i} f_i(x),$$

so $cl_p^n([x]_\top) \xrightarrow{q} x$ for any $x \in X, n \in \mathbb{N}$, i.e., $\gamma^p q$ exists.

Let s denote the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, \gamma^{p_i} q_i))_{i \in I}$. Then

$$\begin{aligned}
\mathbb{F} \xrightarrow{s} x \quad &\Longleftrightarrow \quad \forall i \in I, f_i^{\Rightarrow}(\mathbb{F}) \xrightarrow{\gamma^{p_i} q_i} f_i(x) \overset{\text{Theorem 8}}{\Longleftrightarrow} \forall i \in I, \forall n \in \mathbb{N}, cl_{p_i}^n(f_i^{\Rightarrow}(\mathbb{F})) \xrightarrow{q_i} f_i(x) \\
&\overset{\text{Proposition 2}}{\Longleftrightarrow} \quad \forall i \in I, \forall n \in \mathbb{N}, f_i^{\Rightarrow}(cl_p^n(\mathbb{F})) \xrightarrow{q_i} f_i(x) \\
&\Longleftrightarrow \quad \forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x \overset{\text{Theorem 8}}{\Longleftrightarrow} \mathbb{F} \xrightarrow{\gamma^p q} x. \quad \square
\end{aligned}$$

The following corollary tells us that upper p-regular modification has good compatibility with supremum in the lattice $\top(X)$.

Corollary 3. *Assume that $\{q_i | i \in I\} \subseteq \top(X)$, $p \in \top(X)$ and $q = \sup\{q_i | i \in I\}$. If $\gamma^p q_i$ exists for all $i \in I$ then so does $\gamma^p q$ and $\gamma^p q = \sup\{\gamma^p q_i | i \in I\}$.*

5. Conclusions

In this paper, we studied p-regularity in $\top$-convergence spaces by a diagonal condition and a closure condition about $\top$-filter, respectively. We proved that p-regularity was preserved under the initial and final constructions in the category $\top$-**CS**. We then followed as a conclusion that p-regularity was preserved under the infimum and supremum in the lattice $\top(X)$. Furthermore, we defined and discussed lower (upper) p-regular modifications in $\top$-convergence spaces. In particular, we showed that lower (resp., upper) p-regular modification has good compatibility with final (resp., initial) construction.

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