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Structure of Normed Simple Annihilator Algebras

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Abstract: This article is devoted to normed simple annihilator algebras. Their structure is investigated in the paper. Maximal families of orthogonal irreducible idempotents of normed simple annihilator algebras are scrutinized. Division subalgebras of annihilator algebras are studied. Realizations of these algebras by operator algebras in Banach spaces are described. For this purpose, quasi finite dimensional operators are investigated.

Keywords: operator; algebra; norm; ideal; annihilator; infinite dimension; field

MSC: 12J05; 16D60; 16D80; 46B28; 46H20

1. Introduction

Algebras and operator algebras over the real field $\mathbf{R}$ and the complex field $\mathbf{C}$ have found many-sided applications [1–4]. They compose a large area of mathematics (see, for example, [2,5,6] and references therein). In particular, annihilator algebras play very important roles in structure theory of algebras, decompositions of representations of algebras or groups into irreducible representations, etc. [4,7]. Nevertheless for such algebras over other normed fields little is known. This is caused by their specific features and additional difficulties arising from structure of fields [8–15].

A lot of results in the classical case use the fact that the real field $\mathbf{R}$ has the linear ordering compatible with its additive and multiplicative structure and that the complex field $\mathbf{C}$ is algebraically closed, norm-complete, locally compact, and is the quadratic extension of $\mathbf{R}$. Besides them there are no other commutative fields with Archimedean multiplicative norms which are complete relative to their norms. However, there are many other normed fields apart from $\mathbf{R}$ and $\mathbf{C}$.

It is worth mentioning that the algebraic closure of the field $\mathbf{Q_p}$ of p-adic numbers is not locally compact. In this paper on algebras X over fields F norms $|\cdot|$ are considered which satisfy the strong triangle inequality $|x+y| \le \max(|x|,|y|)$ for each x and y in X. Norms fulfilling the latter condition are also called non-Archimedean norms or ultranorms. Notice that there is not any ordering on an infinite field such as $\mathbf{Q_p}$, $\mathbf{C_p}$ or $\mathbf{F_p}(t)$ compatible with its norm and algebraic structure.

On the other hand, there are fast developing non-Archimedean analysis, functional analysis and representations theory of groups over non-Archimedean fields [14,16–21]. This is stimulated not only by mathematical problems, but also their applications in other sciences such as physics, quantum mechanics, quantum field theory, informatics, etc. (see, for example, [22–29] and references therein).

This article is devoted to normed annihilator algebras over non-Archimedean fields. A structure of simple annihilator algebras is studied in the paper. Necessary Notation 1 and Definitions 1–6 are given. Idempotents of algebras and their orthogonality are investigated in Proposition 1. Extensions of algebras are studied in Proposition 2, Theorem 1, and Corollary 1. Maximal families of orthogonal irreducible idempotents of normed simple annihilator algebras are scrutinized. Division subalgebras associated with idempotents are investigated in Theorems 2 and 3. Relations with operator theory and realizations of these algebras by operator algebras are outlined. For this purpose, quasi-finite dimensional operators are studied.

All main results of this paper are quite new and their applications are discussed in the conclusion section.

2. Normed Annihilator Algebras

To avoid misunderstandings, we first give Definitions and one Notation. Though they follow main lines of that of in [4,7,11,14,30,31], but they also contain some specific ultranormed features. A reader familiar with these sources or the article [32] may skip Notation 1 and Definitions 1–6.

Notation 1. *Assume that F is an infinite field supplied with a multiplicative non-trivial ultranorm $|\cdot|_F$. Assume also that F is non-discrete and $\Gamma_F \subset (0,\infty) = \{r \in \mathbf{R} : 0 < r < \infty\}$, where $\Gamma_F := \{|x|_F : x \in F \setminus \{0\}\}$. Henceforward, fields with multiplicative ultranorms are considered, if something other will not be specified.*

Let $E_n(F)$ denote the class containing F and all ultranormed field extensions G of F so that these G are norm-complete and $|b|_G = |b|_F$ for each b in a given field F. Let also E_n denote the class of all infinite non-trivially ultranormed fields F which are norm-complete.

Henceforth, a commutative field is called shortly a field, while a noncommutative field is called a skew field or a division algebra.

Definition 1. *Suppose that $c_0(\alpha, F)$ is a Banach space consisting of all vectors $x = (x_j : \forall j \in \alpha \; x_j \in F)$ subjected to the condition*

$card\{j \in \alpha : |x_j| > \epsilon\} < \aleph_0$ for each $\epsilon > 0$

and supplied with the norm

(1) $|x| = \sup_{j\in\alpha} |x_j|$,

where α is a set. For locally convex spaces X and Y over F the family of all linear continuous operators $B : X \to Y$ will be denoted by $L(X,Y)$. Particularly, for normed spaces X and Y the linear space $L(X,Y)$ is also normed

(2) $|B| := \sup_{x\in X\setminus\{0\}} |Bx|/|x|$.

For locally convex spaces X and Y over F the space $L(X,Y)$ is topologized by a family of semi-norms

(3) $|B|_{p,q} := \sup_{x\in X, p(x)>0} q(Bx)/p(x)$

for all continuous semi-norms p on X and q on Y.

Speaking about Banach spaces and Banach algebras we stress that a field over which it is defined is ultranorm complete.

Assume that A is an algebra over F, where $F \in E_n$. It is supposed that an ultranorm

$|\cdot|_A$ on A satisfies the conditions:
$|a|_A \in (\Gamma_F \cup \{0\})$ for each $a \in A$, also
$|a|_A = 0$ if and only if $a = 0$ in A,
$|ta|_A = |t|_F|a|_A$ for each $a \in A$ and $t \in F$,
$|a+b|_A \le \max(|a|_A, |b|_A)$ and
$|ab|_A \le |a|_A|b|_A$ for each a and b in A.

Frequently it is shortly written $|\cdot|$ instead of $|\cdot|_F$ or $|\cdot|_A$.

Definition 2. *Assume that F is an infinite field such that $F \in E_n$ and of the characteristic $char(F) \ne 2$. Assume also that $B_2 = B_2(F)$ is the commutative associative algebra with one generator i_1 such that $i_1^2 = -1$ and furnished with the involution $(vi_1)^* = -vi_1$ for each $v \in F$. Suppose that A is a subalgebra in $L(X,X)$ such that A is also a two-sided B_2-module, where $X = c_0(\alpha, F)$ is the Banach space over F, α is a set. Then A is called a $*$-algebra if there is*

(1) *a continuous bijective surjective F-linear operator* $\mathcal{I} : A \to A$ *such that*
(2) $\mathcal{I}(ab) = (\mathcal{I}b)(\mathcal{I}a)$ *and*
(3) $\mathcal{I}(ga) = (\mathcal{I}a)g^*$ *and* $\mathcal{I}(ag) = g^*(\mathcal{I}a)$
(4) $\mathcal{I}\mathcal{I}a = a$
(5) $(\theta(y))(ax) = (\theta((\mathcal{I}a)y))(x)$

for every a and b in A and $g \in B_2$ *and x and y in X, where* $\theta : X \hookrightarrow X'$ *is the canonical embedding of X into the topological dual space* X' *such that* $\theta(y)x = \sum_{j\in\alpha} y_j x_j$. *For the sake of brevity, we can write* a^* *instead of* $\mathcal{I}a$. *The mapping* $\mathcal{I}$ *is called the involution. An element* $a \in A$ *is called self-adjoint if* $a = a^*$.

Definition 3. *For a topological algebra A over a field F and a subset S of A the left annihilator is defined by* $\mathsf{L}(A, S) := \{x \in A : xS = 0\}$ *and the right annihilator by* $\mathsf{R}(A, S) := \{x \in A : Sx = 0\}$. *Shortly they also will be denoted by* $A_l(S) := \mathsf{L}(A, S)$ *and* $A_r(S) := \mathsf{R}(A, S)$ *correspondingly.*

Definition 4. *An algebra A is called an annihilator algebra if it satisfies conditions (1–3):*

(1) $A_l(A) = A_r(A) = 0$ *and*
(2) $A_l(J_r) \neq 0$ *and*
(3) $A_r(J_l) \neq 0$
for all closed right J_r *and left* J_l *ideals in A.*

Definition 5. *Suppose that* w_1 *and* w_2 *are idempotents of an algebra A fulfilling the conditions* $w_1w_2 = 0$ *and* $w_2w_1 = 0$. *Then they are called orthogonal. A family* $\{w_j : j\}$ *of idempotents is called orthogonal, if and only if every two distinct of them are orthogonal. It is said that an idempotent p of the algebra A is irreducible, if it cannot be written as the sum of two mutually orthogonal idempotents.*

Definition 6. *Let A and B be two Banach algebras over an ultranormed field F,* $F \in E_n$. *Let* $A\hat{\otimes}_F B$ *be the completion relative to the projective tensor product topology (see [14,33]) of the tensor product* $A \otimes_F B$ *over the field F.*

Suppose that B is a Banach algebra over an ultranormed field F, $F \in E_n$, *and x is an element in B. It will be said that x has a left core quasi-inverse y if for each* $H \in E_n(F)$ *an element* $y \in B_H$ *exists satisfying the equality* $x + y + yx = 0$, *where* $B_H = B\hat{\otimes}_F H$. *Symmetrically a right core quasi-inverse is defined. In particular, if only the field* $H = F$ *is considered they are called a left quasi-inverse and a right quasi-inverse correspondingly.*

Assume that A is a unital Banach algebra over F, where $F \in E_n$. *Suppose also that an element* $x \in A$ *has the property: for each field extension* $G \in E_n(F)$ *the left inverse* $(1 + yx)_l^{-1}$ *exists in* A_G *for each* $y \in A_G$. *Then we call x a generalized core nil-degree element. The family of all generalized core nil-degree elements of A is called a core radical and it is denoted by* $R_c(A)$. *By* $R(A)$ *a radical of the algebra A is denoted.*

Proposition 1. *Suppose that A is a Banach simple unital annihilator algebra over a field F,* $F \in E_n$, *also* $R_c(A) = R(A)$. *Then a maximal family of orthogonal irreducible idempotents* $\{w_j : j \in J\}$ *exists such that* $\sum_j Aw_j$ *and* $\sum_j w_jA$ *are dense in A.*

Proof. In view of Proposition 8 in [32] there are irreducible idempotents w_j in A. Each right ideal B in A contains a minimal right ideal, consequently, it contains an irreducible idempotent. By virtue of Zorn's lemma (see [34] or [35]) a maximal orthogonal system $\{w_j : j \in J\}$ of irreducible idempotents w_j exists. Let

$$C = \sum_j Aw_j$$

be the sum of all such left ideals. Suppose that $cl_A C \neq A$, where $cl_K P$ denotes the closure of a subset P in a topological space K. Then $cl_A C$ is a closed left ideal. Therefore, $A_r(cl_A C)$ is the right ideal different from zero. This implies that $A_r(cl_A C)$ contains an irreducible idempotent p orthogonal to each w_j.

But this is impossible, since the family $\{w_j : j \in J\}$ is maximal. Thus, it remains that C is dense in A. Similarly, the sum $\sum_j w_j A$ of all such right ideals is dense in A. □

Proposition 2. *Let F be a field and let $\{K_j : j \in P\}$ be a family of division algebras such that F is contained in the center $Z(K_j)$ of K_j for each $j \in P$, where P is a set. Then a minimal division algebra K exists such that $K_j \subseteq K$ for each $j \in P$.*

Proof. Since K_j is a division algebra, then its center $Z(K_j)$ is a field. We take the tensor product

$$T = \bigotimes_{j \in P} K_j$$

of K_j as algebras over the field F. Therefore, T is an algebra over F so that T may be noncommutative if at least one of K_j is noncommutative. For each K_j a natural embedding $h_j : K_j \hookrightarrow K$ exists. Moreover, T contains the unit element which can be identified with the unit of the field F.

For each proper left ideal B in T the intersection $B_j = B \cap h_j(K_j)$ is a left ideal of K_j. In view of Theorem I.9.1 in [7] $B_j = (0)$, since $F \subset T/B$ and the unit is unique in the associative algebra T/B. Particularly, for a maximal proper left ideal B in T this induces the embedding $t \circ h_j$ of K_j into the quotient algebra T/B over the field F for each $j \in P$, where $t : T \to T/B$ denotes the quotient F-linear mapping.

Then equations $a_j x_j = b_j$ and $y_j a_j = b_j$ with $a_j \neq 0$ and b_j in $h_j(K_j)$ have unique solutions x_j and y_j in $h_j(K_j)$ for each $j \in P$. For an arbitrary $a \in T/B$ take an element $c \in t^{-1}(a)$. Then we deduce that $c + B = t^{-1}(a)$ and consequently, $h_j(K_j) \cap t^{-1}(a) = h_j(K_j) \cap \{c\}$, where $\{c\}$ denotes the singleton in T. At the same time, $qu = 0$ for some q and u in T implies $t((q + B)(u + B)) = 0$ in T/B. Therefore, equations $ax = b$ and $ya = b$ with $a \neq 0$ and b in T/B have unique solutions x and y in T/B, since B is the proper maximal left ideal in T and $h_j(K_j) \cap h_i(K_i) = F$ for each $j \neq i$ in P. From Theorem 9.2 and Corollary 9.3 in [36] it follows that an embedding of T/B into a unique-division algebra L over F exists. Taking the intersection of all such algebras L one gets a minimal unique-division algebra K over F containing T/B. Thus the embedding of K_j into the division algebra K exists for each $j \in P$. □

Theorem 1. *If conditions of Proposition 2 are fulfilled and each K_j is a Hausdorff topological division algebra with a topology τ_j such that*

(i) $\tau_j|_{K_i \cap K_j} = \tau_i|_{K_i \cap K_j}$

for each i and j in P, then a Hausdorff topology τ on K exists such that an embedding $h_j : K_j \to K$ is a homeomorphism of (K_j, τ_j) onto $(h_j(K_j), \tau \cap h_j(K_j))$ for each $j \in P$. Moreover, if each K_j is normed and

(ii) $|\cdot|_{K_j}|_{K_i \cap K_j} = |\cdot|_{K_i}|_{K_i \cap K_j}$

for each i and j in P, where $|\cdot|_{K_j}$ denotes a norm on K_j, then K is normed.

Proof. We consider on the weak product

$$S = \prod_{j \in P}{}' K_j$$

the box product topology, where each $s \in S$ has the form

$$s = (s_j : \forall j \, s_j \in K_j, \; card\{j : s_j \neq e_j\} < \aleph_0),$$

where $e_j = 1$ denotes the unit element in K_j. It induces the corresponding topology a_p on the tensor product T, where T is the quotient algebra S/M of S by the submodule M with elements of the form

(1) $(x)+(y)-(z)$ with $x_i+y_i=z_i$ for one index $i\in P$ and with $x_j=y_j=z_j$ for each $j\neq i$ in P;
(2) $(x)-(y)$ with $x_i=by_i$ for one index i in P and $x_j=y_j$ for each $j\neq i$ in P for every $b\in F$, (x), (y) and (z) in S (see also Chapter 3 in [7]). The algebra T is supplied with the multiplication prescribed by the rule

$$(x)(y)=\bigotimes_{j\in P}x_jy_j$$

for each (x) and (y) in T. Due to condition (i) for each i there exists an algebraic topological embedding of K_i into T.

The algebra K over F is obtained as the unique-division algebra K over F containing T/B (see the proof of Proposition 2).

The algebra T is unital, since K_j is unital for each j. There exists a neighborhood W_j of 1 in K_j such that the inversion is continuous on W_j for each j. We take

$$W=\prod_{j\in P}{}'W_j,$$

hence W is a neighborhood of 1 in T such that the inversion is continuous on W, since S is supplied with the box topology. Therefore, if B is a left maximal ideal in T, then B is closed in T, since algebraic operations on T are continuous and T is with the continuous inverse on W.

Therefore, the box topology a_p on T induces the quotient T_1-topology b_p on T/B, since a_p is the Hausdorff topology and B is closed in T. Then we consider a base U of a topology τ on K satisfying the conditions:

(3) $U_x=U_0+x$ for each $x\in K$,
(4) $U_x=xU_1=U_1x$ for each nonzero x in K,
(5) $U_0\cap(T/B)$ is the base of neighborhoods of zero in the b_p topology on T/B, where U_x denotes a base of neighborhoods of an element x in K such that $U_x\subset U$;
(6) for each E and D in U_0 there exists $C\in U_0$ such that $C\subset E\cap D$;
(7) $\bigcap_{V\in U_0}V=\{0\}$;
(8) for each $E\in U_0$ there exists $D\in U_0$ such that $(D+D)\subset E$ and $(D+1)^2\subset(E+1)$;
(9) for each $E\in U_0$ there exists $D\in U_0$ such that $-D\subset E$ and $(D+1)^{-1}\subset(E+1)$,
(10) $U|_F$ provides the base of the $\tau_j|_F$ topology on F.

It exists, since $F\subset K_i$ and condition (i) is fulfilled for each i and j in P and since the b_p topology on T/B satisfies analogous to (3–10) conditions due to Theorem 1.3.12 in [37].

Each element of K is obtained from elements of T/B by a finite number of algebraic operations. Therefore, the intersection of all such bases U satisfying conditions (3)–(9) provides a minimal base possessing these properties. In view of Theorem 1.3.12 in [37] this induces a Hausdorff topology τ on K.

From the construction above it follows that $\tau\cap(T/B)=b_p$, consequently, $\tau\cap h_j(K_j)$ is equivalent with the topology $\tau\cap h_j(K_j)$ on $h_j(K_j)$ inherited from $(T/B,b_p)$ for each $j\in P$, where h_j is the algebraic embedding as in Proposition 2. Therefore h_j is the homeomorphism of (K_j,τ_j) onto $(h_j(K_j),\tau\cap h_j(K_j))$ for each $j\in P$.

In particular, if K_j is normed for each j, then T is normed by $|x|=\sup_j|x_j|_{K_j}$, where $|hx|=|h||x|$ for each $h\in F$ and $x\in T$. Such normed topology is not stronger than the a_p topology. By condition (ii) of this theorem norms $|\cdot|_{K_j}$ and $|\cdot|_{K_i}$ on $K_i\cap K_j$ are equivalent for each i and j in P, hence there exists an algebraic isometric embedding of K_i into T for each i. On the other hand, $F\subset K_i$ for each i. This induces the quotient norm on T/B relative to which h_j is continuous for each j. Therefore,

U_0 and U_1 on K can be chosen countable and such that $V + V \subseteq V$ for each $V \in U_0$, $WW \subseteq W$ for each $W \in U_1$. Thus, the minimal division algebra K is normable:

$$|x + y|_K \leq \max(|x|_K, |y|_K) \text{ and}$$

$$|xy|_K \leq |x|_K |y|_K$$

for each x and y in K. □

Corollary 1. *If conditions of Theorem 1 are satisfied, then a completion $\tilde{K}$ of K relative to a left uniformity l_τ induced by τ exists such that $\tilde{K}$ is a division algebra. Moreover, if condition (ii) of Theorem 1 is fulfilled, then $\tilde{K}$ is the Banach division algebra.*

Proof. Consider on the multiplicative group K^* of nonzero elements of K the left uniformity l_τ induced by τ. In view of §8.1.17 and Theorem 8.3.10 in [34] and conditions $(3, 4)$ (see the proof of Theorem 1) the completion $\tilde{K}$ of K relative to l_τ is the unique-division algebra over F.

If in addition Condition (ii) of Theorem 1 is fulfilled, then we take $\tilde{K}$ as the completion of K relative to its norm. □

Theorem 2. *Let A be a simple unital annihilator Banach algebra over a normed field F, $F \in E_n$, with $R_c(A) = R(A)$, and let $\{w_j\}$ be a maximal orthogonal system of irreducible idempotents in it. Then a Banach division algebra G exists such that $w_j A w_j \subset G$ for each irreducible idempotent w_j in A, also*

$$\sum_{i,j} w_i A w_j \text{ is dense in } A.$$

Proof. Suppose that $\{w_j\}$ is a maximal orthogonal system of irreducible idempotents in the algebra A. For a chosen idempotent w_i one gets the two-sided non-nil ideal Aw_iA. Since A is simple, then $cl_A(Aw_iA) = A$, where $cl_A S$ denotes the closure of a subset S in A. Therefore, $w_j A w_i A w_j \neq (0)$ for each j. Moreover, this implies that $w_j A w_i A w_j \subseteq w_j A w_j = G_j w_j$, where G_j is a division algebra over F according to Theorem 9 in [32].

Consider the algebra $A_j := A_{G_j} = A \hat{\otimes}_F G_j$ obtained from A by extension. For each j elements x_j and y_j in A_j exist such that $w_j x_j w_i y_j w_j = w_j$. We put $w_{j,i} = w_j x_j w_i$ and $w_{i,j} = w_i y_j w_j$ and $w_{j,k} = w_{j,i} w_{i,k}$. Therefore, the elements $w_{j,i}$ and $w_{i,j}$ belong to A_j and $w_{j,j} = w_j$, since $w_i^2 = w_i$. Then we infer that

$$w_{j_1,k_1} w_{k_1,k_2} = w_{j_1} x_{j_1} w_i w_i y_{k_1} w_{k_1} w_{k_1} x_{k_1} w_i w_i y_{k_2} w_{k_2}$$

$$= w_{j_1} x_{j_1} w_i (y_{k_1} w_{k_1} x_{k_1} w_i) y_{k_2} w_{k_2} \text{ and}$$

$$w_{j_1,k_2} = w_{j_1,i} w_{i,k_2} = w_{j_1} x_{j_1} w_i w_i y_{k_2} w_{k_2}$$

$$= w_{j_1} x_{j_1} w_i y_{k_2} w_{k_2}.$$

Notice that this construction implies that $w_{i,j} w_{j,i} \in w_i A w_i = G_i w_i$ and consequently, $w_{i,j} w_{j,i} = b w_i$ for a scalar $b = b_{i,j} \in G_i$. The multiplication of both sides of the latter equality on the left by $w_{j,i}$ and on the right by $w_{i,j}$ leads to

$$w_{j,i} w_{i,j} w_{j,i} w_{i,j} = w_j^2 = b w_{j,i} w_{i,j} = b w_j^2,$$

consequently, $w_j = b w_j$ and hence $b = 1$. Therefore, we deduce that

$$w_{j_1,k_1} w_{k_1,k_2} = w_{j_1,i} w_{i,k_1} w_{k_1,i} w_{i,k_2} = w_{j_1,i} w_i w_{i,k_2} = w_{j_1,k_2},$$

since $w_{j,i}w_i = w_{j,i}$, $w_iw_{i,j} = w_{i,j}$. Then we infer that

$$w_{j_1,k_1}w_{j_2,k_2} = w_{j_1,i}w_{i,k_1}w_{j_2,i}w_{i,k_2} \text{ and}$$

$$w_{i,k_1}w_{j_2,i} = w_iy_{k_1}w_{k_1}w_{j_2}x_{k_2}w_i, \text{ consequently,}$$

$$(1) \quad w_{j_1,k_1}w_{j_2,k_2} = 0 \text{ and}$$

$$(2) \quad w_{k_1}w_{j_2} = 0$$

for each $k_1 \neq j_2$, also

$$(3) \quad w_{j_1,k_1}w_{k_1,k_2} = w_{j_1,k_2}$$

for every j_1, k_1, k_2.

Thus, the set w_jAw_k is composed of elements which are multiples of the element $w_{j,k}$, consequently, $w_jxw_{k,j} \in w_jAw_j = G_jw_j$, where the division algebra G_j is over the field F according to Theorem 9 in [32]. Therefore, a scalar $b \in G_j$ exists such that $w_jxw_{k,j} = bw_j$. Multiplying on the right by $w_{j,k}$ and using identity (3) we infer that $w_jxw_k = bw_{j,k}$, where $b = b(j,k,x) \in G_j$. This implies that

$$\sum_{j,k} w_jAw_k =: B \subset A,$$

where B is an algebra over F.

t By virtue of Theorem 1 and Corollary 1 a Banach division algebra G exists such that $G_j \subset G$ for each j, since G_j is the algebra over F for each j, also since A is the Banach algebra.

We put $A_G = A\hat{\otimes}_FG$, that is A_G is the right G-module and the algebra over F. Thus, this provides the inclusion

$$\sum_j w_jAw_j =: E \subset A_G.$$

On the other hand, $w_jAw_j \subset A$ as the algebra over F for each j, since $w_j \in A$ for each j. Please note that the sum of all w_jAw_k contains the F-linear span Y of the set $(\sum_j w_jA)(\sum_k Aw_k)$. The multiplication and addition are continuous on A, hence Y is dense in the F-linear span X of $(\sum_j w_jA)A$, since $\sum_k Aw_k$ is dense in A. In its turn X is dense in the F-linear span V of A^2, since $\sum_j w_jA$ is dense in A. Therefore, E is dense in A, since V is the two-sided ideal in A which is necessarily dense in A. □

Definition 7. *Let X be a Banach space over a field F, $F \in E_n$, such that X also has the structure of a right G-module, where G is a division algebra over F. An operator $s \in L(X,X)$ will be called (right) quasi-finite dimensional if its range $s(X)$ is contained in a finite direct sum $x_1G \oplus ... \oplus x_nG$ embedded into X and such that s is right G-linear, that is $s(xb) = (sx)b$ for each $x \in X$ and $b \in G$, where x_1,...,x_n are nonzero vectors belonging to X.*

Theorem 3. *Let A be a simple unital annihilator Banach algebra over a normed field F, $F \in E_n$, with $R_c(A) = R(A)$. Then a Banach division algebra G exists such that $A_G := A\hat{\otimes}_FG$ has an embedding T into the algebra $L(X,X)$, where X is a Banach space over F and a right G-module, such that*

(1) $T(A_G)$ contains all (right) quasi-finite dimensional operators so that $T(A_G)$ is a Banach subalgebra in $L(X,X)$ and

(2) a dense subalgebra B in A_G exists whose image $T(B)$ consists of quasi-finite dimensional operators.

Proof. Let a Banach division algebra G be provided by Theorem 2. Then $w_iAw_i = G_i \subseteq G$ for each i and hence $w_iA_Gw_i = G$, since G is the division algebra over F.

We denote A_G by A shortly. Then $cl_A(Aw_iAx) \neq (0)$ for each $x \neq 0$, consequently, $Aw_iAx \neq (0)$ and hence

$$(3) \quad w_iAx \neq (0)$$

for each $x \neq 0$ in A.

Next we consider a left regular representation of the algebra A by operators $\mathbf{L}_x$ for each $x \in A$, where $\mathbf{L}_x y := xy$ for each $y \in A$. From property (3) it follows that the left regular representation $A \ni x \mapsto \mathbf{L}_x$ is the F-linear isomorphism. On the other hand, $G_j \subset A$ and $G_j \subset G$ and $G_j G = G G_j = G$ for each j. The operator $\mathbf{L}_x$ is right G-linear for each $x \in A$, that is $\mathbf{L}_x(yb) = (\mathbf{L}_x y)b$ for each $y \in A$ and $b \in G$, since A and G are associative algebras over the field F, also A has the structure of the right G-module.

In view of Formulas $(2,3)$ in the proof of Theorem 2 the operator $\mathbf{L}_{w_{k,i}}$ maps the one-dimensional over G right module $w_i A w_j$ into $w_k A w_j$, also $\mathbf{L}_{w_{j,k}} w_l A w_i = (0)$ for $k \neq l$. Since the sum $\sum_j w_i A w_j$ is dense in $w_i A$, then $\mathbf{L}_{w_{k,i}} w_i A w_j$ is the one-dimensional over G right module. Therefore, the operator $\mathbf{L}_x$ is quasi-finite dimensional for each x in B, where

$$B := \sum_{j,k} w_j A w_k.$$

Suppose now that V is a one-dimensional operator in $w_i A$ over G and $b \in w_i A$ is an element such that $bA \neq 0$. Therefore, $w_i A = Gb \oplus N(V)$, where

$$N(V) := \{x \in w_i A : Vx = 0\}$$

Assume that $\mathsf{L}(A, N(V)) = (0)$. This implies that $\mathsf{L}(A, N(V)A) = \mathsf{L}(A, N(V)) = (0)$, since the closed right G-module $M_{N(V)A}$ generated by $N(V)A$ has the natural embedding ψ into A and $\psi M_{N(V)A}$ is a right ideal in A. Therefore, this leads to the equality $A = \psi M_{N(V)A}$. On the other hand, $N(V) = w_i N(V)$, consequently,

$$M_{w_i N(V) w_i A} = w_i A.$$

Then the identity $w_i A w_i = w_i G$ would imply that $N(V) = w_i A$ providing the contradiction. This implies that $\mathsf{L}(A, N(V)) \neq (0)$.

We take now $x \neq 0$ in $\mathsf{L}(A, N(V))$. Let $xb = 0$, hence $w_i A x = M_{Gb \oplus N(V)} x = (0)$ contradicting Property (3), consequently, $xb \neq 0$ and hence xbA is a non-null right ideal in A. Then $xbA = w_i A$, since $w_i A$ is the minimal right ideal in A and $xbA \subseteq w_i A$. Thus, an element $y \in A$ exists fulfilling the condition $yxb = Vb$. This implies that the operators V and $\mathbf{L}_{yx}$ coincide on Gb. Mention that $\mathbf{L}_{yx} N(V) = (0) = V N(V)$, since $x \in \mathsf{L}(A, N(V))$, hence $\mathbf{L}_{yx} = V$. Thus, all right G-linear one-dimensional over G operators are among $\mathbf{L}_x$, where $x \in A$.

Assume that A is a Banach algebra, then G provided by Theorem 2 is also a Banach division algebra over F. By the continuity of the multiplication in A we deduce that $w_i A$ is a closed F-linear subspace in A, consequently, $|\mathbf{L}_x| \leq |x|$ for each $x \in A$, since

$$|\mathbf{L}_x y| = |xy| \leq |x||y|$$

for each $y \in A$. Therefore $A \ni x \mapsto \mathbf{L}_x$ is the continuous isomorphism into $L(X, X)$ and each $\mathbf{L}_x$ is the limit relative to the operator norm topology of quasi-finite dimensional operators $\mathbf{L}_{x_n}$ with $x_n \in B$ for each $n \in \mathbf{N}$. $\square$

3. Conclusions

The results obtained in this article can be used for further investigations of normed algebras and operator algebras on non-Archimedean Banach spaces, their cohomologies, spectral theory of operators, non-Archimedean functional analysis, the representation theory of groups, algebraic geometry, PDEs, applications in the sciences, etc. Other possible applications are in mathematical coding theory and its technical implementations [38–40], because frequently codes are based on binary systems and algebras

over non-Archimedean fields. Idempotents and decompositions of operator algebras can be used for an analysis and a classification of flows of information [22,41] and a solution of related PDEs [27,42].

In particular, Propositions 1 and 2, Theorems 2 and 3 can be used for studies of invariant subspaces of operator algebras in ultranormed spaces. They also can be used for studies of decompositions of totally disconnected topological groups representations by operators in ultranormed spaces into irreducible representations. Moreover, their relations with division algebras can be scrutinized with the help of Proposition 2 and Theorems 1 and 2.

It is worth mentioning that it also can serve for advances in non-Archimedean quantum field theory and quantum mechanics, because they are based on algebras and operator algebras over ultranormed fields [23,24,28,29].

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