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# $L_{p}$ Radial Blaschke-Minkowski Homomorphisms and $L_{p}$ Dual Affine Surface Areas 

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#### Abstract

Schuster introduced the notion of radial Blaschke-Minkowski homomorphism and considered the Busemann-Petty problem for volume forms. Whereafter, Wang, Liu and He presented the $L_{p}$ radial Blaschke-Minkowski homomorphisms and extended Schuster's results. In this paper, associated with $L_{p}$ dual affine surface areas, we give an affirmative and a negative form of the Busemann-Petty problem and establish two Brunn-Minkowski inequalities for the $L_{p}$ radial Blaschke-Minkowski homomorphisms.


Keywords: Busemann-Petty problem; $L_{p}$ radial Blaschke-Minkowski homomorphism; $L_{p}$ dual affine surface area; Brunn-Minkowski inequality

MSC: 52A20; 52A39; 52A40

## 1. Introduction

If $K$ is a compact star shaped (about the origin) in $n$-dimensional Euclidean space $\mathbb{R}^{n}$, then its radial function, $\rho_{K}=\rho(K, \cdot): \mathbb{R}^{n} \backslash\{0\} \rightarrow[0, \infty)$, is defined by (see [1])

$$
\rho(K, x)=\max \{\lambda \geq 0: \lambda x \in K\}, x \in \mathbb{R}^{n} \backslash\{0\} .
$$

If $\rho(K, \cdot)$ is positive and continuous, $K$ will be called a star body (about the origin). The set of all star bodies in $\mathbb{R}^{n}$ denotes by $\mathcal{S}_{o}^{n}$. For the set of all origin-symmetric star bodies, we write $\mathcal{S}_{o s}^{n}$. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^{n}$. Two star bodies $K$ and $L$ are said to be dilated (of one another) if $\rho_{K}(u) / \rho_{L}(u)$ is independent of $u \in S^{n-1}$.

Intersection bodies were explicitly defined and named by Lutwak (see [2]). For $K \in \mathcal{S}_{0}^{n}$, the intersection body, $I K$, of $K$ is a star body whose radial function is defined by

$$
\rho(I K, u)=V_{n-1}\left(K \cap u^{\perp}\right)
$$

for all $u \in S^{n-1}$. Here $u^{\perp}$ is the $(n-1)$-dimensional hyperplane orthogonal to $u$ and $V_{n-1}$ denotes the ( $n-1$ )-dimensional volume.

During past three decades, the investigations of intersection bodies have received great attention from many articles (see [1-12]). In particular, intersection bodies led to the following Busemann-Petty problem (see [2]).

Problem 1 (Busemann-Petty problem). For $K, L \in \mathcal{S}_{o}^{n}$, is there the implication

$$
I K \subseteq I L \Rightarrow V(K) \leq V(L) ?
$$

Here $V(K)$ denotes the $n$-dimensional volume of body $K$.
Remark 1. Problem 1 was stated by Lutwak (see [2]). If $K, L \in \mathcal{S}_{o s}^{n}$, then Problem 1 is called the symmetric Busemann-Petty problem. Gardner [13], Zhang [14] showed that the symmetric Busemann-Petty problem has an affirmative answer for $n \leq 4$ and a negative answer for $n \geq 5$.

For Problem 1, Lutwak [2] gave its an affirmative answer if $K$ is restricted to the class of intersection bodies and two negative answers if $K$ is not origin-symmetric star body or $L$ is not intersection body. For more research on solutions to the Busemann-Petty problem, see e.g., [1,13-20].

In 2006, based on the properties of intersection bodies, Schuster [21] introduced the radial Blaschke-Minkowski homomorphism, which is the more general intersection operator as follows:

Definition 1. A map $\Psi: \mathcal{S}_{o}^{n} \rightarrow \mathcal{S}_{o}^{n}$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:
(1) $\Psi$ is continuous;
(2) For all $M, N \in \mathcal{S}_{o}^{n}, \Psi(M \widehat{+} N)=\Psi M \widetilde{+} \Psi N$;
(3) $\Psi(\varphi M)=\varphi \Psi M$ for all $M \in \mathcal{S}_{o}^{n}$ and all $\varphi \in S O(n)$.

Here " $\widehat{+}$ " and " $\widetilde{+}$ " denote radial Blaschke addition and radial Minkowski addition, respectively; $S O(n)$ denotes the group of rotation transformations.

Whereafter, Schuster ([22]) considered the following Busemann-Petty problem for radial Blaschke-Minkowski homomorphisms.

Problem 2. Let $\Psi: \mathcal{S}_{o}^{n} \rightarrow \mathcal{S}_{o}^{n}$ be a radial Blaschke-Minkowski homomorphism. For $K, L \in \mathcal{S}_{o}^{n}$, is there the implication

$$
\Psi K \subseteq \Psi L \Rightarrow V(K) \leq V(L) ?
$$

Obviously, Problem 2 is a more general Busemann-Petty problem compared with Problem 1. For Problem 2, Schuster [22] gave an affirmative answer if $K$ belongs to $\Psi \mathcal{S}_{o}^{n}$ (the range of $\Psi$ ) and two negative forms.

In 2011, Wang, Liu and He [23] extended Schuster's radial Blaschke-Minkowski homomorphisms to $L_{p}$ analogies, and gave the notion of $L_{p}$ radial Blaschke-Minkowski homomorphisms as follows:

Definition 2. For $p>0$, a map $\Psi_{p}: \mathcal{S}_{o}^{n} \rightarrow \mathcal{S}_{o}^{n}$ is called a $L_{p}$ radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:
(1) $\Psi_{p}$ is continuous;
(2) For all $M, N \in \mathcal{S}_{o}^{n}, \Psi_{p}\left(M \widehat{+}_{p} N\right)=\Psi_{p} M \widetilde{+}_{p} \Psi_{p} N$;
(3) $\quad \Psi_{p}(\varphi M)=\varphi \Psi_{p} M$ for all $M \in \mathcal{S}_{o}^{n}$ and all $\varphi \in S O(n)$.

Here " $\hat{+}_{p}$ " and " $\widetilde{+}_{p}$ " denote $L_{p}$ radial Blaschke addition and $L_{p}$ radial Minkowski addition, respectively.
Meanwhile, associated with $L_{p}$ radial Blaschke-Minkowski homomorphisms, Wang et al. [23] extended Schuster's results including the Busemann-Petty Problem 2. In recent years, a lot of important conclusions for the radial Blaschke-Minkowski homomorphisms and their $L_{p}$ analogies were obtained (see e.g., [22-33]).

The $L_{p}$ dual affine surface areas firstly were introduced by Wang, Yuan and He (see [34]). Here, we improve Wang, Yuan and He's definition as follows: For $K \in \mathcal{S}_{o}^{n}$ and $p>0$, the $L_{p}$ dual affine surface area, $\widetilde{\Omega}_{p}(K)$, of $K$ is defined by

$$
\begin{equation*}
n^{-\frac{p}{n}} \widetilde{\Omega}_{p}(K)^{\frac{n+p}{n}}=\sup \left\{n \widetilde{V}_{p}\left(K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \tag{1}
\end{equation*}
$$

Here $\widetilde{V}_{p}(M, N)$ denotes the $L_{p}$ dual mixed volume of $M, N \in \mathcal{S}_{o}^{n}$, and $Q^{*}$ denotes the polar of $Q$ which is defined by (see [1])

$$
Q^{*}=\{x: x \cdot y \leq 1, y \in Q\}, x \in \mathbb{R}^{n}
$$

If $Q$ belongs to the set of convex bodies (or star bodies) whose centroid at the origin, then Equation (1) is just Wang, Yuan and He's (or Wang and Wang's) definition (see [34] or [35]). For the studies of $L_{p}$ dual affine surface areas, also see [36-41].

Remark 2. Recall that Lutwak's $L_{p}$ affine surface area was defined as follows (see [42]): For $K \in \mathcal{K}_{o}^{n}$ and $p \geq 1$, the $L_{p}$ affine surface area, $\Omega_{p}(K)$, of $K$ is defined by

$$
\begin{equation*}
n^{-\frac{p}{n}} \Omega_{p}(K)^{\frac{n+p}{n}}=\inf \left\{n V_{p}\left(K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in S_{o}^{n}\right\} . \tag{2}
\end{equation*}
$$

Here, $\mathcal{K}_{o}^{n}$ denotes the set of convex bodies (compact, convex subsets with nonempty interiors) containing the origin in their interiors in $\mathbb{R}^{n}$ and $V_{p}(M, N)$ is the $L_{p}$ mixed volume of $M$ and $N$ (see [42]). Compare to Equation (1) and Equation (2), we see that Equation (1) is really the duality of Equation (2).

In this paper, associated with $L_{p}$ dual affine surface areas, we research the $L_{p}$ radial Blaschke-Minkowski homomorphisms. We firstly consider the following Busemann-Petty problem of $L_{p}$ radial Blaschke-Minkowski homomorphisms.

Problem 3. Let $p>0$ and $\Psi_{p}: \mathcal{S}_{o}^{n} \rightarrow \mathcal{S}_{o}^{n}$ be a $L_{p}$ radial Blaschke-Minkowski homomorphism. For $K, L \in \mathcal{S}_{o}^{n}$, is there the implication

$$
\Psi_{p} K \subseteq \Psi_{p} L \Rightarrow \widetilde{\Omega}_{p}(K) \leq \widetilde{\Omega}_{p}(L) ?
$$

Let $\Psi_{p} \mathcal{S}_{o}^{n}$ denote the range of $\Psi_{p}$ and $\Psi_{p}^{*} \mathcal{S}_{o}^{n}$ denote the set of polars of all elements in $\Psi_{p} \mathcal{S}_{o}^{n}$, then $\Psi_{p}^{*} \mathcal{S}_{o}^{n} \subseteq \mathcal{S}_{o}^{n}$. From this, we write that

$$
\begin{equation*}
n^{-\frac{p}{n}} \widetilde{\Omega}_{p}^{\circ}(K)^{\frac{n+p}{n}}=\sup \left\{n \widetilde{V}_{p}\left(K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \Psi_{p}^{*} \mathcal{S}_{o}^{n}\right\} \tag{3}
\end{equation*}
$$

For Problem 3, according to Equation (3), we obtain an affirmative form as follows:
Theorem 1. For $p>0$, let $\Psi_{p}: \mathcal{S}_{o}^{n} \rightarrow \mathcal{S}_{o}^{n}$ be a $L_{p}$ radial Blaschke-Minkowski homomorphism. If $K, L \in \mathcal{S}_{o}^{n}$, then

$$
\Psi_{p} K \subseteq \Psi_{p} L \Rightarrow \widetilde{\Omega}_{p}^{\circ}(K) \leq \widetilde{\Omega}_{p}^{\circ}(L)
$$

In addition, $\widetilde{\Omega}_{p}^{\circ}(K)=\widetilde{\Omega}_{p}^{\circ}(L)$ when $\Psi_{p} K=\Psi_{p} L$.
Furthermore, when $K \notin \mathcal{S}_{o s}^{n}$, by Equation (1) we give the following a negative form of Problem 3.
Theorem 2. For $K, L \in \mathcal{S}_{o}^{n}$ and $p>0$, let $\Psi_{p}: \mathcal{S}_{o}^{n} \rightarrow \mathcal{S}_{o}^{n}$ be a $L_{p}$ radial Blaschke-Minkowski homomorphism. If $K \notin \mathcal{S}_{o s}^{n}$, then there exists $L \in \mathcal{S}_{o}^{n}$ such that

$$
\Psi_{p} K \subset \Psi_{p} L
$$

However,

$$
\widetilde{\Omega}_{p}(K)>\widetilde{\Omega}_{p}(L)
$$

Next, associated with $L_{q}$ radial Minkowski sum and $L_{q}$ harmonic Blaschke sum of star bodies, we establish the following $L_{p}$ dual affine surface area forms of Brunn-Minkowski inequalities for the $L_{p}$ radial Blaschke-Minkowski homomorphisms, respectively.

Theorem 3. If $K, L \in \mathcal{S}_{o}^{n}, 0<p<n / 2$ and $0<q \leq n-p$, then

$$
\begin{equation*}
\widetilde{\Omega}_{p}\left(\Psi_{p}\left(K \widetilde{+}_{q} L\right)\right)^{\frac{p q(n+p)}{n(n-p)^{2}}} \leq \widetilde{\Omega}_{p}\left(\Psi_{p} K\right)^{\frac{p q(n+p)}{n(n-p)^{2}}}+\widetilde{\Omega}_{p}\left(\Psi_{p} L\right)^{\frac{p q(n+p)}{n(n-p)^{2}}}, \tag{4}
\end{equation*}
$$

with equality if and only if $K$ and $L$ are dilated.
Let $q=n-p$ in Theorem 3 and notice that $K \widetilde{+}_{n-p} L=K \widehat{+}_{p} L$ (see Equation (7)), we obtain a Brunn-Minkowski inequality for the $L_{p}$ radial Blaschke sum $K \widehat{+}{ }_{p} L$.

Corollary 1. If $K, L \in \mathcal{S}_{o}^{n}, 0<p<n / 2$, then

$$
\widetilde{\Omega}_{p}\left(\Psi_{p}\left(K \widehat{+}_{p} L\right)\right)^{\frac{p(n+p)}{n(n-p)}} \leq \widetilde{\Omega}_{p}\left(\Psi_{p} K\right)^{\frac{p(n+p)}{n(n-p)}}+\widetilde{\Omega}_{p}\left(\Psi_{p} L\right)^{\frac{p(n+p)}{n(n-p)}}
$$

with equality if and only if $K$ and $L$ are dilated.
Theorem 4. If $K, L \in \mathcal{S}_{o}^{n}, 0<p<n / 2$ and $0<p<-q<n$, then

$$
\begin{equation*}
\frac{\widetilde{\Omega}_{p}\left(\Psi_{p}\left(K \mp_{q} L\right)\right)^{\frac{p(n+p)(n+q)}{n(n-p)^{2}}}}{V\left(K \mp_{q} L\right)} \leq \frac{\widetilde{\Omega}_{p}\left(\Psi_{p} K\right)^{\frac{p(n+p)(n+q)}{n(n-p)^{2}}}}{V(K)}+\frac{\widetilde{\Omega}_{p}\left(\Psi_{p} L\right)^{\frac{p(n+p)(n+q)}{n(n-p)^{2}}}}{V(L)} \tag{5}
\end{equation*}
$$

with equality if and only if $K$ and $L$ are dilated. Here $K \mp q L$ denotes the $L_{q}$ harmonic Blaschke sum of $K$ and $L$.
The proofs of Theorem 1 and Theorem 2 are completed in Section 3. In Section 4, we will give the proofs of Theorem 3 and Theorem 4.

## 2. Background Materials

2.1. General $L_{p}$ Radial Blaschke Bodies

For $K, L \in \mathcal{S}_{o}^{n}$, real $p \neq 0$ and $\lambda, \mu \geq 0$ (not both 0 ), the $L_{p}$ radial Minkowski combination, $\lambda \cdot K \widetilde{+}_{p} \mu \cdot L \in \mathcal{S}_{o}^{n}$, of $K$ and $L$ is defined by (see $[43,44]$ )

$$
\rho\left(\lambda \cdot K \widetilde{+}_{p} \mu \cdot L, \cdot\right)^{p}=\lambda \rho(K, \cdot)^{p}+\mu \rho(L, \cdot)^{p} .
$$

Here " $\widetilde{f}_{p}$ " denotes the $L_{p}$ radial Minkowski sum and $\lambda \cdot K=\lambda^{1 / p} K$. The case $p=1$ yields the radial Minkowski combination $\lambda \cdot K \widetilde{+} \mu \cdot L$.

In 2015, Wang and Wang [35] defined the $L_{p}$ radial Blaschke combinations of star bodies as follows: For $K, L \in \mathcal{S}_{0}^{n}, n>p>0$ and $\lambda, \mu \geq 0$ (not both 0 ), the $L_{p}$ radial Blaschke combination, $\lambda \circ K \widehat{+}_{p} \mu \circ L \in \mathcal{S}_{o}^{n}$, of $K$ and $L$ is defined by

$$
\begin{equation*}
\rho\left(\lambda \circ K \widehat{+}_{p} \mu \circ L \cdot \cdot\right)^{n-p}=\lambda \rho(K, \cdot)^{n-p}+\mu \rho(L, \cdot)^{n-p} . \tag{6}
\end{equation*}
$$

Here " $\hat{+}_{p}$ " denotes the $L_{p}$ radial Blaschke sum and $\lambda \circ K=\lambda^{1 /(n-p)} K$. If $p=1$, then $\lambda \circ K \hat{+}_{p} \mu \circ L$ is the radial Blaschke combination $\lambda \circ K \widehat{+} \mu \circ L$ (see [1]).

From the definitions of above two combinations, we easily see

$$
\begin{equation*}
\lambda \cdot K \widetilde{+}_{n-p} \mu \cdot L=\lambda \circ K \widehat{+}_{p} \mu \circ L \tag{7}
\end{equation*}
$$

In Equation (6), let

$$
\begin{equation*}
\lambda=f_{1}(\tau)=\frac{(1+\tau)^{2}}{2\left(1+\tau^{2}\right)^{\prime}}, \quad \mu=f_{2}(\tau)=\frac{(1-\tau)^{2}}{2\left(1+\tau^{2}\right)} \tag{8}
\end{equation*}
$$

with $\tau \in[-1,1]$ and $L=-K$, and write

$$
\begin{equation*}
\widehat{\nabla}_{p}^{\tau} K=f_{1}(\tau) \circ K \widehat{+}_{p} f_{2}(\tau) \circ(-K) \tag{9}
\end{equation*}
$$

We call $\widehat{\nabla}_{p}^{\tau} K$ the general $L_{p}$ radial Blaschke body of $K$. From Equations (8) and (9), we easily see that $\widehat{\nabla}_{p}^{1} K=K, \widehat{\nabla}_{p}^{-1} K=-K$ and

$$
\begin{equation*}
\widehat{\nabla}_{p}^{0} K=\frac{1}{2} \circ K \widehat{+}_{p} \frac{1}{2} \circ(-K) \tag{10}
\end{equation*}
$$

For the general $L_{p}$ radial Blaschke bodies, by Equation (8) we know that $f_{1}(\tau)+f_{2}(\tau)=1$. This and Equation (9) give that if $K \in \mathcal{S}_{o s}^{n}$ then $\widehat{\nabla}_{p}^{\tau} K \in \mathcal{S}_{o s}^{n}$. If $K \notin \mathcal{S}_{o s}^{n}$, we have the following conclusion.

Theorem 5. For $K, L \in \mathcal{S}_{o}^{n}$ and $p>0$. If $K \notin \mathcal{S}_{o s}^{n}$, then for $\tau \in[-1,1]$,

$$
\begin{equation*}
\hat{\nabla}_{p}^{\tau} K \in \mathcal{S}_{o s}^{n} \Leftrightarrow \tau=0 \tag{11}
\end{equation*}
$$

Proof. If $\tau=0$, by Equation (10) we immediately get $\widehat{\nabla}_{p}^{\tau} K \in \mathcal{S}_{o s}^{n}$.
Conversely, since $\rho_{M}(-u)=\rho_{-M}(u)$ for each $M \in \mathcal{S}_{o}^{n}$ and any $u \in S^{n-1}$, thus if $\widehat{\nabla}_{p}^{\tau} K \in \mathcal{S}_{o s}^{n}$, then for all $u \in S^{n-1}$,

$$
\rho_{\widehat{\nabla}_{p}^{\tau} K}^{n-p}(u)=\rho_{-\hat{\nabla}_{p}^{\tau} K}^{n-p}(u)=\rho_{\widehat{\nabla}_{p}^{\tau} K}^{n-p}(-u) .
$$

By Equation (9) we have

$$
\rho_{f_{1}(\tau) \circ K \hat{\mathrm{f}}_{p} f_{2}(\tau) \circ(-K)}^{n-p}(u)=\rho_{f_{1}(\tau) \circ K \hat{\mathrm{f}}_{p} f_{2}(\tau) \circ(-K)}^{n-p}(-u) .
$$

This together with Equation (6) yields

$$
f_{1}(\tau) \rho_{K}^{n-p}(u)+f_{2}(\tau) \rho_{-K}^{n-p}(u)=f_{1}(\tau) \rho_{K}^{n-p}(-u)+f_{2}(\tau) \rho_{-K}^{n-p}(-u)
$$

i.e.,

$$
f_{1}(\tau) \rho_{K}^{n-p}(u)+f_{2}(\tau) \rho_{-K}^{n-p}(u)=f_{1}(\tau) \rho_{-K}^{n-p}(u)+f_{2}(\tau) \rho_{K}^{n-p}(u),
$$

hence

$$
\left[f_{1}(\tau)-f_{2}(\tau)\right]\left[\rho_{K}^{n-p}(u)-\rho_{-K}^{n-p}(u)\right]=0
$$

Since $K \notin \mathcal{S}_{o s}^{n} \operatorname{implies} \rho_{K}^{n-p}(u)-\rho_{-K}^{n-p}(u) \neq 0$ for all $u \in S^{n-1}$, thus we obtain

$$
f_{1}(\tau)-f_{2}(\tau)=0
$$

This and Equation (8) give $\tau=0$.

## 2.2. $L_{p}$ Dual Mixed Volumes

Based on the $L_{p}$ radial Minkowski combinations of star bodies, a class of $L_{p}$ dual mixed volumes were introduced as follows (see [45,46]): For $M, N \in \mathcal{S}_{o}^{n}, p \neq 0$ and $\varepsilon>0$, the $L_{p}$ dual mixed volume, $\widetilde{V}_{p}(M, N)$, of $M$ and $N$ is defined by

$$
\frac{n}{p} \widetilde{V}_{p}(M, N)=\lim _{\varepsilon \rightarrow 0^{+}} \frac{V\left(M \widetilde{+}_{p} \varepsilon \cdot N\right)-V(M)}{\varepsilon}
$$

From the above definition, $L_{p}$ dual mixed volume has the following integral representation (see [45,46]):

$$
\begin{equation*}
\widetilde{V}_{p}(M, N)=\frac{1}{n} \int_{S^{n-1}} \rho_{M}^{n-p}(u) \rho_{N}^{p}(u) d u . \tag{12}
\end{equation*}
$$

## 2.3. $L_{q}$ Harmonic Blaschke Sums

The harmonic Blaschke sums of star bodies were introduced by Lutwak (see [47]). For $M, N \in \mathcal{S}_{o}^{n}$, the harmonic Blaschke sum, $M \mp N \in \mathcal{S}_{o}^{n}$, of $M$ and $N$ is defined by

$$
\frac{\rho(M \mp N, \cdot)^{n+1}}{V(M \mp N)}=\frac{\rho(M, \cdot)^{n+1}}{V(M)}+\frac{\rho(N, \cdot)^{n+1}}{V(N)}
$$

Based on above definition, Feng and Wang ([48]) defined the $L_{q}$ harmonic Blaschke sums as follows: For $M, N \in \mathcal{S}_{o}^{n}$, real $q>-n$, the $L_{q}$ harmonic Blaschke sum, $M \not \mp_{q} N \in \mathcal{S}_{o}^{n}$, of $M$ and $N$ is given by

$$
\begin{equation*}
\frac{\rho\left(M \mp_{q} N, \cdot\right)^{n+q}}{V(M \mp q)}=\frac{\rho(M, \cdot)^{n+q}}{V(M)}+\frac{\rho(N, \cdot)^{n+q}}{V(N)} \tag{13}
\end{equation*}
$$

## 3. A Type of Busemann-Petty Problem

Theorems 1 and 2 show a type of Busemann-Petty Problem of the $L_{p}$ radial Blaschke-Minkowski homomorphisms for the $L_{p}$ dual affine surface areas. In this section, we will prove them. In order to prove Theorem 1, the following lemma is essential.

Lemma 1 ([23]). If $M, N \in \mathcal{S}_{o}^{n}$ and $p>0$, then

$$
\begin{equation*}
\widetilde{V}_{p}\left(M, \Psi_{p} N\right)=\widetilde{V}_{p}\left(N, \Psi_{p} M\right) \tag{14}
\end{equation*}
$$

Proof of Theorem 1. Since $\Psi_{p} K \subseteq \Psi_{p} L$, thus using Equation (12) we know that for $p>0$ and any $M \in \mathcal{S}_{o}^{n}$,

$$
\widetilde{V}_{p}\left(M, \Psi_{p} K\right) \leq \widetilde{V}_{p}\left(M, \Psi_{p} L\right)
$$

This together with Equation (14) yields

$$
\begin{equation*}
\widetilde{V}_{p}\left(K, \Psi_{p} M\right) \leq \widetilde{V}_{p}\left(L, \Psi_{p} M\right) \tag{15}
\end{equation*}
$$

Hence, by Equation (3) we have

$$
\begin{aligned}
& n^{-\frac{p}{n}} \widetilde{\Omega}_{p}^{\circ}(K)^{\frac{n+p}{n}}=\sup \left\{n \widetilde{V}_{p}\left(K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \Psi_{p}^{*} \mathcal{S}_{o}^{n}\right\} \\
&=\sup \left\{n \widetilde{V}_{p}\left(K, \Psi_{p} M\right) V\left(\Psi_{p}^{*} M\right)^{\frac{p}{n}}: \Psi_{p}^{*} M \in \Psi_{p}^{*} \mathcal{S}_{o}^{n}\right\} \\
& \leq \sup \left\{n \widetilde{V}_{p}\left(L, \Psi_{p} M\right) V\left(\Psi_{p}^{*} M\right)^{\frac{p}{n}}: \Psi_{p}^{*} M \in \Psi_{p}^{*} \mathcal{S}_{o}^{n}\right\} \\
& \quad=n^{-\frac{p}{n}} \widetilde{\Omega}_{p}^{\circ}(L)^{\frac{n+p}{n}},
\end{aligned}
$$

this gives

$$
\widetilde{\Omega}_{p}^{\circ}(K) \leq \widetilde{\Omega}_{p}^{\circ}(L)
$$

Obviously, we see that $\widetilde{\Omega}_{p}^{\circ}(K)=\widetilde{\Omega}_{p}^{\circ}(L)$ when $\Psi_{p} K=\Psi_{p} L$.
The proof of Theorem 2 needs the following lemmas.
Lemma 2. If $K, L \in \mathcal{S}_{o}^{n}, \lambda, \mu \geq 0($ not both zero) and $0<p<n$, then

$$
\begin{equation*}
\widetilde{\Omega}_{p}\left(\lambda \circ K \widehat{+}_{p} \mu \circ L\right)^{\frac{n+p}{n}} \leq \lambda \widetilde{\Omega}_{p}(K)^{\frac{n+p}{n}}+\mu \widetilde{\Omega}_{p}(L)^{\frac{n+p}{n}} \tag{16}
\end{equation*}
$$

with equality in Equation (16) for $\lambda, \mu>0$ if and only if $K$ and L are dilated. For $\lambda=0$ or $\mu=0$, Equation (16) becomes an equality.

Proof. From Equations (2) and (6), we have

$$
\begin{aligned}
& n^{-\frac{p}{n}} \widetilde{\Omega}_{p}\left(\lambda \circ K \widehat{+}_{p} \mu \circ L\right)^{\frac{n+p}{n}} \\
&= \sup \left\{n \widetilde{V}_{p}\left(\lambda \circ K \widehat{+}_{p} \mu \circ L, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
&= \sup \left\{\left[\int_{S^{n-1}} \rho_{\lambda \circ K \widehat{+} p \mu \circ L}^{n-p}(u) \rho_{Q^{*}}^{p}(u) d u\right] V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
&=\sup \left\{\left[\int_{S^{n-1}}\left[\lambda \rho_{K}^{n-p}(u)+\mu \rho_{L}^{n-p}(u)\right] \rho\left(Q^{*}, u\right)^{p} d u\right] V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
&= \sup \left\{\lambda\left[\int_{S^{n-1}} \rho_{K}^{n-p}(u) \rho_{Q^{*}}^{p}(u) d u\right] V(Q)^{\frac{p}{n}}\right. \\
&= \sup \left\{n \lambda \widetilde{V}_{p}\left(K, Q^{*}\right) V(Q)^{\frac{p}{n}}+n \mu \widetilde{V}_{p}\left(L, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
& \leq \sup \left\{n \lambda \widetilde{V}_{p}\left(K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
&+\sup \left\{n \mu \widetilde{V}_{p}\left(L, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
&= \lambda n^{-\frac{p}{n}} \widetilde{\Omega}_{p}(K)^{\frac{n+p}{n}}+\mu n^{-\frac{p}{n}} \widetilde{\Omega}_{p}(L)^{\frac{n+p}{n}} .
\end{aligned}
$$

This gives inequality Equation (16).
We easily know that equality holds in Equation (16) for $\lambda, \mu>0$ if and only if $K$ and $L$ are dilated. For $\lambda=0$ or $\mu=0$, Equation (16) becomes an equality.

Lemma 3. If $K \in \mathcal{S}_{o}^{n}, 0<p<n$ and $\tau \in[-1,1]$, then

$$
\begin{equation*}
\widetilde{\Omega}_{p}\left(\widehat{\nabla}_{p}^{\tau} K\right) \leq \widetilde{\Omega}_{p}(K) \tag{17}
\end{equation*}
$$

Equality holds in Equation (17) for $\tau \in(-1,1)$ if and only if $K$ is origin-symmetric. For $\tau= \pm 1$, Equation (17) becomes an equality.

Proof. Taking $\lambda=f_{1}(\tau), \mu=f_{2}(\tau)$ and $L=-K$ in Equation (16), these and Equation (9) yield

$$
\begin{equation*}
\widetilde{\Omega}_{p}\left(\bar{\nabla}_{p}^{\tau} K\right) \leq f_{1}(\tau) \widetilde{\Omega}_{p}(K)+f_{2}(\tau) \widetilde{\Omega}_{p}(-K) \tag{18}
\end{equation*}
$$

However, by Equations (2) and (12) we have

$$
\begin{aligned}
n^{-\frac{p}{n}} \widetilde{\Omega}_{p}(-K)^{\frac{n+p}{n}}= & \sup \left\{n \widetilde{V}_{p}\left(-K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
= & \sup \left\{\left[\int_{S^{n-1}} \rho_{-K}^{n-p}(u) \rho_{Q^{*}}^{p}(u) d u\right] V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
= & \sup \left\{\left[\int_{S^{n-1}} \rho_{K}^{n-p}(-u) \rho_{-Q^{*}}^{p}(-u) d u\right] V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
= & \sup \left\{n \widetilde{V}_{p}\left(K,-Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
= & \sup \left\{n \widetilde{V}_{p}\left(K,(-Q)^{*}\right) V(-Q)^{\frac{p}{n}}:-Q \in \mathcal{S}_{o}^{n}\right\} \\
& =n^{-\frac{p}{n}} \widetilde{\Omega}_{p}(K)^{\frac{n+p}{n}},
\end{aligned}
$$

i.e.,

$$
\widetilde{\Omega}_{p}(-K)=\widetilde{\Omega}_{p}(K)
$$

This together with Equation (18), and notice that $f_{1}(\tau)+f_{2}(\tau)=1$, we obtain Equation (17).
According to the equality conditions of Equation (16), we see that equality holds in Equation (17) for $\tau \in(-1,1)$ if and only if $K$ and $-K$ are dilated, i.e., $K$ is origin-symmetric. For $\tau= \pm 1$, Equation (17) becomes an equality.

Lemma 4 ([23]). For $p>0$, a map $\Psi_{p}: \mathcal{S}_{o}^{n} \rightarrow \mathcal{S}_{o}^{n}$ is a $L_{p}$ radial Blaschke-Minkowski homomorphism if and only if there is a non-negative measure $\mu \in \mathcal{M}\left(S^{n-1}, \widehat{e}\right)$ such that for $K \in \mathcal{S}_{0}^{n}, \rho\left(\Psi_{p} K, \cdot\right)^{p}$ is the convolution of $\rho(K, \cdot)^{n-p}$ and $\mu$, namely

$$
\begin{equation*}
\rho\left(\Psi_{p} K, \cdot\right)^{p}=\rho(K, \cdot)^{n-p} * \mu \tag{19}
\end{equation*}
$$

Here $\widehat{e}$ denotes the pole point of $S^{n-1}$ and $\mathcal{M}\left(S^{n-1}, \widehat{e}\right)$ denotes the signed finite Borel measure space on $S^{n-1}$ (see [21]).

Obviously, Equation (19) gives that for $c>0$,

$$
\begin{equation*}
\Psi_{p}(c K)=c^{\frac{n-p}{p}} \Psi_{p} K . \tag{20}
\end{equation*}
$$

Lemma 5. For $0<p<n$, let $\Psi_{p}$ be an even $L_{p}$ radial Blaschke-Minkowski homomorphism. If $K \in \mathcal{S}_{o}^{n}$ and $\tau \in[-1,1]$, then

$$
\begin{equation*}
\Psi_{p}\left(\hat{\nabla}_{p}^{\tau} K\right)=\Psi_{p} K \tag{21}
\end{equation*}
$$

Proof. Since $\Psi_{p}$ is an even $L_{p}$ radial Blaschke-Minkowski homomorphism, thus for any $K \in \mathcal{S}_{o}^{n}$, $\Psi_{p}(-K)=\Psi_{p} K$. From this, according to Equation (19), Equation (6) and Equation (9), we have for $0<p<n$ and $\mu \in \mathcal{M}\left(S^{n-1}, \widehat{e}\right)$,

$$
\begin{aligned}
\rho\left(\Psi_{p}\left(\widehat{\nabla}_{p}^{\tau} K\right)\right. & , \cdot)=\rho\left(\widehat{\nabla}_{p}^{\tau} K, \cdot\right)^{n-p} * \mu \\
& =\left[f_{1}(\tau) \rho(K, \cdot)^{n-p}+f_{2}(\tau) \rho(-K, \cdot)^{n-p}\right] * \mu \\
& =f_{1}(\tau) \rho(K, \cdot)^{n-p} * \mu+f_{2}(\tau) \rho(-K, \cdot)^{n-p} * \mu \\
& =f_{1}(\tau) \rho\left(\Psi_{p} K, \cdot\right)+f_{2}(\tau) \rho\left(\Psi_{p}(-K), \cdot\right) \\
& =f_{1}(\tau) \rho\left(\Psi_{p} K, \cdot\right)+f_{2}(\tau) \rho\left(\Psi_{p} K, \cdot\right)=\rho\left(\Psi_{p} K, \cdot\right) .
\end{aligned}
$$

This gives Equation (21).
Proof of Theorem 2. Since $K \notin \mathcal{S}_{o s}^{n}$, thus for $0<p<n$, by Equation (17) we know that for $\tau \in(-1,1)$,

$$
\widetilde{\Omega}_{p}\left(\widehat{\nabla}_{p}^{\tau} K\right)<\widetilde{\Omega}_{p}(K)
$$

Choose $\varepsilon>0$ such that

$$
\widetilde{\Omega}_{p}\left((1+\varepsilon) \widehat{\nabla}_{p}^{\tau} K\right)<\widetilde{\Omega}_{p}(K)
$$

From this, let $L=(1+\varepsilon) \hat{\nabla}_{p}^{\tau} K$, then $L \in \mathcal{S}_{o}^{n}$ (Theorem 5 gives that for $\tau=0, L \in \mathcal{S}_{o s}^{n}$; for $\tau \in(-1,1)$ and $\tau \neq 0, L \in \mathcal{S}_{o}^{n} \backslash \mathcal{S}_{o s}^{n}$ ) and satisfies $\widetilde{\Omega}_{p}(L)<\widetilde{\Omega}_{p}(K)$.

However, by Equations (20) and (21) we obtain for $0<p<n$,

$$
\Psi_{p} L=\Psi_{p}\left((1+\varepsilon) \widehat{\nabla}_{p}^{\tau} K\right)=(1+\varepsilon)^{\frac{n-p}{p}} \Psi_{p}\left(\widehat{\nabla}_{p}^{\tau} K\right)=(1+\varepsilon)^{\frac{n-p}{p}} \Psi_{p} K \supset \Psi_{p} K .
$$

## 4. Brunn-Minkowski Inequalities for the $L_{p}$ Radial Blaschke-Minkowski Homomorphisms

Associated with $L_{q}$ radial Minkowski sum and $L_{q}$ harmonic Blaschke sum of star bodies, Theorems 3 and 4 respectively give the $L_{p}$ dual affine surface area forms of Brunn-Minkowski inequalities for the $L_{p}$ radial Blaschke-Minkowski homomorphisms. In this section, we will complete their proofs. For the proof of Theorem 3, the following lemmas are essential.

Lemma 6. If $K, L \in \mathcal{S}_{o}^{n}, p>0$ and $0<q \leq n-p$, then for any $u \in S^{n-1}$,

$$
\begin{equation*}
\rho_{\Psi_{p}\left(K \tilde{+}_{q} L\right)}^{\frac{p q}{n-p}}(u) \leq \rho_{\Psi_{p} K}^{\frac{p q}{n-p}}(u)+\rho_{\Psi_{p} L}^{\frac{p q}{n-p}}(u) \tag{22}
\end{equation*}
$$

with equality for $0<q<n-p$ if and only if $K$ and $L$ are dilated. For $q=n-p$, Equation (22) becomes an equality.

Proof. Because of $0<q<n-p$ implies $\frac{n-p}{q}>1$, thus by Equation (19) and the Minkowski integral inequality we have for $\mu \in \mathcal{M}\left(S^{n-1}, \widehat{e}\right)$ and any $u \in S^{n-1}$,

$$
\begin{aligned}
\rho_{\Psi_{p}\left(K \tilde{f}_{q} L\right)}^{\frac{p q}{n-p}}(u)= & \left(\rho_{\Psi_{p}\left(K \tilde{+}_{q} L\right)}^{p}(u)\right)^{\frac{q}{n-p}}=\left(\rho_{K \tilde{\Psi}_{q} L}^{n-p}(u) * \mu\right)^{\frac{q}{n-p}} \\
= & {\left[\left(\rho_{K \tilde{\Psi}_{q} L}^{q}(u)\right)^{\frac{n-p}{q}} * \mu\right]^{\frac{q}{n-p}} } \\
= & {\left[\left(\rho_{K}^{q}(u)+\rho_{L}^{q}(u)\right)^{\frac{n-p}{q}} * \mu\right]^{\frac{q}{n-p}} } \\
\leq & \left(\rho_{K}^{n-p}(u) * \mu\right)^{\frac{q}{n-p}}+\left(\rho_{L}^{n-p}(u) * \mu\right)^{\frac{q}{n-p}} \\
& =\rho_{\Psi_{p} K}^{\frac{p q}{n-p}}(u)+\rho_{\Psi_{p} L}^{\frac{p q}{n-p}}(u) .
\end{aligned}
$$

This yields inequality Equation (22).
From the equality condition of Minkowski integral inequality, we know that equality holds in Equation (22) for $0<q<n-p$ if and only if $K$ and $L$ are dilated. Clearly, if $q=n-p$, Equation (22) becomes an equality.

Lemma 7. If $K, L \in \mathcal{S}_{o}^{n}, 0<p<n / 2$ and $0<q \leq n-p$, then for any $M \in \mathcal{S}_{o}^{n}$,

$$
\begin{equation*}
\widetilde{V}_{p}\left(\Psi_{p}\left(K \widetilde{+}_{q} L\right), M\right)^{\frac{p q}{(n-p)^{2}}} \leq \widetilde{V}_{p}\left(\Psi_{p} K, M\right)^{\frac{p q}{(n-p)^{2}}}+\widetilde{V}_{p}\left(\Psi_{p} L, M\right)^{\frac{p q}{(n-p)^{2}}} \tag{23}
\end{equation*}
$$

with equality if and only if $K$ and $L$ are dilated.
Proof. Since $0<p<n / 2$ and $0<q \leq n-p$, thus $\frac{(n-p)^{2}}{p q}>1$. Using Equation (12), Equation (22) and the Minkowski integral inequality, we have for any $M \in \mathcal{S}_{o}^{n}$,

$$
\begin{gathered}
\widetilde{V}_{p}\left(\Psi_{p}\left(K \widetilde{+}_{q} L\right), M\right)^{\frac{p q}{(n-p)^{2}}}=\left[\frac{1}{n} \int_{S^{n-1}} \rho_{\Psi_{p\left(K \tilde{+}_{q} L\right)}^{n-p}}^{n-p}(u) \rho_{M}^{p}(u) d u\right]^{\frac{p q}{(n-p)^{2}}} \\
=\left[\frac{1}{n} \int_{S^{n-1}}\left(\rho_{\Psi_{p}\left(K \tilde{+}_{q} L\right)}^{\frac{p q}{n-p}}(u)\right)^{\frac{(n-p)^{2}}{p q}} \rho_{M}^{p}(u) d u\right]^{\frac{p q}{(n-p)^{2}}}
\end{gathered}
$$

$$
\begin{aligned}
\leq & {\left[\frac{1}{n} \int_{S^{n-1}}\left(\rho_{\Psi_{p} K}^{\frac{p q}{n-p}}(u)+\rho_{\Psi_{p} L}^{\frac{p q}{n-p}}(u)\right)^{\frac{(n-p)^{2}}{p q}} \rho_{M}^{p}(u) d u\right]^{\frac{p q}{(n-p)^{2}}} } \\
\leq & {\left[\frac{1}{n} \int_{S^{n-1}} \rho_{\Psi_{p} K}^{n-p}(u) \rho_{M}^{p}(u) d u\right]^{\frac{p q}{(n-p)^{2}}} } \\
& +\left[\frac{1}{n} \int_{S^{n-1}} \rho_{\Psi_{p} L}^{n-p}(u) \rho_{M}^{p}(u) d u\right]^{\frac{p q}{(n-p)^{2}}} \\
= & \widetilde{V}_{p}\left(\Psi_{p} K, M\right)^{\frac{p q}{(n-p)^{2}}}+\widetilde{V}_{p}\left(\Psi_{p} L, M\right)^{\frac{p q}{(n-p)^{2}}}
\end{aligned}
$$

This yields inequality Equation (23).
For $0<q<n-p$, according to the equality conditions of Equation (22) and Minkowski integral inequality, we see that equality holds in Equation (23) if and only if $K$ and $L$ are dilated, and $\Psi_{p} K$ and $\Psi_{p} L$ are dilated. However, by Equation (19) we know that $K$ and $L$ are dilated is equivalent to $\Psi_{p} K$ and $\Psi_{p} L$ are dilated. Therefore, equality holds in Equation (23) if and only if $K$ and $L$ are dilated.

For $q=n-p$, from equality condition of Minkowski integral inequality, we are aware of equality holds in Equation (23) if and only if $\Psi_{p} K$ and $\Psi_{p} L$ are dilated, this is equivalent to $K$ and $L$ are dilated.

Proof of Theorem 3. Because of $\frac{p q}{(n-p)^{2}}>0$, thus by Equations (1) and (23), we obtain that

$$
\begin{aligned}
& {\left[n^{-\frac{p}{n}} \widetilde{\Omega}_{p}\left(\Psi_{p}\left(K \widetilde{f}_{q} L\right)\right)^{\frac{n+p}{n}}\right]^{\frac{p p}{(n-p)^{2}}} } \\
&= {\left[\sup \left\{n \widetilde{V}_{p}\left(\Psi_{p}\left(K \widetilde{\Psi}_{q} L\right), Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\}\right]^{\frac{p q}{(n-p)^{2}}} } \\
&= \sup \left\{n^{\frac{p q}{(n-p)^{2}}} \widetilde{V}_{p}\left(\Psi_{p}\left(K \widetilde{f}_{q} L\right), Q^{*}\right)^{\frac{p q}{(n-p)^{2}}} V(Q)^{\frac{p^{2} q}{n(n-p)^{2}}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
& \leq \sup \left\{n^{\frac{p q}{(n-p)^{2}}}\left[\widetilde{V}_{p}\left(\Psi_{p} K, Q^{*}\right)^{\frac{p q}{(n-p)^{2}}}+\widetilde{V}_{p}\left(\Psi_{p} L, Q^{*}\right)^{\frac{p q}{(n-p)^{2}}}\right] V(Q)^{\frac{p^{2} q}{n(n-p)^{2}}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
& \leq \sup \left\{n^{\left.\frac{p q}{(n-p)^{2}} \widetilde{V}_{p}\left(\Psi_{p} K, Q^{*}\right)^{\frac{p q}{(n-p)^{2}}} V(Q)^{\frac{p^{2} q}{n(n-p)^{2}}}: Q \in \mathcal{S}_{o}^{n}\right\}}\right. \\
&+\sup \left\{n^{\left.\frac{p q}{(n-p)^{2}} \widetilde{V}_{p}\left(\Psi_{p} L, Q^{*}\right)^{\frac{p q}{(n-p)^{2}}} V(Q)^{\frac{p^{2} q}{n(n-p)^{2}}}: Q \in \mathcal{S}_{o}^{n}\right\}}\right. \\
&= {\left[\sup \left\{n \widetilde{V}_{p}\left(\Psi_{p} K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\}\right]^{\frac{p q}{(n-p)^{2}}} } \\
&+\left[\sup \left\{n \widetilde{V}_{p}\left(\Psi_{p} L, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\}\right]^{\frac{p q}{(n-p)^{2}}} \\
&= {\left[n^{-\frac{p}{n}} \widetilde{\Omega}_{p}\left(\Psi_{p} K\right)^{\frac{n+p}{n}}\right]^{\frac{p q}{(p-p)^{2}}}+\left[n^{-\frac{p}{n}} \widetilde{\Omega}_{p}\left(\Psi_{p} L\right)^{\frac{n+p}{n}}\right]^{\frac{p p}{(n-p)^{2}}} . }
\end{aligned}
$$

From this, we get inequality Equation (4), and equality holds in Equation (4) if and only if $K$ and $L$ are dilated.

The proof of Theorem 1 requires the following lemmas.

Lemma 8. If $K, L \in \mathcal{S}_{o}^{n}, 0<p<-q<n$, then for any $u \in S^{n-1}$,

$$
\begin{equation*}
\frac{\rho_{\Psi_{p}\left(K \not \mp_{q} L\right)}^{\frac{p(n+q)}{n-p}}(u)}{V(K \mp q L)} \leq \frac{\rho_{\Psi_{p} K}^{\frac{p(n+q)}{n-p}}(u)}{V(K)}+\frac{\rho_{\Psi_{p} L}^{\frac{p(n+q)}{n-p}}(u)}{V(L)} \tag{24}
\end{equation*}
$$

with equality if and only if $K$ and $L$ are dilated.
Proof. Since $0<p<-q<n$, thus $\frac{n-p}{n+q}>1$. Hence, by Equation (13), Equation (19) and the Minkowski integral inequality we have for $\mu \in \mathcal{M}\left(S^{n-1}, \widehat{e}\right)$ and any $u \in S^{n-1}$,

$$
\begin{aligned}
\frac{\rho_{\Psi_{p}\left(K \not \mp_{q} L\right)}^{\frac{p(n+q)}{n-p}}(u)}{V(K \mp q L)} & =\frac{\left[\rho_{\Psi_{p}\left(K \mp_{q} L\right)}^{p}(u)\right]^{\frac{n+q}{n-p}}}{V\left(K \mp_{q} L\right)}=\frac{\left[\rho_{K \neq q}^{n-p}(u) * \mu\right]^{\frac{n+q}{n-p}}}{V(K \mp q L)} \\
& =\left[\left(\frac{\rho_{K \mp q L}^{n+q}(u)}{V\left(K \mp_{q} L\right)}\right)^{\frac{n-p}{n+q}} * \mu\right]^{\frac{n+q}{n-p}} \\
= & {\left[\left(\frac{\rho_{K}^{n+q}(u)}{V(K)}+\frac{\rho_{L}^{n+q}(u)}{V(L)}\right)^{\frac{n-p}{n+q}} * \mu\right]^{\frac{n+q}{n-p}} } \\
\leq & {\left[\frac{\rho_{K}^{n-p}(u) * \mu}{V(K)^{\frac{n-p}{n+q}}}\right]^{\frac{n+q}{n-p}}+\left[\frac{\rho_{L}^{n-p}(u) * \mu}{\left.V(L)^{\frac{n-p}{n+q}}\right]^{\frac{n+q}{n-p}}}\right.} \\
& =\frac{\rho_{\Psi_{p} K}^{\frac{p(n+q)}{n-p}}(u)}{V(K)}+\frac{\rho_{\Psi_{p} L}^{\frac{p(n+q)}{n-p}}(u)}{V(L)} .
\end{aligned}
$$

This deduces Equation (24).
According to the equality condition of Minkowski integral inequality, we know that equality holds in Equation (24) if and only if $K$ and $L$ are dilated.

Lemma 9. If $K, L \in \mathcal{S}_{o}^{n}, 0<p<n / 2$ and $0<p<-q<n$, then for any $M \in \mathcal{S}_{o}^{n}$,

$$
\begin{equation*}
\frac{\widetilde{V}_{p}\left(\Psi_{p}\left(K \mp_{q} L\right), M\right)^{\frac{p(n+q)}{(n-p)^{2}}}}{V\left(K \mp_{q} L\right)} \leq \frac{\widetilde{V}_{p}\left(\Psi_{p} K, M\right)^{\frac{p(n+q)}{(n-p)^{2}}}}{V(K)}+\frac{\widetilde{V}_{p}\left(\Psi_{p} L, M\right)^{\frac{p(n+q)}{(n-p)^{2}}}}{V(L)} \tag{25}
\end{equation*}
$$

with equality if and only if $K$ and $L$ are dilated.
Proof. Since $0<p<n / 2$ and $0<p<-q<n$, thus $\frac{(n-p)^{2}}{p(n+q)}>1$. Using Equation (12), Equation (24) and the Minkowski integral inequality, we have for any $M \in \mathcal{S}_{o}^{n}$,

$$
\begin{aligned}
& \frac{\widetilde{V}_{p}\left(\Psi_{p}\left(K \mp_{q} L\right), M\right)^{\frac{p(n+q)}{(n-p)^{2}}}}{V\left(K \mp_{q} L\right)} \\
= & \frac{\left[\frac{1}{n} \int_{S^{n-1}} \rho_{\Psi_{p}\left(K \mp_{q} L\right)}^{n-p}(u) \rho_{M}^{p}(u) d u\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V\left(K \mp_{q} L\right)} \\
= & {\left[\frac{1}{n} \int_{S^{n-1}}\left(\frac{\rho_{\Psi_{p}\left(K \mp_{q} L\right)}^{\frac{p(n+q)}{n-p}}(u)}{V\left(K \mp_{q} L\right)}\right)^{\frac{(n-p)^{2}}{p(n+q)}} \rho_{M}^{p}(u) d u\right]^{\frac{p(n+q)}{(n-p)^{2}}} }
\end{aligned}
$$

$$
\begin{aligned}
& \leq\left[\frac{1}{n} \int_{S^{n-1}}\left(\frac{\rho_{\Psi_{p} K}^{\frac{p(n+q)}{n-p}}(u)}{V(K)}+\frac{\rho_{\Psi_{p} L}^{\frac{p(n+q)}{n-p}}(u)}{V(L)}\right)^{\frac{(n-p)^{2}}{p(n+q)}} \rho_{M}^{p}(u) d u\right]^{\frac{p(n+q)}{(n-p)^{2}}} \\
& \leq\left[\frac{1}{n} \int_{S^{n-1}} \frac{\rho_{\Psi_{p} K}^{n-p}(u) \rho_{M}^{p}(u)}{V(K)^{\frac{(n-p)^{2}}{p(n+q)}}} d u\right]^{\frac{p(n+q)}{(n-p)^{2}}}+\left[\frac{1}{n} \int_{S^{n-1}} \frac{\rho_{\Psi_{p} L}^{n-p}(u) \rho_{M}^{p}(u)}{V(L)^{\frac{(n-p)^{2}}{p(n+q)}}} d u\right]^{\frac{p(n+q)}{(n-p)^{2}}} \\
& \quad=\frac{\widetilde{V}_{p}\left(\Psi_{p} K, M\right)^{\frac{p(n+q)}{(n-p)^{2}}}}{V(K)}+\frac{\widetilde{V}_{p}\left(\Psi_{p} L, M\right)^{\frac{p(n+q)}{(n-p)^{2}}}}{V(L)} .
\end{aligned}
$$

From this, inequality Equation (25) is obtained.
By the equality conditions of Equation (24) and the Minkowski integral inequality, we see that equality holds in Equation (25) if and only if $K$ and $L$ are dilated.

Proof of Theorem 1. From $0<p<n / 2$ and $0<p<-q<n$, we know that $\frac{p(n+q)}{(n-p)^{2}}>0$. Thus by Equations (1) and (25), we obtain that

$$
\begin{aligned}
& \frac{\left[n^{-\frac{p}{n}} \widetilde{\Omega}_{p}\left(\Psi_{p}\left(K \mp_{q} L\right)\right)^{\frac{n+p}{n}}\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V\left(K \mp_{q} L\right)} \\
& =\frac{\left[\sup \left\{n \widetilde{V}_{p}\left(\Psi_{p}\left(K \mp_{q} L\right), Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\}\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V\left(K \mp_{q} L\right)} \\
& =\sup \left\{\frac{\left[n \widetilde{V}_{p}\left(\Psi_{p}\left(K \mp_{q} L\right), Q^{*}\right)\right]^{\frac{p(n+q)}{(n-p)^{2}}} V(Q)^{\frac{p^{2}(n+q)}{n(n-p)^{2}}}}{V\left(K \mp_{q} L\right)}: Q \in \mathcal{S}_{o}^{n}\right\} \\
& \leq \sup \left\{\left[\frac{\left[n \widetilde{V}_{p}\left(\Psi_{p} K, Q^{*}\right)\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V(K)}+\frac{\left[n \widetilde{V}_{p}\left(\Psi_{p} L, Q^{*}\right)\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V(L)}\right] V(Q)^{\frac{p^{2}(n+q)}{n(n-p)^{2}}}: Q \in \mathcal{S}_{o}^{n}\right\} \\
& \leq \sup \left\{\frac{\left[n \widetilde{V}_{p}\left(\Psi_{p} K, Q^{*}\right)\right]^{\frac{p(n+q)}{(n-p)^{2}}} V(Q)^{\frac{p^{2}(n+q)}{n(n-p)^{2}}}}{V(K)}: Q \in \mathcal{S}_{o}^{n}\right\} \\
& +\sup \left\{\frac{\left[n \widetilde{V}_{p}\left(\Psi_{p} L, Q^{*}\right)\right]^{\frac{p(n+q)}{(n-p)^{2}}} V(Q)^{\frac{p^{2}(n+q)}{n(n-p)^{2}}}}{V(L)}: Q \in \mathcal{S}_{o}^{n}\right\} \\
& =\frac{\left[\sup \left\{\widetilde{V}_{p}\left(\Psi_{p} K, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\}\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V(K)} \\
& +\frac{\left[\sup \left\{\widetilde{V}_{p}\left(\Psi_{p} L, Q^{*}\right) V(Q)^{\frac{p}{n}}: Q \in \mathcal{S}_{o}^{n}\right\}\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V(L)} \\
& =\frac{\left[n^{-\frac{p}{n}} \widetilde{\Omega}_{p}\left(\Psi_{p} K\right)^{\frac{n+p}{n}}\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V(K)}+\frac{\left[n^{-\frac{p}{n}} \widetilde{\Omega}_{p}\left(\Psi_{p} L\right)^{\frac{n+p}{n}}\right]^{\frac{p(n+q)}{(n-p)^{2}}}}{V(L)} .
\end{aligned}
$$

This gives inequality Equation (5). In addition, equality holds in Equation (5) if and only if $K$ and $L$ are dilated.

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