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Generalized Steffensen's Inequality by Fink's Identity

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Abstract: By using Fink's Identity, Green functions, and Montgomery identities we prove some identities related to Steffensen's inequality. Under the assumptions of n-convexity and n-concavity, we give new generalizations of Steffensen's inequality and its reverse. Generalizations of some inequalities (and their reverse), which are related to Hardy-type inequality. New bounds of Grüss and Ostrowski-type inequalities have been proved. Moreover, we formulate generalized Steffensen's-type linear functionals and prove their monotonicity for the generalized class of $(n+1)$-convex functions at a point. At the end, we present some applications of our study to the theory of exponentially convex functions.

Keywords: Steffensen's inequality; higher order convexity; Green functions; Montgomery identity; Fink's identity

1. Introduction

Integral inequalities such as Hardy's inequality, Steffensen's inequality, and Ostrowski's inequality are topics of interest of many Mathematicians since their pronouncement. Several generalizations of these inequalities have been proved for different classes of functions, such as convex functions, n-convex functions, and other types of functions, for example see [1–4]. Moreover, integral inequalities have been proved for different integrals, such as Jensen-steffensen inequality for diamond integral and bounds of related identities have been obtained in [5]. Other than that, Hardy's inequality for fractional integral on general domains have been proved in [6].

Steffensen's inequality was proved in [7]: if $\psi, f : [c,d] \to \mathbb{R}$, with ψ be a decreasing function and function f having range in $[0,1]$, then

$$\int_c^d \psi(z) f(z)\, dt \le \int_c^{c+\theta} \psi(z)\, dz, \qquad \text{where } \theta = \int_c^d f(z)\, dz. \tag{1}$$

A massive literature dealing with several variants and improvements of Steffensen's inequality can be seen in [8,9] and references therein. A well known generalization of Steffensen's inequality has been presented in [4]. Several results of [4] have been recently generalized by using non-bounded Montgomery's identity in [10]. To proceed further, we recall a nice generalization of Steffensen's inequality proved by Pečarić, see [11].

Theorem 1. *Let $\psi : J \to \mathbb{R}$ be a increasing function (J is an interval in $\mathbb{R}$ such that $c, d, f(c), f(d) \in J$) and $f : [c,d] \to \mathbb{R}$ be increasing and differentiable function.*

(*i*) *If* $f(t) \leq t$, *then*

$$\int_{f(c)}^{f(d)} \psi(z)\,dz \leq \int_c^d \psi(z) f'(z)\,dz. \tag{2}$$

(*ii*) *If* $f(t) \geq t$, *then* (2) *holds in reverse direction.*

Remark 1. *We can consider* f *to be absolute continuous instead of differentiable function and the suppositions of Theorem 1 can also be weakened. In fact for an increasing function* ψ, *the function* $\Psi(x) = \int_c^x \psi(z)\,dz$ *is well defined and satisfies* $\Psi' = \psi$ *at all except the set of points with measure zero. One can substitute* $x = f(z)$ *in* (2) *(see [12] (Corollary 20.5)), provided that* f *is absolutely continuous increasing function, therefore*

$$\Psi(f(d)) - \Psi(f(c)) = \int\limits_{f(c)}^{f(d)} \psi(x)\,dx = \int\limits_c^d \psi(f(z)) f'(z)\,dz \leq \int\limits_c^d \psi(z) f'(z)\,dz, \tag{3}$$

where the last inequality holds when $f(z) \leq z$. *In [1], substitutions presented conclude that (3) yields (2) and generalization of a result proved by Rabier in [4], which gives (1).*

Recently, Fahad et al. introduced new generalization [1] of (1) by extending the results of [4,11]. By using Hermite interpolation, several inequalities related to the results of [1,4,11] have also been proved in [13]. We consider the important conclusions given in [1].

Corollary 1. *Suppose* $\psi : J \to \mathbb{R}$, $f : [c,d] \to \mathbb{R}$ *two differentiable functions with* f *non-decreasing as well, where* J *is an interval containing* $[c,d]$, $f(c)$ *and* $f(d)$. *If* ψ *is convex, then:*

(*i*) *If* f *satisfies condition* (i) *given in Theorem 1, then*

$$\psi(f(d)) \leq \psi(f(c)) + \int\limits_c^d \psi'(z) f'(z)\,dz. \tag{4}$$

(*ii*) (4) *holds in reverse direction, if* f *satisfies condition* (ii) *given in Theorem 1.*

Corollary 1 gives (3) and therefore leads to (1), (2) and generalization of Rabier's result in [4]. Next we narrate some further important results of [1].

Corollary 2. *Consider* $\psi : [0,d] \to \mathbb{R}$ *be differentiable convex function with* $\psi(0) = 0$ *and* $f : [0,d] \to [0,+\infty)$ *be another function.*

(*i*) *If* $\int\limits_0^t f(z)\,dz \leq t$ *for every* $t \in [0,b]$, *then*

$$\psi\left(\int\limits_0^d f(z)\,dz\right) \leq \int\limits_0^d \psi'(z) f(z)\,dz. \tag{5}$$

(*ii*) (5) *holds reversely if* $t \leq \int\limits_0^t f(z)\,dz$ *for every* $t \in [0,d]$.

Corollary 3. *Consider* ψ *and* f *as defined in Corollary 2 and let* $\lambda : [0,d] \to [0,+\infty)$ *and denote* $\Lambda(z) = \int\limits_z^d \lambda(t)\,dt$.

(*i*) *If* $\int_0^t f(z)\,dz \le t$ *for every* $t \in [0, d]$, *then*

$$\int_0^d \lambda(t)\psi\left(\int_0^t f(z)\,dz\right)dt \le \int_0^d \Lambda(z)\psi'(z)f(z)\,dz. \tag{6}$$

(*ii*) (6) *holds reversely if* $t \le \int_0^t f(z)\,dz$ *for every* $t \in [0, d]$.

Following two lemmas will be useful in our construction as well, see [14,15].

Lemma 1. *For a function* $\psi \in C^2([c, d])$, *we have:*

$$\psi(\xi) = \frac{d-\xi}{d-c}\psi(c) + \frac{\xi-c}{d-c}\psi(d) + \int_c^d G_{*,1}(\xi, u)\psi''(u)\,du, \tag{7}$$

$$\psi(\xi) = \psi(c) + (\xi - c)\psi'(d) + \int_c^d G_{*,2}(\xi, u)\psi''(u)\,du, \tag{8}$$

$$\psi(\xi) = \psi(d) + (d - \xi)\psi'(c) + \int_c^d G_{*,3}(\xi, u)\psi''(u)\,du, \tag{9}$$

$$\psi(\xi) = \psi(d) - (d - c)\psi'(d) + (\xi - c)\psi'(c) + \int_c^d G_{*,4}(\xi, u)\psi''(u)\,du, \tag{10}$$

$$\psi(\xi) = \psi(c) + (d - c)\psi'(c) - (d - \xi)\psi'(d) + \int_c^d G_{*,5}(\xi, u)\psi''(u)\,du, \tag{11}$$

where

$$G_{*,1}(\xi, u) = \begin{cases} \frac{(\xi-d)(u-c)}{d-c}, & \text{if } c \le u \le \xi, \\ \frac{(u-d)(\xi-c)}{d-c}, & \text{if } \xi < u \le d. \end{cases} \tag{12}$$

$$G_{*,2}(\xi, u) = \begin{cases} c - u, & \text{if } c \le u \le \xi, \\ c - \xi, & \text{if } \xi < u \le d. \end{cases} \tag{13}$$

$$G_{*,3}(\xi, u) = \begin{cases} \xi - d, & \text{if } c \le u \le \xi, \\ u - d, & \text{if } \xi < u \le d. \end{cases} \tag{14}$$

$$G_{*,4}(\xi, u) = \begin{cases} \xi - c, & \text{if } c \le u \le \xi, \\ u - c, & \text{if } \xi < u \le d. \end{cases} \tag{15}$$

and

$$G_{*,5}(\xi, u) = \begin{cases} d - u, & \text{if } c \le u \le \xi, \\ d - \xi, & \text{if } \xi < u \le d. \end{cases} \tag{16}$$

Lemma 2. *Let* $\psi \in C^1[c, d]$, *then*

$$\psi(\xi) = \frac{1}{d-c}\int_c^d \psi(u)\,du + \int_c^d p_1(\xi, u)\psi'(u)\,du, \tag{17}$$

$$\psi(\xi) = \psi(d) + \int_c^d p_2(\xi, u)\psi'(u)\,du \tag{18}$$

and

$$\psi(\xi) = \psi(c) + \int_c^d p_3(\xi, u)\psi'(u)\,du, \tag{19}$$

where

$$p_1(\xi, u) = \begin{cases} \frac{u-c}{d-c}, & \text{if } c \leq u \leq \xi, \\ \frac{u-d}{d-c}, & \text{if } \xi < u \leq d. \end{cases} \tag{20}$$

$$p_2(\xi, u) = \begin{cases} 0, & \text{if } c \leq u \leq \xi, \\ -1, & \text{if } \xi < u \leq d. \end{cases} \tag{21}$$

$$p_3(\xi, u) = \begin{cases} 1, & \text{if } c \leq u \leq \xi, \\ 0, & \text{if } \xi < u \leq d. \end{cases} \tag{22}$$

Clearly,

$$p_i(\xi, u) = \frac{\partial G_{*,i}(\xi, u)}{\partial \xi} \quad for \;\; all \;\; i = 1, 2, 3,$$
$$p_2(\xi, u) = \frac{\partial G_{*,5}(\xi, u)}{\partial \xi} \quad and \quad p_3(\xi, u) = \frac{\partial G_{*,4}(\xi, u)}{\partial \xi}. \tag{23}$$

Throughout the calculations in the main results, we will use $p_i(\xi, u)$ corresponding to $\frac{\partial G_{*,i}(\xi,u)}{\partial \xi}$ for $i = 1, 2, 3$, and for $\frac{\partial G_{*,4}(\xi,u)}{\partial \xi}$, $\frac{\partial G_{*,5}(\xi,u)}{\partial \xi}$ we use $p_3(\xi, s)$ and $p_2(\xi, s)$, respectively.

We also require the classical Fink's identity given in [16]:

Lemma 3. *Let $c, d \in \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$, $n \geq 1$ and $\psi^{(n-1)}$ is absolutely continuous on $[c, d]$.*

$$\psi(u) = \frac{n}{d-c} \int_c^d \psi(s) ds - \sum_{w=1}^{n-1} \left(\frac{n-w}{(d-c)w!} \right) \left(\psi^{(w-1)}(c)(u-c)^w - \psi^{(w-1)}(d)(u-d)^w \right)$$
$$+ \frac{1}{(n-1)!(d-c)} \int_c^d (u-t)^{n-1} W^{[c,d]}(t, u) \psi^{(n)}(t) dt, \tag{24}$$

where $W^{[c,d]}(t, u)$ is given by:

$$W^{[c,d]}(t, u) = \begin{cases} t - c, & \text{if } c \leq t \leq u \leq d, \\ t - d, & \text{if } c \leq u < t \leq d. \end{cases} \tag{25}$$

Divided differences are fairly ascribed to Newton, and the term "divided difference" was used by Augustus de Morgan in 1842. Divided differences are found to be very helpful when we are dealing with functions having different degrees of smoothness. The following definition of divided difference is given in [8] (p. 14).

Definition 1. *The nth-order divided difference of a function $\psi : [c, d] \to \mathbb{R}$ at mutually distinct points $z_0, ..., z_n \in [c, d]$ is defined recursively by*

$$[z_i; \psi] = \psi(z_i), \quad i = 0, \ldots, n,$$
$$[z_0, \ldots, z_n; \psi] = \frac{[z_1, \ldots, z_n; \psi] - [z_0, \ldots, z_{n-1}; \psi]}{z_n - z_0}. \tag{26}$$

It is easy to see that (26) is equivalent to

$$[z_0, \ldots, z_n; \psi] = \sum_{i=0}^{n} \frac{\psi(z_i)}{q'(z_i)}, \quad where \; q(z) = \prod_{j=0}^{n} (z - z_j).$$

The following definition of a real valued convex function is characterized by nth-order divided difference (see [8] (p. 15)).

Definition 2. *A function $\psi : [c,d] \to \mathbb{R}$ is said to be n-convex $(n \geq 0)$ if and only if for all choices of $(n+1)$ distinct points $z_0, \ldots, z_n \in [c,d]$, $[z_0, \ldots, z_n; \psi] \geq 0$ holds.*

If this inequality is reversed, then ψ is said to be n-concave. If the inequality is strict, then ψ is said to be a strictly n-convex (n-concave) function.

Remark 2. *Note that 0-convex functions are non-negative functions, 1-convex functions are increasing functions, and 2-convex functions are simply the convex functions.*

The following theorem gives an important criteria to examine the n-convexity of a function ψ (see [8] (p. 16)).

Theorem 2. *If $\psi^{(n)}$ exists, then ψ is n-convex if and only if $\psi^{(n)} \geq 0$.*

In this article, we use Fink's identity, Montgomery identities, and Green functions to prove some identities related to Steffensen's inequality. By using these identities we obtain a generalization of (4). In addition, we construct new identities which enable us to prove generalizations of inequalities (5) and (6) as one can obtain Classical Hardy-type inequalities from them, see [1]. We use Čebyšev functional to construct new bounds of Grüss and Ostrowski-type inequalities. Finally, we give several applications of our work.

2. Main Results

For our convenience, we use the following notations and assumptions:

$$\mathbb{S}_1(\psi, f, c, d) = \psi(f(c)) + \int_c^d \psi'(z) f'(z)\, dz - \psi(f(d)).$$

$$\mathbb{S}_2(\psi, f, d) = \int_0^d \psi'(z) f(z)\, dz - \psi\left(\int_0^d f(z)\, dz\right).$$

$$\mathbb{S}_3(\psi, f, w, d) = \int_0^d \Lambda(z)\psi'(z) f(z)\, dz - \int_0^d \lambda(t)\psi\left(\int_0^t f(z)\, dz\right) dt.$$

(A_1) For $n \in \mathbb{N}$, $n \geq 3$, let $\psi : [c,d] \to \mathbb{R}$ be n times differentiable function with $\psi^{(n-1)}$ absolutely continuous on $[c,d]$.

(A_2) For $n \in \mathbb{N}$, $n \geq 3$, let $\psi : [0,d] \to \mathbb{R}$ be n times differentiable function with $\psi(0) = 0$ and $\psi^{(n-1)}$ absolutely continuous on $[0,d]$.

The first part of this section is the generalization of (4). For this, we start with the following theorem:

Theorem 3. *Consider (A_1) with f be as in Corollary 1 (i) then:*

(*a*) *For* $j = 1, 2, 4, 5$, *we have:*

$$\mathbb{S}_1(\psi, f, c, d) = \frac{(n-2)(\psi'(d)-\psi'(c))}{d-c} \int_c^d \mathbb{S}_1(G_{*,j}(.,u), f, c, d) du + \sum_{w=1}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \times$$
$$\left(\psi^{(w+1)}(d) \int_c^d \mathbb{S}_1(G_{*,j}(.,u), f, c, d)(u-d)^w du - \psi^{(w+1)}(c) \int_c^d \mathbb{S}_1(G_{*,j}(.,u), f, c, d)(u-c)^w du\right) \tag{27}$$
$$+ \frac{1}{(n-3)!(d-c)} \int_c^d \psi^{(n)}(t) \left(\int_c^d \mathbb{S}_1(G_{*,j}(.,u), f, c, d)(u-t)^{n-3} W^{[c,d]}(t,u) du\right) dt.$$

(*b*) *If* $\psi'(c) = 0$, *then*

$$\mathbb{S}_1(\psi, f, c, d) = \frac{(n-2)(\psi'(d)-\psi'(c))}{d-c} \int_c^d \mathbb{S}_1(G_{*,3}(.,u), f, c, d) du + \sum_{w=1}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \times$$
$$\left(\psi^{(w+1)}(d) \int_c^d \mathbb{S}_1(G_{*,3}(.,u), f, c, d)(u-d)^w du - \psi^{(w+1)}(c) \int_c^d \mathbb{S}_1(G_{*,3}(.,u), f, c, d)(u-c)^w du\right)$$
$$+ \frac{1}{(n-3)!(d-c)} \int_c^d f^{(n)}(t) \left(\int_c^d \mathbb{S}_1(G_{*,3}(.,u), f, c, d)(u-t)^{n-3} W^{[c,d]}(t,u) du\right) dt.$$

Proof. (*a*) We first prove by fixing $j = 1$, other cases for $j = 2, 4, 5$ can be treated analogously. Utilizing (7) and (17) for ψ and ψ' respectively, we get

$$\mathbb{S}_1(\psi, f, c, d) = \psi(f(c)) - \psi(f(d)) + \int_c^d \psi'(t) f'(t)\, dt = \frac{d-f(c)}{d-c}\psi(c) + \frac{f(c)-c}{d-c}\psi(d) +$$
$$\int_c^d G_{*,1}(f(c), u)\psi''(u)\, du - \frac{d-f(d)}{d-c}\psi(c) - \frac{f(d)-c}{d-c}\psi(d) - \int_c^d G_{*,1}(f(d), u)\psi''(u)\, du$$
$$+ \int_c^d \left[\frac{\psi(d)-\psi(c)}{d-c} + \int_c^d p_1(t,u)\psi''(u)\, du\right] f'(t)\, dt.$$

Simplifying and employing Fubini's theorem, we get

$$\begin{aligned} \mathbb{S}_1(\psi, f, c, d) &= \frac{f(d)-f(c)}{d-c}\psi(c) - \frac{f(d)-f(c)}{d-c}\psi(d) \\ &+ \int_c^d \left[G_{*,1}(f(c), u) - G_{*,1}(f(d), u)\right] \psi''(u)\, du \\ &+ \frac{\psi(d)-\psi(c)}{d-c}\left(f(d) - f(c)\right) + \int_c^d \int_c^d p_1(t,u) f'(t) \psi''(u)\, dt\, du \\ &= \int_c^d S_1(G_{*,1}(.,u), f, c, d)\psi''(u)\, du. \end{aligned}$$

Now by replacing n with $n-2$ in (24) for ψ'', we have:

$$\mathbb{S}_1(\psi, f, c, d) = \int_c^d \mathbb{S}_1(G_{*,1}(.,u), f, c, d) \left(\frac{(n-2)(\psi'(d)-\psi'(c))}{d-c} + \sum_{w=1}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \times\right.$$
$$\left.\left(\psi^{(w+1)}(d)(u-d)^w - \psi^{(w+1)}(c)(u-c)^w\right) + \frac{1}{(n-3)!(d-c)} \int_c^d (u-t)^{n-3} W^{[c,d]}(t,u)\psi^{(n)}(t) dt\right) du.$$

Rest follows from simplification and Fubini's theorem.

(*b*) Using assumption $\psi'(c) = 0$ and employing a similar method as in (*a*).

□

From the next two theorems we get a generalization of Steffensen's inequality and its reverse by generalizing (4) and its reverse.

Theorem 4. *Consider (A_1) with f be as in Corollary 1 (i) and let*

$$(u-t)^{n-3}W^{[c,d]}(t,u) \geq 0. \tag{28}$$

(a) *If ψ is n-convex, then for each $j \in \{1,2,3,4,5\}$ (where $\psi'(0)=0$ for $j=3$), we have:*

$$\begin{gathered}
\mathbb{S}_1(\psi, f, c, d) \geq \\
\frac{(n-2)(\psi'(d)-\psi'(c))}{d-c} \int_c^d \mathbb{S}_1(G_{*,1}(.,u), f, c, d)du + \sum_{w=1}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \times \\
\left(\psi^{(w+1)}(d) \int_c^d \mathbb{S}_1(G_{*,1}(.,u), f, c, d)(u-d)^w du - \psi^{(w+1)}(c) \int_c^d \mathbb{S}_1(G_{*,1}(.,u), f, c, d)(u-c)^w du\right).
\end{gathered} \tag{29}$$

(b) *If $-\psi$ is n-convex, then for each j, (29) holds in the reverse direction.*

Proof. For each j, the function $G_{*,j}(.,u)$ is convex and differentiable. Since f is non-decreasing with $f(z) \leq z$, therefore Corollary 1 (i) gives $\mathbb{S}_1(G_{*,1}(.,u), f, c, d) \geq 0$. On the other hand, if ψ is n-convex $(-\psi$ is n-convex), then $\psi^{(n)}(z) \geq (\leq) 0$. Therefore, given assumption together with n-convexity of ψ $(-\psi)$ implies $\int_c^d \psi^{(n)}(t) \left(\int_c^d \mathbb{S}_1(G_{*,j}(.,u), f, c, d)(u-t)^{n-3}W^{[c,d]}(t,u)du\right) dt \geq (\leq) 0$. The rest follows from (27). □

Theorem 5. *Consider (A_1) for even n and f as in Corollary 1 (i). Then*

(a) *If ψ is n-convex, then (29) holds.*
(b) *If $-\psi$ is n-convex, then the reverse of (29) holds.*
(c) *Let (29) (reverse of (29)) holds and*

$$\sum_{w=0}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \left(\psi^{(w+1)}(d)(u-d)^w du - \psi^{(w+1)}(c)(u-c)^w du\right) \geq (\leq) 0. \tag{30}$$

Then $\mathbb{S}_1(\psi, f, c, d) \geq (\leq)\ 0$.

Proof.
$(a), (b)$ We define

$$H(u,t) = (u-t)^{n-3}W^{[c,d]}(t,u) = \begin{cases} (u-t)^{n-3}(t-c), & \text{if } c \leq t \leq u \leq d, \\ (u-t)^{n-3}(t-d), & \text{if } c \leq u < t \leq d. \end{cases}$$

Clearly $H(u,t) \geq 0$ for even n. Consequently, we get (28), n-convexity of ψ $(-\psi)$, and Theorem 4 (a) (Theorem 4 (b)) yields (29) (and its reverse).

(c) By definition of $G_{*,j}(.,u)$ and assumption on f, Corollary 1 (i) gives $\mathbb{S}_1(G_{*,j}(.,u), f, c, d) \geq 0$. Therefore, by using (30) and $\mathbb{S}_1(G_{*,j}(.,u), f, c, d) \geq 0$ in (29) (and its reverse), we get $\mathbb{S}_1(\psi, f, c, d) \geq (\leq)$ $(\leq)0$, which completes the proof. □

Now, we prove the following theorem which enables us to prove a generalization of (5).

Theorem 6. *Consider (A_2) and let f be as in Corollary 2 (i) then:*

(a)

$$\mathbb{S}_2(\psi, f, d) = \tfrac{(n-2)(\psi'(d)-\psi'(0))}{d} \int_0^d \mathbb{S}_2(G_{*,j}(.,u), f, d)du + \sum_{w=1}^{n-3} \left(\tfrac{n-w-2}{dw!}\right)$$
$$\times \left(\psi^{(w+1)}(d) \int_0^d S_2(G_{*,j}(.,u), f, d)(u-d)^w du - \psi^{(w+1)}(0) \int_0^d \mathbb{S}_2(G_{*,j}(.,u), f, d)u^w du \right)$$
$$+ \tfrac{1}{d(n-3)!} \int_0^d \psi^{(n)}(t) \Big(\int_0^d \mathbb{S}_2(G_{*,j}(.,u), f, d)(u-t)^{n-3} W^{[0,d]}(t,u) du \Big) dt$$

for $j = 1, 2$.

(b) *If* $\psi'(0) = 0$, *then*

$$\mathbb{S}_2(\psi, f, d) + \psi(d) = \tfrac{(n-2)(\psi'(d)-\psi'(0))}{d} \int_0^d \mathbb{S}_2(G_{*,3}(.,u), f, d)du + \sum_{w=1}^{n-3} \left(\tfrac{n-w-2}{dw!}\right)$$
$$\times \left(\psi^{(w+1)}(d) \int_0^d \mathbb{S}_2(G_{*,3}(.,u), f, d)(u-d)^w du - f^{(w+1)}(0) \int_0^d \mathbb{S}_2(G_{*,3}(.,u), f, d)u^w du \right)$$
$$+ \tfrac{1}{d(n-3)!} \int_0^d \psi^{(n)}(t) \Big(\int_0^d \mathbb{S}_2(G_{*,3}(.,u), f, d)(u-t)^{n-3} W^{[0,d]}(t,u) du \Big) dt.$$

(c)

$$\mathbb{S}_2(\psi, f, d) + \psi(d) - d\psi'(d) =$$
$$\tfrac{(n-2)(\psi'(d)-\psi'(0))}{d} \int_0^d \mathbb{S}_2(G_{*,4}(.,u), f, d)du + \sum_{w=1}^{n-3} \left(\tfrac{n-w-2}{dw!}\right)$$
$$\times \left(\psi^{(w+1)}(d) \int_0^d \mathbb{S}_2(G_{*,4}(.,u), f, d)(u-d)^w du - f^{(w+1)}(0) \int_0^d \mathbb{S}_2(G_{*,4}(.,u), f, d)u^w du \right)$$
$$+ \tfrac{1}{d(n-3)!} \int_0^d \psi^{(n)}(t) \Big(\int_0^d \mathbb{S}_2(G_{*,4}(.,u), f, d)(u-t)^{n-3} W^{[0,d]}(t,u) du \Big) dt.$$

(d) *If* $\psi'(0) = 0$, *then*

$$\mathbb{S}_2(\psi, f, d) - d\psi'(d) = \tfrac{(n-2)(\psi'(d)-\psi'(0))}{d} \int_0^d \mathbb{S}_2(G_{*,5}(.,u), f, d)du + \sum_{w=1}^{n-3} \left(\tfrac{n-w-2}{dw!}\right)$$
$$\times \left(\psi^{(w+1)}(d) \int_0^d \mathbb{S}_2(G_{*,5}(.,u), f, d)(u-d)^w du - f^{(w+1)}(0) \int_0^d \mathbb{S}_2(G_{*,5}(.,u), f, d)u^w du \right)$$
$$+ \tfrac{1}{d(n-3)!} \int_0^d \psi^{(n)}(t) \Big(\int_0^d \mathbb{S}_2(G_{*,5}(.,u), f, d)(u-t)^{n-3} W^{[0,d]}(t,u) du \Big) dt.$$

Proof. We give proof of our results by fixing $j = 1$, and other cases can be proved in the similar way. By using (7) and (17) for ψ and ψ' respectively and applying assumption $\psi(0) = 0$, we get

$$\mathbb{S}_2(\psi, f, d) = \int_0^d \psi'(t) f(t)\, dt - \psi\left(\int_0^d f(t)\, dt\right) =$$
$$\int_0^d \tfrac{1}{d}\psi(d) f(t)\, dt + \int_0^d \left[\int_0^d \tfrac{\partial G_{*,1}(t,u)}{\partial t} \psi''(u)\, du\right] f(t)\, dt - \frac{\int_0^d f(t)\, dt}{d} \psi(d)$$
$$- \int_0^d G_{*,1}\left(\int_0^d f(t)\, dt, u\right) \psi''(u)\, du$$
$$= \int_0^d S_2(G_{*,1}(.,u), f, d)\psi''(u)\, du.$$

Now replacing n with $n-2$ in (24) for ψ'' and simplifying we get the required identities. □

Our next result gives a generalization of (5).

Theorem 7. *Consider* (A_2), f *as in Corollary 2* (i) *and let*

$$(u-t)^{n-3}W^{[0,d]}(t,u) \geq 0, \tag{31}$$

then the following hold:

(a) *If* ψ *is n-convex, then*
(i)

$$\begin{aligned}\mathbb{S}_2(\psi,f,d) \geq& \frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_2(G_{*,j}(.,u),f,d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)\\ &\times\left(\psi^{(w+1)}(d)\int_0^d \mathbb{S}_2(G_{*,j}(.,u),f,d)(u-d)^w du - f^{(w+1)}(0)\int_0^d \mathbb{S}_2(G_{*,j}(.,u),f,d)u^w du\right)\end{aligned} \tag{32}$$

for $j=1,2$.
(ii) *If* $\psi'(0)=0$, *then*

$$\begin{aligned}\mathbb{S}_2(\psi,f,d)+\psi(d) \geq& \frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_2(G_{*,3}(.,u),f,d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)\\ &\times\left(\psi^{(w+1)}(d)\int_0^d \mathbb{S}_2(G_{*,3}(.,u),f,d)(u-d)^w du - f^{(w+1)}(0)\int_0^d \mathbb{S}_2(G_{*,3}(.,u),f,d)u^w du\right).\end{aligned} \tag{33}$$

(iii)

$$\begin{aligned}&\mathbb{S}_2(\psi,f,d)+\psi(d)-d\psi'(d) \geq\\ &\frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_2(G_{*,4}(.,u),f,d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)\\ &\times\left(\psi^{(w+1)}(d)\int_0^d \mathbb{S}_2(G_{*,4}(.,u),f,d)(u-d)^w du - f^{(w+1)}(0)\int_0^d \mathbb{S}_2(G_{*,4}(.,u),f,d)u^w du\right).\end{aligned} \tag{34}$$

(iv) *If* $\psi'(0)=0$, *then*

$$\begin{aligned}&\mathbb{S}_2(\psi,f,d)-d\psi'(d) \geq\\ &\frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_2(G_{*,5}(.,u),f,d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)\\ &\times\left(\psi^{(w+1)}(d)\int_0^d \mathbb{S}_2(G_{*,5}(.,u),f,d)(u-d)^w du - f^{(w+1)}(0)\int_0^d \mathbb{S}_2(G_{*,5}(.,u),f,d)u^w du\right).\end{aligned} \tag{35}$$

(b) *Inequalities* (32)–(35) *are reversed provided that* $-\psi$ *is n-convex.*

Proof. The proof is similar to that of Theorem 4 except using Theorem 6 and Corollary 2 (i). □

Theorem 8. *Consider* (A_2) *for even n and* f *be as in Corollary 2* (i). *Then*

(a) *If* ψ *is n-convex, then* (32)–(35) *hold.*
(b) *If* $-\psi$ *is n-convex, then the reverse of* (32)–(35) *holds.*
(c) *If any of* (32)–(35) *(reverse of* (32)–(35)*) hold and*

$$\sum_{w=0}^{n-3}\left(\frac{n-w-2}{dw!}\right)\left(\psi^{(w+1)}(d)(u-d)^w du - \psi^{(w+1)}(0)u^w du\right) \geq (\leq) 0. \tag{36}$$

Then $\mathbb{S}_2(\psi, f, d) \geq (\leq)\ 0.$

Proof. The proof is similar to that of Theorem 5 except using Theorem 7 and Corollary 2 (i). □

Next we give some generalized identities considering (6).

Theorem 9. *Consider* (A_2) *and let* f, λ *and* Λ *be as in Corollary* 3 (i) *then:*

(a) *For* $j = 1, 2$, *we have*

$$\mathbb{S}_3(\psi, f, \lambda, d) = \frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_3(G_{*,j}(.,u), f, \lambda, d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)$$
$$\times\left(\psi^{(w+1)}(d)\int_0^d S_3(G_{*,j}(.,u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0)\int_0^d \mathbb{S}_3(G_{*,j}(.,u), f, \lambda, d)u^w du\right)$$
$$+\frac{1}{d(n-3)!}\int_0^d \psi^{(n)}(t)\left(\int_0^d \mathbb{S}_3(G_{*,j}(.,u), f, \lambda, d)(u-t)^{n-3}W^{[0,d]}(t,u)du\right)dt.$$

(b) *If* $\psi'(0) = 0$, *then*

$$\mathbb{S}_3(\psi, f, \lambda, d) + \psi(d)\int_0^d \lambda(x)\,dx =$$
$$\frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_3(G_{*,3}(.,u), f, \lambda, d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)$$
$$\times\left(\psi^{(w+1)}(d)\int_0^d S_3(G_{*,3}(.,u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0)\int_0^d \mathbb{S}_3(G_{*,3}(.,u), f, \lambda, d)u^w du\right)$$
$$+\frac{1}{d(n-3)!}\int_0^d \psi^{(n)}(t)\left(\int_0^d \mathbb{S}_3(G_{*,3}(.,u), f, \lambda, d)(u-t)^{n-3}W^{[0,d]}(t,u)du\right)dt.$$

(c)

$$\mathbb{S}_3(\psi, f, \lambda, d) + (\psi(d) - d\psi'(d))\int_0^d \lambda(x)\,dx =$$
$$\frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_3(G_{*,4}(.,u), f, \lambda, d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)$$
$$\times\left(\psi^{(w+1)}(d)\int_0^d S_3(G_{*,4}(.,u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0)\int_0^d \mathbb{S}_3(G_{*,4}(.,u), f, \lambda, d)u^w du\right)$$
$$+\frac{1}{d(n-3)!}\int_0^d \psi^{(n)}(t)\left(\int_0^d \mathbb{S}_3(G_{*,4}(.,u), f, \lambda, d)(u-t)^{n-3}W^{[0,d]}(t,u)du\right)dt.$$

(d) *If* $\psi'(0) = 0$, *then*

$$\mathbb{S}_3(\psi, f, \lambda, d) - d\psi'(d)\int_0^d \lambda(x)\,dx =$$
$$\frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_3(G_{*,5}(.,u), f, \lambda, d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right)$$
$$\times\left(\psi^{(w+1)}(d)\int_0^d S_3(G_{*,5}(.,u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0)\int_0^d \mathbb{S}_3(G_{*,5}(.,u), f, \lambda, d)u^w du\right)$$
$$+\frac{1}{d(n-3)!}\int_0^d \psi^{(n)}(t)\left(\int_0^d \mathbb{S}_3(G_{*,5}(.,u), f, \lambda, d)(u-t)^{n-3}W^{[0,d]}(t,u)du\right)dt.$$

Proof. We give a proof of our results by fixing $j = 1$, and other cases can be proved in a similar way. By using (7) and (17) for ψ and ψ' respectively and applying assumption $\psi(0) = 0$, we get:

$$\begin{aligned}
&\mathbb{S}_3(\psi, f, \lambda, d) = \int_0^d \Lambda(t)\psi'(t)f(t)\,dt - \int_0^d \lambda(x)\psi\left(\int_0^x f(t)\,dt\right)dx = \\
&\int_0^d \Lambda(t)f(t)\left[\frac{1}{d}\psi(d) + \int_0^d \frac{\partial G_{*,1}(t,u)}{\partial t}\psi''(u)\,du\right]dt - \int_0^d \lambda(x)\left[\frac{1}{d}\psi(d)\int_0^x f(t)\,dt\right. \\
&\left.+ \int_0^d G_{*,1}\left(\int_0^x f(t)\,dt, u\right)\psi''(u)\,du\right]dx = \frac{1}{d}\psi(d)\left[\int_0^d \Lambda(t)f(t)\,dt - \int_0^d \lambda(x)\int_0^x f(t)\,dt\,dx\right] \\
&+ \int_0^d \Lambda(t)f(t)\int_0^d \frac{\partial G_{*,1}(t,u)}{\partial t}\psi''(u)\,du\,dt - \int_0^d \lambda(x)\int_0^d G_{*,1}\left(\int_0^x f(t)\,dt, u\right)\psi''(u)\,du\,dx.
\end{aligned}$$

Since $\int_0^d \lambda(x)\int_0^x f(t)\,dt\,dx = \int_0^d f(t)\left(\int_t^d \lambda(x)\,dx\right)dt = \int_0^d \Lambda(t)f(t)dt$, therefore

$$\begin{aligned}
&\mathbb{S}_3(\psi, f, \lambda, d) \\
&= \int_0^d \left[\int_0^d \Lambda(t)f(t)\frac{\partial G_{*,1}(t,u)}{\partial t}\,dt - \int_0^d \lambda(x)G_{*,1}\left(\int_0^x f(t)\,dt, u\right)dx\right]\psi''(u)\,du \\
&= \int_0^d S_3(G_{*,1}(., u), f, \lambda, d)\psi''(u)\,du.
\end{aligned}$$

The rest follows from (24). □

Next, we present a generalization of (6).

Theorem 10. *Consider (A_2) and let f, λ, Λ be as in Corollary 3 (i) and (31) holds, then:*

(a) If ψ is n-convex, then

(i) For $j = 1, 2$, we have

$$\begin{aligned}
&\mathbb{S}_3(\psi, f, \lambda, d) \geq \frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_3(G_{*,j}(., u), f, \lambda, d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right) \\
&\times\left(\psi^{(w+1)}(d)\int_0^d S_3(G_{*,j}(., u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0)\int_0^d \mathbb{S}_3(G_{*,j}(., u), f, \lambda, d)u^w du\right).
\end{aligned} \tag{37}$$

(ii) If $\psi'(0) = 0$, then

$$\begin{aligned}
&\mathbb{S}_3(\psi, f, \lambda, d) + \psi(d)\int_0^d \lambda(x)\,dx \geq \\
&\frac{(n-2)(\psi'(d)-\psi'(0))}{d}\int_0^d \mathbb{S}_3(G_{*,3}(., u), f, \lambda, d)du + \sum_{w=1}^{n-3}\left(\frac{n-w-2}{dw!}\right) \\
&\times\left(\psi^{(w+1)}(d)\int_0^d S_3(G_{*,3}(., u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0)\int_0^d \mathbb{S}_3(G_{*,3}(., u), f, \lambda, d)u^w du\right).
\end{aligned} \tag{38}$$

(iii)

$$\begin{aligned}&\mathbb{S}_3(\psi, f, \lambda, d) + (\psi(d) - d\psi'(d)) \int_0^d \lambda(x)\, dx \geq \\ &\tfrac{(n-2)(\psi'(d)-\psi'(0))}{d} \int_0^d \mathbb{S}_3(G_{*,4}(.,u), f, \lambda, d) du + \sum_{w=1}^{n-3} \left(\tfrac{n-w-2}{dw!}\right) \\ &\times \left(\psi^{(w+1)}(d) \int_0^d S_3(G_{*,4}(.,u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0) \int_0^d \mathbb{S}_3(G_{*,4}(.,u), f, \lambda, d) u^w du \right).\end{aligned} \tag{39}$$

(iv) *If* $\psi'(0) = 0$, *then*

$$\begin{aligned}&\mathbb{S}_3(\psi, f, \lambda, d) - d\psi'(d) \int_0^d \lambda(x)\, dx \geq \\ &\tfrac{(n-2)(\psi'(d)-\psi'(0))}{d} \int_0^d \mathbb{S}_3(G_{*,5}(.,u), f, \lambda, d) du + \sum_{w=1}^{n-3} \left(\tfrac{n-w-2}{dw!}\right) \\ &\times \left(\psi^{(w+1)}(d) \int_0^d S_3(G_{*,5}(.,u), f, \lambda, d)(u-d)^w du - \psi^{(w+1)}(0) \int_0^d \mathbb{S}_3(G_{*,5}(.,u), f, \lambda, d) u^w du \right).\end{aligned} \tag{40}$$

(b) *Inequalities* (37)–(40) *are reversed provided that* $-\psi$ *is n-convex.*

Proof. The proof is similar to that of Theorem 4 except using Theorem 9 and Corollary 3 (i). □

Theorem 11. *Consider* (A_2) *for even n and let* f, λ, *and* Λ *be as in Corollary* 3 (i). *Then*

(a) *If* ψ *is n-convex, then* (37)–(40) *hold.*
(b) *If* $-\psi$ *is n-convex, then the reverses of* (37)–(40) *hold.*
(c) *If any of* (37)–(40) *(reverse of* (37)–(40)*) hold and* (36) *is valid. Then* $\mathbb{S}_3(\psi, f, \lambda, d) \geq (\leq)\ 0$.

Proof. The proof is similar to that of Theorem 5 except using Theorem 10 and Corollary 3 (i). □

3. New Upper Bounds Via Čebyšev Functional

Consider the Čebyšev functional for two Lebesgue integrable functions $\mathbb{F}_1, \mathbb{F}_2 : [c,d] \to \mathbb{R}$ given as:

$$T(\mathbb{F}_1, \mathbb{F}_2) = \frac{1}{d-c} \int_c^d \mathbb{F}_1(\xi)\mathbb{F}_2(\xi) d\xi - \frac{1}{d-c} \int_c^d \mathbb{F}_1(\xi) d\xi . \frac{1}{d-c} \int_c^d \mathbb{F}_2(\xi) d\xi.$$

Cerone and Dragomir in [17] proposed new bounds utilizing Čebyšev functional given as:

Theorem 12. *For* $\mathbb{F}_1 \in L[c,d]$ *and* $\mathbb{F}_2 : [c,d] \to \mathbb{R}$ *be an absolutely continuous function along with* $(.-c)(d-.)[\mathbb{F}_2']^2 \in L[c,d]$. *The following inequality holds*

$$|T(\mathbb{F}_1, \mathbb{F}_2)| \leq \frac{1}{\sqrt{2}} \left[\frac{T(\mathbb{F}_1, \mathbb{F}_1)}{(d-c)} \right]^{\frac{1}{2}} \left(\int_c^d (\xi - c)(d-\xi)[\mathbb{F}_2'(\xi)]^2 d\xi \right)^{\frac{1}{2}}. \tag{41}$$

Theorem 13. *For* $\mathbb{F}_1 : [c,d] \to \mathbb{R}$ *be an absolutely continuous with* $\mathbb{F}_1' \in L_\infty[c,d]$ *and* $\mathbb{F}_2 : [c,d] \to \mathbb{R}$ *is an increasing function. The following inequality holds*

$$|T(\mathbb{F}_1, \mathbb{F}_2)| \leq \frac{||\mathbb{F}_1'||_\infty}{2(d-c)} \int_c^d (\xi - c)(d-\xi) d\mathbb{F}_2(\xi). \tag{42}$$

The constants $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are the optimal constants.

Now we utilize the above theorems to construct new upper bounds for our obtained generalized identities. For our convenience we denote

$$\mathfrak{O}_j(t) = \int_c^d \mathbb{S}_1(G_{*,j}(.,u), f, c, d)(u-t)^{n-3} W^{[c,d]}(t,u)du, \quad t \in [c,d], \tag{43}$$

for $\{j = 1, \ldots, 5\}$. Consider the Čebyšev functional $T_j(\mathfrak{O}_j, \mathfrak{O}_j)$ $\{j = 1, \ldots, 5\}$ given as:

$$T_j(\mathfrak{O}_j, \mathfrak{O}_j) = \frac{1}{d-c}\int_c^d \mathfrak{O}_j{}^2(\xi)d\xi - \left(\frac{1}{d-c}\int_c^d \mathfrak{O}_j(\xi)d\xi\right)^2. \tag{44}$$

Grüss type inequalities associated with Theorems 12 and 13 can be given as:

Theorem 14. *Under the assumptions of Theorem 3, let $\psi : [c,d] \to \mathbb{R}$ be absolutely continuous along with $(.-c)(d-.)[\psi^{(n+1)}]^2 \in L[c,d]$ and $\mathfrak{O}_j$ $\{j = 1,2,3,4,5\}$ be defined as in (43). Then*

$$\begin{aligned}
&\mathbb{S}_1(\psi, f, c, d) - \sum_{w=0}^{n-3}\left(\tfrac{n-w-2}{(d-c)w!}\right) \times \\
&\left(\psi^{(w+1)}(d)\int_c^d \mathbb{S}_1(G_{*,1}(.,u), f, c, d)(u-d)^w du - \psi^{(w+1)}(c)\int_c^d \mathbb{S}_1(G_{*,1}(.,u), f, c, d)(u-c)^w du\right) \\
&- \tfrac{\psi^{(n-1)}(d)-\psi^{(n-1)}(c)}{(d-c)^2(n-3)!}\int_c^d \mathfrak{O}_j(t)dt = Rem(c,d,\mathfrak{O}_j,\psi^{(n)})
\end{aligned} \tag{45}$$

where

$$|Rem(c,d,\mathfrak{O}_j,\psi^{(n)})| \leq \frac{1}{\sqrt{2}\,(n-3)!}\left[\frac{T_j(\mathfrak{O}_j,\mathfrak{O}_j)}{(d-c)}\right]^{\frac{1}{2}}\left|\int_c^d (t-c)(d-t)[\psi^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.$$

Proof. Fix $\{j = 1, \ldots, 5\}$. Using Čebyšev functional for $\mathbb{F}_1 = \mathfrak{O}_j$, $\mathbb{F}_2 = \psi^{(n)}$ and by comparing (45) with (27), we have

$$Rem(c,d,\mathfrak{O}_j,\psi^{(n)}) = \frac{1}{(n-3)!}T_j(\mathfrak{O}_j,\psi^{(n)}).$$

Employing Theorem 12 for the new functions, we get the required bound. □

Theorem 15. *Under the assumptions of Theorem 3, let $\psi : [c,d] \to \mathbb{R}$ be absolutely continuous along with $\psi^{(n+1)} \geq 0$ and $\mathfrak{O}_j$ $\{j = 1,2,3,4,5\}$ be defined as in (44). Then $Rem(c,d,\mathfrak{O}_j,\psi^{(n)})$ in (45) satisfies a bound*

$$|Rem(c,d,\mathfrak{O}_j,\psi^{(n)})| \leq \frac{||\mathfrak{O}_j{}'||_\infty}{(n-3)!}\left[\frac{\psi^{(n-1)}(d)+\psi^{(n-1)}(c)}{2} - \frac{\psi^{(n-2)}(d)-\psi^{(n-2)}(c)}{d-c}\right]. \tag{46}$$

Proof. In the proof of Theorem 14, we have established that

$$Rem(c,d,\mathfrak{O}_j,\psi^{(n)}) = \frac{1}{(n-3)!}T_j(\mathfrak{O}_j,\psi^{(n)}).$$

Now applying Theorem 13 for $\mathbb{F}_1 = \mathfrak{O}_j$, $\mathbb{F}_2 = \psi^{(n)}$, we have

$$|Rem(c,d,\mathfrak{O}_j,\psi^{(n)})| = \frac{1}{(n-3)!}|T_j(\mathfrak{O}_j,\psi^{(n)})|$$

$$\leq \frac{||\mathfrak{O}_j{}'||_\infty}{2(d-c)\,(n-3)!}\int_c^d (t-c)(d-t)\psi^{(n+1)}(t)dt$$

Now since

$$\int_c^d (t-c)(d-t)\psi^{(n+1)}(t)dt = \int_c^d [2t-(c+d)]\psi^{(n)}(t)dt$$
$$= (d-c)\left[\psi^{(n-1)}(d)+\psi^{(n-1)}(c)\right] - 2\left(\psi^{(n-2)}(d) - f^{(n-2)}(c)\right)$$

therefore the required bound in (46) follows. □

Ostrowski-type inequalities associated with generalized Steffensen's inequality can be given as:

Theorem 16. *Under the assumptions of Theorem 3, let $|\psi^{(n)}|^s : [c,d] \to \mathbb{R}$ be a R-integrable function and consider (s,s') pair of conjugate exponents from $[1,\infty]$ such that $\frac{1}{s}+\frac{1}{s'}=1$. Then, we have*

$$\left| \begin{array}{c} \mathbb{S}_1(\psi,f,c,d) - \sum_{w=0}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \times \\ \left(\psi^{(w+1)}(d)\int_c^d \mathbb{S}_1(G_{*,1}(.,u),f,c,d)(u-d)^w du - \psi^{(w+1)}(c)\int_c^d \mathbb{S}_1(G_{*,1}(.,u),f,c,d)(u-c)^w du\right) \end{array} \right|$$
$$\leq \frac{||\psi^{(n)}||_s}{(n-3)!(d-c)} \left(\int_c^d \left| \int_c^d \mathbb{S}_1(G_{*,j}(.,u),f,c,d)(u-t)^{n-3} W^{[c,d]}(t,u)du \right|^{s'} dt \right)^{1/s'}. \tag{47}$$

The constant on the R.H.S. of (47) is sharp for $1 < s \leq \infty$ and the best possible for $s=1$.

Proof. Fix $\{j=1,\ldots,5\}$. Let us denote by

$$\Im_j = \frac{1}{(n-3)!(d-c)} \left(\int_c^d \mathbb{S}_1(G_{*,j}(.,u),f,c,d)(u-t)^{n-3} W^{[c,d]}(t,u)du \right), \quad t \in [c,d].$$

Using identity (27), we find

$$\left| \begin{array}{c} \mathbb{S}_1(\psi,f,c,d) - \sum_{w=0}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \times \\ \left(\psi^{(w+1)}(d)\int_c^d \mathbb{S}_1(G_{*,1}(.,u),f,c,d)(u-d)^w du - \psi^{(w+1)}(c)\int_c^d \mathbb{S}_1(G_{*,1}(.,u),f,c,d)(u-c)^w du\right) \end{array} \right|$$
$$= \left| \int_c^d \Im_j(t)\psi^{(n)}(t)dt \right|. \tag{48}$$

Applying Hölder's inequality for integrals on the R. H. S. of (48), we obtain

$$\left| \int_c^d \Im_j(t)\psi^{(n)}(t)dt \right| \leq \left(\int_c^d \left|\psi^{(n)}(t)\right|^s dt \right)^{\frac{1}{s}} \left(\int_c^d |\Im_j(t)|^{s'} dt \right)^{\frac{1}{s'}},$$

which combined together with (48) gives (47).

For sharpness of the constant $\left(\int_c^d |\Im_j(t)|^{s'} dt \right)^{1/s'}$ let us define the function ψ for which the equality in (47) holds.

For $1 < s \leq \infty$ let ψ be such that

$$\psi^{(n)}(t) = \operatorname{sgn}\Im_j(t)|\Im_j(t)|^{\frac{1}{s-1}}$$

and for $s=\infty$ let $\psi^{(n)}(t) = \operatorname{sgn}\Im_j(t)$.

For $s = 1$, we shall show that

$$\left| \int_c^d \Im_j(t)\psi^{(n)}(t)dt \right| \le \max_{t\in[c,d]} |\Im_j(t)| \left(\int_c^d \psi^{(n)}(t)dt \right) \tag{49}$$

is the best possible inequality. Suppose that $|\Im_j(t)|$ attains its maximum at $t_0 \in [c,d]$. To start with first we assume that $\Im_j(t_0) > 0$. For Θ small enough we define $\psi_\Theta(t)$ by

$$\psi_\Theta(t) = \begin{cases} 0\,, & c \le t \le t_0\,, \\ \frac{1}{\Theta n!}(t-t_0)^n\,, & t_o \le t \le t_0+\Theta\,, \\ \frac{1}{n!}(t-t_0)^{n-1}\,, & t_0+\Theta \le t \le d\,. \end{cases}$$

Then for Θ small enough

$$\left| \int_c^d \Im_j(t)\psi^{(n)}(t)dt \right| = \left| \int_{t_0}^{t_0+\Theta} \Im_j(t)\frac{1}{\Theta}dt \right| = \frac{1}{\Theta}\int_{t_0}^{t_0+\Theta} \Im_j(t)dt.$$

Now from inequality (49), we have

$$\frac{1}{\Theta}\int_{t_0}^{t_0+\Theta} \Im_j(t)dt \le \Im_j(t_0)\int_{t_0}^{t_0+\Theta} \frac{1}{\Theta}dt = \Im_j(t_0).$$

Since

$$\lim_{\Theta\to 0}\frac{1}{\Theta}\int_{t_0}^{t_0+\Theta} \Im_j(t)dt = \Im_j(t_0),$$

the statement follows. In the case when $\Im_j(t_0) < 0$, we define $f_\Theta(t)$ by

$$\psi_\Theta(t) = \begin{cases} \frac{1}{n!}(t-t_0-\Theta)^{n-1}\,, & c \le t \le t_0\,, \\ \frac{-1}{\Theta n!}(t-t_0-\Theta)^n\,, & t_o \le t \le t_0+\Theta\,, \\ 0\,, & t_0+\Theta \le t \le d, \end{cases}$$

then the rest of the proof is the same as above. □

Remark 3. *Similar bounds of Grüss and Ostrowski-type inequalities can be obtained by using Theorems 6 and 9.*

4. Monotonic Steffensen's-Type Functionals

The notion of $(n+1)$-convex function at a point was introduced in [18]. In the current section, we define some linear functionals from the differences of the generalized Steffensen's-type inequalities. By proving monotonicity of these functionals, we obtain new inequalities which contribute to the theory of more generalized class of functions, i.e., $(n+1)$-convex functions at a point. Below is the definition of $(n+1)$-convex function at point, see [18].

Definition 3. *Let $I \subseteq \mathbb{R}$ be an interval, $\xi \in I^0$ and $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is said to be $(n+1)$-convex at point ξ if there exists a constant K_ξ such that the function*

$$F(x) = f(x) - K_\xi \frac{x^n}{n!}$$

is n-concave on $I \cap (-\infty, \xi]$ and n-convex on $I \cap [\xi, \infty)$.

Pečarić et al. in [18] studied necessary and sufficient conditions on two linear functionals $\Omega : C([\delta_1, \xi]) \to \mathbb{R}$ and $\Gamma : C([\xi, \delta_2] \to \mathbb{R}$ so that the inequality $\Omega(f) \le \Gamma(f)$ holds for every function f that is $(n+1)$-convex at point ξ. In this section, we define some linear functionals and obtained certain inequalities associated with these linear functionals. Let $n \in \mathbb{N}$ be even, $\psi : [c,d] \to \mathbb{R}$ be n times differentiable function with $\psi^{(n-1)}$ absolutely continuous on $[c,d]$. Let $c_1, c_2 \in [c,d]$ and $\xi \in (c,d)$, where $c_1 < \xi < c_2$. Let $f_1 : [c_1, \xi] \to \mathbb{R}$ and $f_2 : [\xi, c_2] \to \mathbb{R}$ be increasing with $f_i(t) \le t$ for $i = 1,2$. For $j = 1,2,\ldots,5$, we construct:

$$\begin{aligned}&\Omega_{1,j}(\psi) = \mathbb{S}_1(\psi, f_1, c_1, \xi) - \textstyle\sum_{w=0}^{n-3} \left(\frac{n-w-2}{(\xi-c_1)w!}\right) \times \\ &\left(\psi^{(w+1)}(\xi) \int\limits_{c_1}^{\xi} \mathbb{S}_1(G_{*,1}(.,u), f_1, c_1, \xi)(u-\xi)^w du - \psi^{(w+1)}(c_1) \int\limits_{c_1}^{\xi} \mathbb{S}_1(G_{*,1}(.,u), f_1, c_1, \xi)(u-c_1)^w du\right)\end{aligned} \tag{50}$$

and

$$\begin{aligned}&\Gamma_{1,j}(\psi) = \mathbb{S}_1(\psi, f_2, \xi, c_2) - \textstyle\sum_{w=0}^{n-3} \left(\frac{n-w-2}{(c_2-\xi)w!}\right) \times \\ &\left(\psi^{(w+1)}(c_2) \int\limits_{\xi}^{c_2} \mathbb{S}_1(G_{*,1}(.,u), f_2, \xi, c_2)(u-c_2)^w du - \psi^{(w+1)}(\xi) \int\limits_{\xi}^{c_2} \mathbb{S}_1(G_{*,1}(.,u), f_2, \xi, c_2)(u-\xi)^w du\right).\end{aligned} \tag{51}$$

Theorem 5 (a) enables $\Gamma_{1,j}(\psi) \ge 0$ for $j = 1,2,\ldots,5$ (and $\psi'(0) = 0$ for $j = 3$), provided that ψ is n-convex. Furthermore, Theorem 5 (b) enables $\Omega_{1,j}(\psi) \le 0$ for $j = 1,2,\ldots,5$ (and $f'(0) = 0$ for $j = 3$), provided that $-\psi$ is n-convex.

Theorem 17. *Let ψ, f_1, f_2 be as defined above and $\psi : [c,d] \to \mathbb{R}$ be $(n+1)$-convex at a point ξ for even $n > 3$. If $\Omega_{1,j}(P_n) = \Gamma_{1,j}(P_n)$, for all $j = 1,2,\ldots,5$ and $\psi'(0) = 0$ for $j = 3$, where $P_n(u) = u^n$ then:*

$$\Omega_{1,j}(\psi) \le \Gamma_{1,j}(\psi),$$

for $j = 1,2,\ldots,5$.

Proof. Since ψ is $(n+1)$-convex, it follows from Definition 3 that there exist K_ξ such that $\Psi(u) = \psi(u) - \frac{K_\xi u^n}{n!}$ is n-concave on $[c_1, \xi]$ and n-convex on $[\xi, c_2]$. Therefore, for each $j = 1,2,\ldots,5$, we have

$$\Omega_{1,j}(\psi) - \frac{K_\xi}{n!}\Omega_{1,j}(P_n) = \Omega_{1,j}(\Psi) \le 0 \le \Gamma_{1,j}(\Psi) = \Gamma_{1,j}(\psi) - \frac{K_\xi}{n!}\Gamma_{1,j}(P_n).$$

Since $\Omega_{1,j}(P_n) = \Gamma_{1,j}(P_n)$, therefore $\Omega_{1,j}(\psi) \le \Gamma_{1,j}(\psi)$, which completes the proof. □

Remark 4. *We may proceed further by defining linear functionals with the inequalities proved in Theorems 8 and 11. Moreover, by proving monotonicity of new functionals we extend the inequalities in Theorems 8 and 11.*

5. Application to Exponentially Convex Functions

We start this section by an important Remark given as:

Remark 5. *By the virtue of Theorem 4 (a), for $j = 1,2,\ldots,5$, we define the positive linear functionals with respect to n-convex function ψ as follows*

$$\begin{aligned}&\Delta_{1,j}(\psi) := \mathbb{S}_1(\psi, f, c, d) - \textstyle\sum_{w=0}^{n-3} \left(\frac{n-w-2}{(d-c)w!}\right) \times \\ &\left(\psi^{(w+1)}(d) \int\limits_{c}^{d} \mathbb{S}_1(G_{*,1}(.,u), f, c, d)(u-d)^w du - \psi^{(w+1)}(c) \int\limits_{c}^{d} \mathbb{S}_1(G_{*,1}(.,u), f, c, d)(u-c)^w du\right) \ge 0.\end{aligned} \tag{52}$$

Next we construct the non trivial examples of exponentially convex functions (see [19]) from positive linear functionals $\Delta_{1,j}(\psi)$ for $(j = 1,2,\ldots,5)$.

For this consider the family of real valued functions on $[0, \infty)$ given as

$$\psi_s(u) = \begin{cases} \frac{u^s}{s(s-1)\cdots(s-n+1)}, & s \notin \{0, 1, \ldots, n-1\}; \\ \frac{u^t \ln u}{(-1)^{n-1-t} t!(n-1-t)!}, & s = t \in \{0, 1, \ldots, n-1\}. \end{cases} \tag{53}$$

It is interesting to note that this is a family of n-convex functions as

$$\frac{d^n}{du^n} \psi_s(u) = u^{s-n} \geq 0.$$

Since $s \mapsto u^{s-n} = e^{(s-n)\ln u}$ is exponentially convex function, therefore the mapping $s \mapsto \Delta_{1,j}(\psi_s)$ is exponential convex and as a special case, it is also log-convex mapping. The log-convexity of this mapping enables us to construct the known Lyapunov inequality given as

$$\left(\Delta_{1,j}(\psi_s)\right)^{t-r} \leq \left(\Delta_{1,j}(\psi_r)\right)^{t-s} \left(\Delta_{1,j}(\psi_t)\right)^{s-r} \tag{54}$$

for $r, s, t \in \mathbb{R}$ such that $r < s < t$ where $j = 1, 2, \ldots, 5$.

Remark 6. *We have not given the proof of the above mentioned results (see [19] for details). The Lyapunov inequality empowered us to refine lower (upper) bound for action of the functional on the class of functions given in (53) because if exponentially convex mapping attains zero value at some point it is zero everywhere (see [19]).*

One can also consider some other classes of n-convex functions given in the paper [19,20] and can get similar estimations. A similar technique can also be employed by considering the results of Theorems 7 and 10.

6. Conclusions and Outlooks

In this article, we extended the pool of inequalities by proving generalizations of well-known Steffensen's inequalities and their reverses. The inequalities proved in the main results provide generalizations of the results from [1,4,7,11]. Moreover, Hardy's inequality is also one of the well-known inequalities. In this article, we also proved generalizations of inequalities, from [1], which are closely related to Hardy's inequality. We also developed new bounds of Grüss and Ostrowski-type inequalities. Further, the contribution of these inequalities to the theory of $(n+1)$-convex functions has been presented by defining functionals from new inequalities and describing their properties. Lastly, new inequalities related to exponentially convex functions and log-convex functions, such as the Lyapunov inequality, have been developed. In the future, it can be investigated whether we can use other interpolations, such as Hermite interpolation, to prove new generalizations of Steffensen's inequality and related results.

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