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Existence and Unique Coupled Solution in S_b-Metric Spaces by Rational Contraction with Application

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Abstract: In this paper, we prove a unique common coupled fixed point theorem for two pairs of w-compatible mappings in S_b-metric spaces. We also furnish an example to support our main result.

Keywords: S_b-metric space; w-compatible pairs; S_b-completeness.

1. Introduction

In 2012, Sedghi et al. [1] introduced the notion of S-metric space and proved several results. Some other authors also worked on this (e.g., [2–6]). On the other hand, the concept of b-metric space was introduced by Bakhtin [7] and Czerwik [8] (see also [9–11]).

Recently, Sedghi et al. [1] defined S_b-metric spaces using the concepts of S and b-metric spaces and proved common fixed point theorem for four maps in S_b-metric spaces (see also [12]).

Bhaskar and Lakshmikantham [13] introduced the notion of coupled fixed point and proved some coupled fixed point results as well.

The aim of this paper is to prove a unique common coupled fixed point theorem for four mappings in S_b-metric spaces. Throughout this paper, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of all non-negative real numbers and positive integers, respectively.

First, we recall some definitions, lemmas and examples.

Definition 1. *Ref. [4] Let X be a nonempty set. A S-metric on X is a function $S : X^3 \to \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$:*

(S1) $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then, the pair (X, S) is called a S-metric space.

Definition 2. *([8]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric if the following axioms are satisfied for all $x, y, z \in X$:*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$;

(b_2) $d(x,y) = d(y,x)$; *and*
(b_3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) *is called a b b-metric space.*

Definition 3. *([1]) Let* X *be a nonempty set and* $b \geq 1$ *be a given real number. Suppose that a mapping* $S_b : X^3 \to \mathbb{R}^+$ *is a function satisfying the following properties:*

(S_b1) $S_b(x,y,z) = 0$ *if and only if* $x = y = z$; *and*
(S_b2) $S_b(x,y,z) \leq b(S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a))$ *for all* $x,y,z,a \in X$.

Then, the function S_b *is called a* S_b*-metric on* X *and the pair* (X, S_b) *is called a* S_b*-metric space.*

Remark 1. *([1]) It should be noted that the class of* S_b*-metric spaces is effectively larger than that of* S*-metric spaces. Indeed, each* S*-metric space is a* S_b*-metric space with* $b = 1$.

The following example shows that a S_b-metric on X need not be a S-metric on X.

Example 1. *([1]) Let* (X, S_b) *be a* S_b*-metric space and* $S_b(x,y,z) = S^p(x,y,z)$, *where* $p > 1$ *is a real number. Note that* S_b *is a* S_b*-metric with* $b = 2^{2(p-1)}$. *In addition,* (X, S_b) *is not necessarily a* S*-metric space.*

Definition 4. *([1]) Let* (X, S_b) *be a* S_b*-metric space. Then, for* $x \in X$, $r > 0$, *we defined the open ball* $B_{S_b}(x,r)$ *and closed ball* $B_{S_b}[x,r]$ *with center* x *and radius* r *as follows, respectively:*

$$\begin{aligned} B_{S_b}(x,r) &= \{y \in X : S_b(y,y,x) < r\}, \\ B_{S_b}[x,r] &= \{y \in X : S_b(y,y,x) \leq r\}. \end{aligned}$$

Lemma 1. *([1]) In a* S_b*-metric space, we have*

$$S_b(x,x,y) \leq bS_b(y,y,x)$$

and

$$S_b(y,y,x) \leq bS_b(x,x,y).$$

Lemma 2. *([1]) In a* S_b*-metric space we have*

$$S_b(x,x,z) \leq 2bS_b(x,x,y) + b^2 S_b(y,y,z).$$

Definition 5. *([1]) Let* (X, S_b) *be a* S_b*-metric space. A sequence* $\{x_n\}$ *in* X *is said to be:*

(1) S_b*-Cauchy sequence if, for each* $\epsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $S_b(x_n, x_n, x_m) < \epsilon$ *for each* $m, n \geq n_0$.
(2) S_b*-convergent to a point* $x \in X$ *if, for each* $\epsilon > 0$, *there exists a positive integer* n_0 *such that* $S_b(x_n, x_n, x) < \epsilon$ *or* $S_b(x,,x,x_n) < \epsilon$ *for all* $n \geq n_0$ *and we denote that by* $\lim\limits_{n\to\infty} x_n = x$.

Definition 6. *([1]) A* S_b*-metric space* (X, S_b) *is called complete if every* S_b*-Cauchy sequence is* S_b*-convergent in* X.

Lemma 3. *([14]) If* (X, S_b) *is a* S_b*-metric space with* $b \geq 1$ *and* $\{x_n\}$ *is a* S_b*-convergent to* x, *then for all* $y \in X$, *we have*

(i) $\frac{1}{2b} S_b(y,x,x) \leq \liminf\limits_{n\to\infty} S_b(y,y,x_n) \leq \limsup\limits_{n\to\infty} S_b(y,y,x_n) \leq 2bS_b(y,y,x)$; *and*

(ii) $\frac{1}{b^2} S_b(x,x,y) \leq \liminf\limits_{n\to\infty} S_b(x_n,x_n,y) \leq \limsup\limits_{n\to\infty} S_b(x_n,x_n,y) \leq b^2 S_b(x,x,y)$.

In particular, if $x = y$, then we have $\lim_{n\to\infty} S_b(x_n, x_n, y) = 0$.

Definition 7. *([13]) Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.*

Definition 8. *([15]) Let X be a nonempty set. An element $(x, y) \in X \times X$ is called:*

(i) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $f(x) = F(x, y)$ and $f(y) = F(y, x)$; and
(ii) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = f(x) = F(x, y)$ and $y = f(y) = F(y, x)$.

Definition 9. *([16]) Let X be a nonempty set and $F : X \times X \to X$ and $f : X \to X$. The $\{F, f\}$ is said to be w-compatible pair if $f(F(x, y)) = F(f(x), f(y))$ and $f(F(y, x)) = F(f(y), f(x))$ whenever there exist $x, y \in X$ with $f(x) = F(x, y)$ and $f(y) = F(y, x)$.*

For more details of other generalized metric spaces as well as on some rational contraction, see [9,17–19].

Now, we give our main result.

2. Main Results

Let Φ denote the class of all functions $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is a non-decreasing, continuous, $\phi(t) < \frac{t}{4b^4}$ for all $t > 0$ and $\phi(0) = 0$.

Theorem 1. *Let (X, S_b) be a S_b-metric space. Suppose that $A, B : X \times X \to X$ and $P, Q : X \to X$ satisfy:*

(1) $A(X \times X) \subseteq Q(X), B(X \times X) \subseteq P(X)$;
(2) $\{A, P\}$ and $\{B, Q\}$ are w-compatible pairs;
(3) one of $P(X)$ or $Q(X)$ is S_b-complete subspace of X; and
(4) $2b^5\, S_b(A(x, y), A(x, y), B(u, v))$

$$\le \phi\left(\max\left\{\begin{array}{c} S_b(P(x), P(x), Q(u)), S_b(P(y), P(y), Q(v)), S_b(A(x, y), A(x, y), P(x)), \\ S_b(A(y, x), A(y, x), P(y)), S_b(B(u, v), B(u, v), Q(u)), S_b(B(v, u), B(v, u), Q(v)), \\ \dfrac{S_b(A(x, y), A(x, y), Q(u))\, S_b(B(u, v), B(u, v), P(x))}{1 + S_b(P(x), P(x), Q(u))}, \\ \dfrac{S_b(A(y, x), A(y, x), Q(v))\, S_b(B(v, u), B(v, u), P(y))}{1 + S_b(P(y), P(y), Q(v))} \end{array}\right\}\right),$$

for all $x, y, u, v \in X$, $\phi \in \Phi$.

Then, A, B, P and Q have a unique common coupled fixed point in $X \times X$.

Proof of Theorem. Let $x_0, y_0 \in X$. From Equation (1), we can construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ such that

$$\begin{aligned} &A(x_{2n}, y_{2n}) = Q(x_{2n+1}) = z_{2n}, \\ &A(y_{2n}, x_{2n}) = Q(y_{2n+1}) = w_{2n}, \\ &B(x_{2n+1}, y_{2n+1}) = P(x_{2n+2}) = z_{2n+1}, \\ &B(y_{2n+1}, x_{2n+1}) = P(y_{2n+2}) = w_{2n+1}, \quad n = 0, 1, 2, \cdots. \end{aligned}$$

Case (i). Suppose $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$ for some m. Assume that $z_{2m+1} \neq z_{2m+2}$ or $w_{2m+1} \neq w_{2m+2}$.

From Equation (4), we have

$$
\begin{aligned}
& S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}) \\
& \leq 2b^5 S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), B(x_{2m+1}, y_{2m+1})) \\
& \leq \phi\left(\max\left\{\begin{array}{c}
S_b(P(x_{2m+2}), P(x_{2m+2}), Q(x_{2m+1})), S_b(P(y_{2m+2}), P(y_{2m+2}), Q(y_{2m+1})), \\
S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), P(x_{2m+2})), \\
S_b(A(y_{2m+2}, x_{2m+2}), A(y_{2m+2}, x_{2m+2}), P(y_{2m+2})), \\
S_b(B(x_{2m+1}, y_{2m+1}), B(x_{2m+1}, y_{2m+1}), Q(x_{2m+1})), \\
S_b(B(y_{2m+1}, x_{2m+1}), B(y_{2m+1}, x_{2m+1}), Q(y_{2m+1})), \\
\frac{S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), Q(x_{2m+1}))S_b(B(x_{2m+1}, y_{2m+1}), B(x_{2m+1}, y_{2m+1}), P(x_{2m+2}))}{1 + S_b(P(x_{2m+2}), P(x_{2m+2}), Q(x_{2m+1}))} \\
\frac{S_b(A(y_{2m+2}, x_{2m+2}), A(y_{2m+2}, x_{2m+2}), Q(y_{2m+1}))S_b(B(y_{2m+1}, x_{2m+1}), B(y_{2m+1}, x_{2m+1}), P(y_{2m+2}))}{1 + S_b(P(y_{2m+2}), P(y_{2m+2}), Q(y_{2m+1}))}
\end{array}\right\}\right) \\
& = \phi\left(\max\left\{\begin{array}{l}
S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\
S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\
\frac{S_b(z_{2m+2}, z_{2m+2}, z_{2m+1})\ S_b(z_{2m+1}, z_{2m+1}, z_{2m})}{1 + S_b(z_{2m+1}, z_{2m+1}, z_{2m})}, \frac{S_b(w_{2m+2}, w_{2m+2}, w_{2m+1})\ S_b(w_{2m+1}, w_{2m+1}, w_{2m})}{1 + S_b(w_{2m+1}, w_{2m+1}, w_{2m})}
\end{array}\right\}\right) \\
& = \phi\left(\max\left\{\ 0, 0, S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), 0, 0, 0, 0\ \right\}\right) \\
& = \phi\left(\max\left\{\ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1})\ \right\}\right).
\end{aligned}
$$

Similarly, we can prove that

$$S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \leq \phi\left(\max\left\{S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1})\right\}\right).$$

It follows that

$$
\begin{aligned}
& \max\left\{S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1})\right\} \\
& \leq \phi\left(\max\left\{S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1})\right\}\right).
\end{aligned}
$$

Hence, $z_{2m+2} = z_{2m+1}$ and $w_{2m+2} = w_{2m+1}$.
Continuing in this process, we can conclude that $z_{2m+k} = z_{2m}$ and $w_{2m+k} = w_{2m}$ for all $k \geq 0$.
It follows that $\{z_m\}$ and $\{w_m\}$ are Cauchy sequences.

Case (ii). Assume that $z_{2n} \neq z_{2n+1}$ or $w_{2n} \neq w_{2n+1}$ for all n.

Put $S_n = \max\left\{S_b(z_{n+1}, z_{n+1}, z_n), S_b(w_{n+1}, w_{n+1}, w_n)\right\}$.
From Equation (4), we have

$$
\begin{aligned}
& S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) \\
& \leq 2b^5 S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), B(x_{2n+1}, y_{2n+1})) \\
& \leq \phi\left(\max\left\{\begin{array}{c}
S_b(P(x_{2n+2}), P(x_{2n+2}), Q(x_{2n+1})), S_b(P(y_{2n+2}), P(y_{2n+2}), Q(y_{2n+1})), \\
S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), P(x_{2n+2})), \\
S_b(A(y_{2n+2}, x_{2n+2}), A(y_{2n+2}, x_{2n+2}), P(y_{2n+2})), \\
S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), Q(x_{2n+1})), \\
S_b(B(y_{2n+1}, x_{2n+1}), B(y_{2n+1}, x_{2n+1}), Q(y_{2n+1})), \\
\frac{S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), Q(x_{2n+1}))S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), P(x_{2n+2}))}{1 + S_b(P(x_{2n+2}), P(x_{2n+2}), Q(x_{2n+1}))}, \\
\frac{S_b(A(y_{2n+2}, x_{2n+2}), A(y_{2n+2}, x_{2n+2}), Q(y_{2n+1}))S_b(B(y_{2n+1}, x_{2n+1}), B(y_{2n+1}, x_{2n+1}), P(y_{2n+2}))}{1 + S_b(P(y_{2n+2}), P(y_{2n+2}), Q(y_{2n+1}))}
\end{array}\right\}\right)
\end{aligned}
$$

$$= \phi\left(\max\left\{\begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), \\ S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{S_b(z_{2n+2}, z_{2n+2}, z_{2n})\, S_b(z_{2n+1}, z_{2n+1}, z_{2n+1})}{1 + S_b(z_{2n+1}, z_{2n+1}, z_{2n})}, \frac{S_b(w_{2n+2}, w_{2n+2}, w_{2n})\, S_b(w_{2n+1}, w_{2n+1}, w_{2n+1})}{1 + S_b(w_{2n+1}, w_{2n+1}, w_{2n})} \end{array}\right\}\right)$$
$$= \phi\left(\max\left\{\begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \end{array}\right\}\right)$$
$$= \phi\left(\max\left\{ S_{2n+1}, S_{2n} \right\}\right).$$

Similarly, we can prove

$$S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \leq \phi\left(\max\left\{S_{2n+1}, S_{2n}\right\}\right).$$

Thus,

$$S_{2n+1} \leq \phi(\max\{S_{2n}, S_{2n+1}\}).$$

If S_{2n+1} is maximum, then we get contradiction so that S_{2n} is maximum.
Thus,

$$S_{2n+1} \leq \phi(S_{2n}) < S_{2n}. \tag{1}$$

Similarly, we can conclude that $S_{2n} < S_{2n-1}$.

It is clear that $\{S_n\}$ is a non-increasing sequence of non-negative real numbers and must converge to a real number, say $r \geq 0$.

Suppose $r > 0$. Letting $n \to \infty$, in Equation (1), we have $r \leq \phi(r) \leq r$.

It is a contradiction. Hence, $r = 0$.

Thus,

$$\lim_{n\to\infty} S_b(z_{n+1}, z_{n+1}, z_n) = 0 \tag{2}$$

and

$$\lim_{n\to\infty} S_b(w_{n+1}, w_{n+1}, w_n) = 0. \tag{3}$$

Now, we prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in (X, S_b). On teh contrary, we suppose that $\{z_{2n}\}$ or $\{w_{2n}\}$ is not Cauchy. Then, there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that for $n_k > m_k$

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \geq \epsilon \tag{4}$$

and

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\} < \epsilon. \tag{5}$$

From Equations (4) and (5), we have

$$
\begin{aligned}
\epsilon &\leq \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\leq 2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+2}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+2})\} \\
&\quad + b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k+2}), S_b(w_{2n_k+2}, w_{2n_k}, w_{2m_k+2})\} \\
&\leq 2b \left(2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\}\right) \\
&\quad + 2b \left(b \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\}\right) \\
&\quad + b \left(2b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k+1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k+1})\}\right) \\
&\quad + b \left(b \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}\right) \\
&\leq 4b^3 \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + 2b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\} \\
&\quad + 2b^3 \max\{S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k})\} \\
&\quad + b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}.
\end{aligned} \tag{6}
$$

Now, from Equation (4), we have

$$
\begin{aligned}
&2b^5 S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}) \\
&\leq \phi\left(\max\left\{\begin{array}{l}
S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), \\
S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\
\dfrac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\, S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1 + S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\
\dfrac{S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\, S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1 + S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})}
\end{array}\right\}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&2b^5 S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \\
&\leq \phi\left(\max\left\{\begin{array}{l}
S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), \\
S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\
\dfrac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\, S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1 + S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\
\dfrac{S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\, S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1 + S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})}
\end{array}\right\}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&2b^5 \max\left\{\; S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \;\right\} \\
&\leq \phi\left(\max\left\{\begin{array}{l}
S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\
S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\
S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\
\dfrac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\, S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1 + S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\
\dfrac{S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\, S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1 + S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})}
\end{array}\right\}\right).
\end{aligned} \tag{7}
$$

However,

$$\begin{aligned}
&\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \\
&\leq 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k})\} \\
&\leq 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + b^2\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\leq 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + b^2\left(2b\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\}\right) \\
&\quad + b^2\left(b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_{k-2}}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_{k-2}})\}\right) \\
&< 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + 2b^3\epsilon + b^3\left(2b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k-1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k-1})\}\right) \\
&\quad + b^3\left(b\max\{S_b(z_{2n_{k-2}}, z_{2n_k-2}, z_{2n_k-1}), S_b(w_{2n_k-2}, w_{2n_k-2}, w_{2n_k-1})\}\right) \\
&\leq 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + 2b^3\epsilon + 2b^4\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k-1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k-1})\} \\
&\quad + b^5\max\{S_b(z_{2n_k-1}, z_{2n_k-1}, z_{2n_k-2}), S_b(w_{2n_k-1}, w_{2n_k-1}, w_{2n_k-2})\}.
\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty}\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \leq 2b^3\epsilon.$$

In addition,

$$\begin{aligned}
&\lim_{k\to\infty}\frac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1 + S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})} \\
&\leq \lim_{k\to\infty}\frac{1}{1 + S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}\left[2bS_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}) + bS_b(z_{2n_k}, z_{2n_k}, z_{2m_k+1})\right]\cdot \\
&\left[2bS_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}) + bS_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})\right] \\
&\leq \lim_{k\to\infty}\frac{b^3S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})\ S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}{1 + S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})} \\
&\leq \lim_{k\to\infty} b^3S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) \\
&\leq 2b^6\epsilon.
\end{aligned}$$

Similarly,

$$\lim_{k\to\infty}\frac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1 + S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \leq 2b^6\epsilon.$$

Letting $k \to \infty$ in Equation (7), we have

$$\begin{aligned}
\lim_{k\to\infty}\quad &\max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\} \\
&\leq \frac{1}{2b^5}\phi\left(\max\{2b^3\epsilon, 0, 0, 0, 0, 2b^6\epsilon, 2b^6\epsilon\}\right) \\
&= \frac{1}{2b^5}\phi(2b^6\epsilon).
\end{aligned} \tag{8}$$

Now, letting $n \to \infty$ in Equation (6), from Equations (2), (3) and (8), we have

$$\epsilon \le 0 + 0 + 0 + b^2 \frac{1}{2b^5} \phi(2b^6 \epsilon) < \epsilon.$$

It is a contradiction. Hence, $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b-Cauchy sequences in (X, S_b).
In addition,

$$\begin{aligned}
&\max\{S_b(z_{2n+1}, z_{2n+1}, z_{2m+1}), S_b(w_{2n+1}, w_{2n+1}, w_{2m+1})\} \\
&\qquad \le 2b \max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\
&\qquad\quad + b \max\{S_b(z_{2m+1}, z_{2m+1}, z_{2n}), S_b(w_{2m+1}, w_{2m+1}, w_{2n})\} \\
&\qquad \le 2b \max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\
&\qquad\quad + 2b^2 \max\{S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m})\} \\
&\qquad\quad + b^2 \max\{S_b(z_{2n}, z_{2n}, z_{2m}), S_b(w_{2n}, w_{2n}, w_{2m})\}.
\end{aligned}$$

Since $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b-Cauchy sequences, from Equations (2) and (3), it follows that $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are also S_b-Cauchy sequences in (X, S_b). Hence, $\{z_n\}$ and $\{w_n\}$ are S_b-Cauchy sequences in (X, S_b).

Suppose $P(X)$ is a S_b-complete subspace of (X, S_b). Then, the sequences $\{z_n\}$ and $\{w_n\}$ converge to α and β in $P(X)$. Thus, there exist a and b in $P(X)$ such that

$$\lim_{n\to\infty} z_n = \alpha = P(a) \text{ and } \lim_{n\to\infty} w_n = \beta = P(b). \tag{9}$$

Now, we have to prove that $A(a,b) = \alpha$ and $A(b,a) = \beta$. On the contrary, suppose that $A(a,b) \neq \alpha$ or $A(b,a) \neq \beta$.

From Equation (4) and Lemma 3, we obtain that

$$\begin{aligned}
&\frac{1}{2b} S_b(A(a,b), A(a,b), \alpha) \\
&\le \liminf_{n\to\infty} 2b^5\, S_b(A(a,b), A(a,b), B(x_{2n+1}, y_{2n+1})) \\
&\le \liminf_{n\to\infty} \phi\left(\max\left\{ \begin{array}{c}
S_b(P(a), P(a), Q(x_{2n+1})), S_b(P(b), P(b), Q(y_{2n+1})), \\
S_b(A(a,b), A(a,b), P(a)), S_b(A(b,a), A(b,a), P(b)), \\
S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), Q(x_{2n+1})), \\
S_b(B(y_{2n+1}, x_{2n+1}), B(y_{2n+1}, x_{2n+1}), Q(y_{2n+1})), \\
\frac{S_b(A(a,b), A(a,b), Q(x_{2n+1}))\ S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), P(a))}{1 + S_b(P(a), P(a), Q(x_{2n+1}))}, \\
\frac{S_b(A(b,a), A(b,a), Q(y_{2n+1}))\ S_b(B(y_{2n+1}, x_{2n+1}), B(y_{2n+1}, x_{2n+1}), P(b))}{1 + S_b(P(b), P(b), Q(y_{2n+1}))}
\end{array} \right\} \right) \\
&\le \liminf_{n\to\infty} \phi\left(\max\left\{ \begin{array}{c}
S_b(\alpha, \alpha, z_{2n}), S_b(\beta, \beta, w_{2n}), \\
S_b(A(a,b), A(a,b), \alpha), S_b(A(b,a), A(b,a), \beta), \\
S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\
\frac{S_b(A(a,b), A(a,b), Q(x_{2n+1}))\ S_b(z_{2n+1}, z_{2n+1}, \alpha)}{1 + S_b(\alpha, \alpha, Q(x_{2n+1}))}, \\
\frac{S_b(A(b,a), A(b,a), w_{2n}) S_b(w_{2n+1}, w_{2n+1}, \beta)}{1 + S_b(\beta, \beta, Q(y_{2n+1}))}
\end{array} \right\} \right) \\
&= \phi\left(\max\left\{ 0, 0, S_b(A(a,b), A(a,b), \alpha), S_b(A(b,a), A(b,a), \beta), 0, 0, 0, 0 \right\} \right) \\
&= \phi\left(\max\left\{ S_b(A(a,b), A(a,b), \alpha), S_b(A(b,a), A(b,a), \beta) \right\} \right).
\end{aligned}$$

Similarly,

$$\frac{1}{2b}S_b(A(b,a),A(b,a),\beta) \leq \phi\left(\max\left\{\ S_b(A(a,b),A(a,b),\alpha), S_b(A(b,a),A(b,a),\beta)\ \right\}\right).$$

Thus,

$$\frac{1}{2b}\max\left\{\begin{array}{c} S_b(A(a,b),A(a,b),\alpha), \\ S_b(A(b,a),A(b,a),\beta) \end{array}\right\} \leq \phi\left(\max\left\{\begin{array}{c} S_b(A(a,b),A(a,b),\alpha), \\ S_b(A(b,a),A(b,a),\beta) \end{array}\right\}\right).$$

By the definition of ϕ, it follows that $A(a,b) = \alpha = P(a)$ and $A(b,a) = \beta = P(b)$. Since (A,P) is w-compatible pair, we have that $A(\alpha,\beta) = P(\alpha)$ and $A(\beta,\alpha) = P(\beta)$.

From Equation (4) and Lemma 3, we have

$$\begin{aligned}
&\frac{1}{2b}S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha) \\
&\leq \lim_{n\to\infty}\sup 2b^5 S_b(A(\alpha,\beta),A(\alpha,\beta),B(x_{2n+1},y_{2n+1})) \\
&\leq \lim_{n\to\infty}\sup\ \phi\left(\max\left\{\begin{array}{c}
S_b(P(\alpha),P(\alpha),Q(x_{2n+1})), S_b(P(\beta),P(\beta),Q(y_{2n+1})), \\
S_b(A(\alpha,\beta),A(\alpha,\beta),P(\alpha)), \\
S_b(A(\beta,\alpha),A(\beta,\alpha),P(\beta)), \\
S_b(B(x_{2n+1},y_{2n+1}),B(x_{2n+1},y_{2n+1}),Q(x_{2n+1})), \\
S_b(B(y_{2n+1},x_{2n+1}),B(y_{2n+1},x_{2n+1}),Q(y_{2n+1})), \\
\dfrac{S_b(A(\alpha,\beta),A(\alpha,\beta),Q(x_{2n+1}))\ S_b(B(x_{2n+1},y_{2n+1}),B(x_{2n+1},y_{2n+1}),P(\alpha))}{1+S_b(P(\alpha),P(\alpha),Q(x_{2n+1}))}, \\
\dfrac{S_b(A(\beta,\alpha),A(\beta,\alpha),Q(y_{2n+1}))\ S_b(B(y_{2n+1},x_{2n+1}),B(y_{2n+1},x_{2n+1}),P(\beta))}{1+S_b(P(\beta),P(\beta),Q(y_{2n+1}))}
\end{array}\right\}\right) \\
&= \lim_{n\to\infty}\sup\ \phi\left(\max\left\{\begin{array}{c}
S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n}), S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n}), \\
0, 0, S_b(z_{2n+1},z_{2n+1},z_{2n}), S_b(w_{2n+1},w_{2n+1},w_{2n}), \\
\dfrac{S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n})\ S_b(z_{2n+1},z_{2n+1},A(\alpha,\beta))}{1+S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n})}, \\
\dfrac{S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n})\ S_b(w_{2n+1},w_{2n+1},A(\beta,\alpha))}{1+S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n})}
\end{array}\right\}\right) \\
&\leq \lim_{n\to\infty}\sup\ \phi\left(\max\left\{\begin{array}{c}
S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n}), S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n}), \\
S_b(z_{2n+1},z_{2n+1},z_{2n}), S_b(w_{2n+1},w_{2n+1},w_{2n}), \\
S_b(z_{2n+1},z_{2n+1},A(\alpha,\beta)), S_b(w_{2n+1},w_{2n+1},A(\beta,\alpha))
\end{array}\right\}\right) \\
&\leq \phi\left(\max\left\{\begin{array}{c}
2bS_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), 2bS_b(A(\beta,\alpha),A(\beta,\alpha),\beta), \\
0, 0, b^2 S_b(\alpha,\alpha,A(\alpha,\beta)), b^2 S_b(\beta,\beta,A(\beta,\alpha))
\end{array}\right\}\right) \\
&\leq \phi\left(2b^2\max\left\{\ S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), S_b(A(\beta,\alpha),A(\beta,\alpha),\beta)\ \right\}\right).
\end{aligned}$$

Similarly,

$$\frac{1}{2b}S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \leq \phi\left(2b^2\max\left\{\ S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), S_b(A(\beta,\alpha),A(\beta,\alpha),\beta)\ \right\}\right).$$

Thus,

$$\frac{1}{2b}\max\left\{\begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array}\right\} \leq \phi\left(2b^2\max\left\{\begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array}\right\}\right).$$

By the definition of ϕ, it follows that $A(\alpha,\beta) = \alpha = P(\alpha)$ and $A(\beta,\alpha) = \beta = P(\beta)$.

Therefore, (α,β) is a common coupled fixed point of A and P.

Since $A(X \times X) \subseteq Q(X)$, there exist x and y in X such that $A(\alpha,\beta) = \alpha = Q(x)$ and $A(\beta,\alpha) = \beta = Q(y)$.

From Equation (4), we have

$$\begin{aligned}
S_b(\alpha,\alpha,B(x,y)) &= S_b(A(\alpha,\beta),A(\alpha,\beta),B(x,y)) \\
&\leq 2b^5 S_b(A(\alpha,\beta),A(\alpha,\beta),B(x,y)) \\
&\leq \phi\left(\max\left\{\begin{array}{c} S_b(P(\alpha),P(\alpha),Q(x)), S_b(P(\beta),P(\beta),Q(y)), \\ S_b(A(\alpha,\beta),A(\alpha,\beta),P(\alpha)), S_b(A(\beta,\alpha),A(\beta,\alpha),P(\beta)), \\ S_b(B(x,y),B(x,y),Q(x)), S_b(B(y,x),B(y,x),Q(y)), \\ \frac{S_b(A(\alpha,\beta),A(\alpha,\beta),Q(x))\; S_b(B(x,y),B(x,y),P(\alpha))}{1+S_b(P(\alpha),P(\alpha),Q(x))}, \\ \frac{S_b(A(\beta,\alpha),A(\beta,\alpha),Q(y)) S_b(B(y,x),B(y,x),P(\beta))}{1+S_b(P(\beta),P(\beta),Q(y))} \end{array}\right\}\right) \\
&= \phi\left(\max\left\{ 0,0,0,0,S_b(B(x,y),B(x,y),\alpha), S_b(B(y,x),B(y,x),\beta),0,0 \right\}\right) \\
&\leq \phi\left(b\max\left\{ S_b(\alpha,\alpha,B(x,y)), S_b(\beta,\beta,B(y,x)) \right\}\right).
\end{aligned}$$

Similarly,

$$S_b(\beta,\beta,B(y,x)) \leq \phi\left(b\max\left\{ S_b(\alpha,\alpha,B(x,y)), S_b(\beta,\beta,B(y,x)) \right\}\right).$$

Thus,

$$\max\left\{ S_b(\alpha,\alpha,B(x,y)), S_b(\beta,\beta,B(y,x)) \right\} \leq \phi\left(b\max\left\{ S_b(\alpha,\alpha,B(x,y)), S_b(\beta,\beta,B(y,x)) \right\}\right).$$

It follows that $B(x,y) = \alpha = Q(x)$ and $B(y,x) = \beta = Q(y)$.

Since (B,Q) is w-compatible pair, we have $B(\alpha,\beta) = Q(\alpha)$ and $B(\beta,\alpha) = Q(\beta)$.

From Equation (4), we have

$$\begin{aligned}
S_b(\alpha,\alpha,B(\alpha,\beta)) &= S_b(A(\alpha,\beta),A(\alpha,\beta),B(\alpha,\beta)) \\
&\leq 2b^5 S_b(A(\alpha,\beta),A(\alpha,\beta),B(\alpha,\beta)) \\
&\leq \phi\left(\max\left\{\begin{array}{c} S_b(P(\alpha),P(\alpha),Q(\alpha)), S_b(P(\beta),P(\beta),Q(\beta)), \\ S_b(A(\alpha,\beta),A(\alpha,\beta),P(\alpha)), S_b(A(\beta,\alpha),A(\beta,\alpha),P(\beta)), \\ S_b(B(\alpha,\beta),B(\alpha,\beta),Q(\alpha)), S_b(B(\beta,\alpha),B(\beta,\alpha),Q(\beta)), \\ \frac{S_b(A(\alpha,\beta),A(\alpha,\beta),Q(\alpha)) S_b(B(\alpha,\beta),B(\alpha,\beta),P(\alpha))}{1+S_b(P(\alpha),P(\alpha),Q(\alpha))}, \\ \frac{S_b(A(\beta,\alpha),A(\beta,\alpha),Q(\beta)) S_b(B(\beta,\alpha),B(\beta,\alpha),P(\beta))}{1+S_b(P(\beta),P(\beta),Q(\beta))} \end{array}\right\}\right) \\
&= \phi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,B(\alpha,\beta)), S_b(\beta,\beta,B(\beta,\alpha)), \\ S_b(B(\alpha,\beta),B(\alpha,\beta),\alpha), S_b(B(\beta,\alpha),B(\beta,\alpha),\beta) \end{array}\right\}\right) \\
&\leq \phi\left(b\max\left\{ S_b(\alpha,\alpha,B(\alpha,\beta)), S_b(\beta,\beta,B(\beta,\alpha)) \right\}\right).
\end{aligned}$$

Similarly,

$$S_b(\beta,\beta,B(\beta,\alpha)) \leq \phi\left(b\max\left\{ S_b(\alpha,\alpha,B(\alpha,\beta)), S_b(\beta,\beta,B(\beta,\alpha)) \right\}\right).$$

Thus,

$$\max\left\{ S_b(\alpha,\alpha,B(\alpha,\beta)), S_b(\beta,\beta,B(\beta,\alpha)) \right\} \leq \phi\left(b\max\left\{ S_b(\alpha,\alpha,B(\alpha,\beta)), S_b(\beta,\beta,B(\beta,\alpha)) \right\}\right).$$

It follows that $B(\alpha,\beta) = \alpha = Q(\alpha)$ and $B(\beta,\alpha) = \beta = Q(\beta)$.

Therefore, (α, β) is a common coupled fixed point of A, B, P and Q.

To prove uniqueness, let us take that (α^1, β^1) is another common coupled fixed point of A, B, P and Q.

From Equation (4), we have

$$\begin{aligned} S_b(\alpha,\alpha,\alpha^1) &\leq 2b^5 S_b(\alpha,\alpha,\alpha^1) \\ &= 2b^5 S_b(A(\alpha,\beta), A(\alpha,\beta), B(\alpha^1,\beta^1)) \\ &\leq \phi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1), S_b(\alpha,\alpha,\alpha), \\ S_b(\beta,\beta,\beta), S_b(\alpha^1,\alpha^1,\alpha^1), S_b(\beta^1,\beta^1,\beta^1), \\ \dfrac{S_b(\alpha,\alpha,\alpha^1)S_b(\alpha^1,\alpha^1,\alpha)}{1+S_b(\alpha,\alpha,\alpha^1)}, \dfrac{S_b(\beta,\beta,\beta^1)S_b(\beta^1,\beta^1,\beta)}{1+S_b(\beta,\beta,\beta^1)} \end{array}\right\}\right) \\ &\leq \phi(b\max\{S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1)\}). \end{aligned}$$

Similarly,

$$S_b(\beta,\beta,\beta^1) \leq \phi(\max\{bS_b(\alpha,\alpha,\alpha^1), bS_b(\beta,\beta,\beta^1)\}).$$

Thus,

$$\max\left\{\ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1)\ \right\} \leq \phi\left(b\max\left\{\ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1)\ \right\}\right).$$

It follows that $\alpha = \alpha^1$ and $\beta = \beta^1$. Hence, (α, β) is a unique common coupled fixed point of A, B, P and Q.

□

Now, we give one example which support our main theoretical result.

Example 2. *Let $X = [0,1]$ and let $S_b : X \times X \times X \to \mathbb{R}^+$ be defined by $S_b(x,y,z) = (|y+z-2x| + |y-z|)^2$. Then, S_b is a S_b-metric space with $b = 4$. Define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = \dfrac{t}{4^6}$, $A, B : X \times X \to X$ and $P, Q : X \to X$ by $A(x,y) = \dfrac{x^2+y^2}{4^8}$, $B(x,y) = \dfrac{x^2+y^2}{4^9}$, $P(x) = \dfrac{x^2}{4}$ and $Q(x) = \dfrac{x^2}{16}$, respectively. Then, we have*

$$
\begin{aligned}
&2b^5 S_b(A(x,y), A(x,y), B(u,v)) \\
&= 2\left(4^5\right)(|A(x,y) + B(u,v) - 2A(x,y)| + |A(x,y) - B(u,v)|)^2 \\
&= 2\left(4^5\right)(2|A(x,y) - B(u,v)|)^2 \\
&= 2\left(4^6\right)(|A(x,y) - B(u,v)|)^2 \\
&= 2(4^6)\left|\frac{x^2+y^2}{4^8} - \frac{u^2+v^2}{4^9}\right|^2 \\
&= 2(4^6)\left|\frac{4x^2-u^2}{4^9} + \frac{4y^2-v^2}{4^9}\right|^2 \\
&= \frac{2\left(4^6\right)}{(4^6)^2}\left(\frac{1}{4}\left\{\left|\frac{4x^2-u^2}{16}\right| + \left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2 \\
&\leq \frac{2}{4\left(4^6\right)}\left(\frac{1}{2}\left\{\left|\frac{4x^2-u^2}{16}\right| + \left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2 \\
&\leq \frac{2}{4^7}\left(\max\left\{\left|\frac{4x^2-u^2}{16}\right|, \left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2 \\
&\leq \frac{2}{(4^7)4}\left(\max\left\{4\left|\frac{4x^2-u^2}{16}\right|, 4\left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2 \\
&= \frac{1}{2(4^7)}\max\{S_b(P(x), P(x), Q(u)), S_b(P(y), P(y), Q(v))\} \\
&\leq \frac{1}{2(4^7)}\max\left\{\begin{array}{c} S_b(P(x), P(x), Q(u)), S_b(P(y), P(y), Q(v)), S_b(A(x,y), A(x,y), P(x)), \\ S_b(A(y,x), A(y,x), P(y)), S_b(B(u,v), B(u,v), Q(u)), S_b(B(v,u), B(v,u), Q(v)), \\ \frac{S_b(A(x,y), A(x,y), Q(u))\ S_b(B(u,v), B(u,v), P(x))}{1+S_b(P(x), P(x), Q(u))}, \frac{S_b(A(y,x), A(y,x), Q(v))\ S_b(B(v,u), B(v,u), P(y))}{1+S_b(P(y), P(y), Q(v))} \end{array}\right\} \\
&< \phi\left(\max\left\{\begin{array}{c} S_b(P(x), P(x), Q(u)), S_b(P(y), P(y), Q(v)), S_b(A(x,y), A(x,y), P(x)), \\ S_b(A(y,x), A(y,x), P(y)), S_b(B(u,v), B(u,v), Q(u)), S_b(B(v,u), B(v,u), Q(v)), \\ \frac{S_b(A(x,y), A(x,y), Q(u))\ S_b(B(u,v), B(u,v), P(x))}{1+S_b(P(x), P(x), Q(u))}, \frac{S_b(A(y,x), A(y,x), Q(v))\ S_b(B(v,u), B(v,u), P(y))}{1+S_b(P(y), P(y), Q(v))} \end{array}\right\}\right).
\end{aligned}
$$

It is clear that all conditions of Theorem 1 are satisfied and $(0,0)$ is a unique common coupled fixed point of A, B, P and Q.

Putting $A = B = P = Q$ in Theorem 1, we obtain the next important result on unique fixed point.

Theorem 2. *Let (X, S_b) be a complete S_b-metric space. Suppose that $A : X \times X \to X$ satisfies condition* $2b^5 S_b(A(x,y), A(x,y), A(u,v))$

$$
\leq \phi\left(\max\left\{\begin{array}{c} S_b(x,x,u), S_b(y,y,v), S_b(A(x,y), A(x,y), x), \\ S_b(A(y,x), A(y,x), y), S_b(A(u,v), A(u,v), u), S_b(A(v,u), A(v,u), v), \\ \frac{S_b(A(x,y), A(x,y), u)\ S_b(B(u,v), B(u,v), x)}{1+S_b(x,x,u)}, \frac{S_b(A(y,x), A(y,x), v)\ S_b(B(v,u), B(v,u), y)}{1+S_b(y,y,v)} \end{array}\right\}\right)
$$

for all $x, y, u, v \in X$, $\phi \in \Phi$. Then, A has a unique coupled fixed point in $X \times X$.

3. Application

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 2. Consider the initial value problem:

$$
x^1(t) = f(t, x(t), x(t)), \quad t \in I = [0,1], \quad x(0) = x_0 \tag{10}
$$

where $f : I \times \left[\frac{x_0}{4}, \infty\right) \times \left[\frac{x_0}{4}, \infty\right) \to \left[\frac{x_0}{4}, \infty\right)$ and $x_0 \in \mathbb{R}$.

Theorem 3. *Consider the initial value problem in Equation (10) with* $f \in C\left(I \times \left[\frac{x_0}{4}, \infty\right) \times \left[\frac{x_0}{4}, \infty\right)\right)$ *and*

$$\int_0^t f(s, x(s), y(s))ds = \frac{1}{\sqrt{5}} \min\left\{ \int_0^t f(s, x(s), x(s))ds, \int_0^t f(s, y(s), y(s))ds \right\}.$$

Then, there exists a unique solution in $C\left(I \times \left[\frac{x_0}{4}, \infty\right) \times \left[\frac{x_0}{4}, \infty\right)\right)$ *for the initial value problem in Equation (10).*

Proof of Theorem. The integral equation corresponding to the initial value problem in Equation (10) is

$$x(t) = x_0 + \int_0^t f(s, x(s), x(s))ds.$$

Let $X = C\left(I \times \left[\frac{x_0}{4}, \infty\right) \times \left[\frac{x_0}{4}, \infty\right)\right)$ and let $S(x, y, z) = (|y + z - 2x| + |y - z|)^2$ *for* $x, y \in X$. Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{4t}{5}$ and $A : X \times X \to X$ by

$$A(x, y)(t) \quad = \quad x_0 + \int_0^t f(s, x(s), y(s))ds. \tag{11}$$

Now, we have

$$\begin{aligned}
&S\left(A\left(x,y\right)\left(t\right), A\left(x,y\right)\left(t\right), A\left(u,v\right)\left(t\right)\right) \\
&= \{|A(x,y)(t) + A(u,v)(t) - 2A(x,y)(t)| + |A(x,y)(t) - A(u,v)(t)|\}^2 \\
&= 4\ |A(x,y)(t) - A(u,v)(t)|^2 \\
&= 4 \left| \int_0^t f(s, x(s), y(s))ds - \int_0^t f(s, u(s), v(s))ds \right|^2 \\
&= 4 \left| \frac{1}{\sqrt{5}} \min \left\{ \begin{array}{l} \int_0^t f(s, x(s), x(s))ds, \\ \int_0^t f(s, y(s), y(s))ds \end{array} \right\} - \frac{1}{\sqrt{5}} \min \left\{ \begin{array}{l} \int_0^t f(s, u(s), u(s))ds, \\ \int_0^t f(s, v(s), v(s))ds \end{array} \right\} \right|^2 \\
&\leq \frac{4}{5} \left| \max \left\{ \begin{array}{l} \int_0^t f(s, x(s), x(s))ds - \int_0^t f(s, u(s), u(s))ds, \\ \int_0^t f(s, y(s), y(s))ds - \int_0^t f(s, v(s), v(s))ds \end{array} \right\} \right|^2 \\
&= \frac{4}{5} \max \left\{ \begin{array}{l} \left| \int_0^t f(s, x(s), x(s))ds - \int_0^t f(s, u(s), u(s))ds \right|^2, \\ \left| \int_0^t f(s, y(s), y(s))ds - \int_0^t f(s, v(s), v(s))ds \right|^2 \end{array} \right\} \\
&= \frac{1}{5} \max\left\{ 4 \mid x(t) - u(t) \mid^2, 4 \mid y(t) - v(t) \mid^2 \right\} \\
&= \frac{1}{5} \max\left\{S(x, x, u), S(y, y, v)\right\} \\
&\leq \phi(M(x, u, y, v)).
\end{aligned}$$

Hence, from Theorem 2, we conclude that A has a unique coupled fixed point in X. □

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