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A Proposal for Revisiting Banach and Caristi Type Theorems in b -Metric Spaces

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Abstract: In this paper, we revisit the renowned fixed point theorems belongs to Caristi and Banach. We propose a new fixed point theorem which is inspired from both Caristi and Banach. We also consider an example to illustrate our result.

Keywords: b -metric; Banach fixed point theorem; Caristi fixed point theorem

MSC: 46T99; 47H10; 54H25

1. Introduction and Preliminaries

In fixed point theory, the approaches of the renowned results of Caristi [1] and Banach [2] are quite different and the structures of the corresponding proofs varies. In this short note, we propose a new fixed point theorem that is inspired from these two famous results.

We aim to present our results in the largest framework, b -metric space, instead of standard metric space. The concept of b -metric has been discovered several times by different authors with distinct names, such as quasi-metric, generalized metric and so on. On the other hand, this concept became popular after the interesting papers of Bakhtin [3] and Czerwik [4]. For more details in b -metric space and advances in fixed point theory in the setting of b -metric spaces, we refer e.g., [5–17].

Definition 1. Let X be a nonempty set and $s \geq 1$ be a real number. We say that $d : X \times X \rightarrow [0, 1)$ is a b -metric with coefficient s when, for each $x, y, z \in X$,

$$(b1) \quad d(x, y) = d(y, x);$$

$$(b2) \quad d(x, y) = 0 \text{ if and only if } x = y;$$

$$(b3) \quad d(x, z) \leq s[d(x, y) + d(y, z)] \text{ (Expanded triangle inequality).}$$

In this case, the triple (X, d, s) is called a b -metric space with coefficient s .

The classical examples and crucial examples of b -metric spaces are $l^p(\mathbb{R})$ and $L^p[0, 1]$, $p \in (0, 1)$.

The topological notions (such as, convergence, Cauchy criteria, completeness, and so on) are defined by verbatim of the corresponding notions for standard metric. On the other hand, we should underline the fact that b -metric does need to be continuous, for certain details, see e.g., [3,4].

We recollect the following basic observations here.

Lemma 1. [14] For a sequence $(\theta_n)_{n \in \mathbb{N}}$ in a b -metric space (X, d, s) , there exists a constant $\gamma \in [0, 1)$ such that

$$d(\theta_{n+1}, \theta_n) \leq \gamma d(\theta_n, \theta_{n-1}), \text{ for all } n \in \mathbb{N}.$$

Then, the sequence $(\theta_n)_{n \in \mathbb{N}}$ is fundamental (Cauchy).

The aim of this paper is to correlate the Banach type fixed point result with Caristi type fixed point results in b -metric spaces.

2. Main Result

Theorem 1. Let (X, d, s) be a complete metric space and $T : X \rightarrow X$ be a map. Suppose that there exists a function $\varphi : X \rightarrow \mathbb{R}$ with

- (i) φ is bounded from below ($\inf \varphi(X) > -\infty$),
- (ii) $d(x, Tx) > 0$ implies $d(Tx, Ty) \leq (\varphi(x) - \varphi(Tx))d(x, y)$.

Then, T has at least one fixed point in X .

Proof. Let $\theta_0 \in X$. If $T\theta_0 = \theta_0$, the proof is completed. Herewith, we assume $d(\theta_0, T\theta_0) > 0$. Without loss of generality, keeping the same argument in mind, we assume that $\theta_{n+1} = T\theta_n$ and hence

$$d(\theta_n, \theta_{n+1}) = d(\theta_n, T\theta_n) > 0. \quad (1)$$

For that sake of convenience, suppose that $a_n = d(\theta_n, \theta_{n-1})$. From (ii), we derive that

$$\begin{aligned} a_{n+1} &= d(\theta_n, \theta_{n+1}) = d(T\theta_{n-1}, T\theta_n) \\ &\leq (\varphi(\theta_{n-1}) - \varphi(T\theta_{n-1}))d(\theta_{n-1}, \theta_n) \\ &= (\varphi(\theta_{n-1}) - \varphi(\theta_n))a_n. \end{aligned}$$

So we have,

$$0 < \frac{a_{n+1}}{a_n} \leq \varphi(\theta_{n-1}) - \varphi(\theta_n) \text{ for each } n \in \mathbb{N}.$$

Thus the sequence $\{\varphi(\theta_n)\}$ is necessarily positive and non-increasing. Hence, it converges to some $r \geq 0$. On the other hand, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^n \frac{a_{k+1}}{a_k} &\leq \sum_{k=1}^n (\varphi(\theta_{k-1}) - \varphi(\theta_k)) \\ &= (\varphi(\theta_0) - \varphi(\theta_1)) + (\varphi(\theta_1) - \varphi(\theta_2)) + \dots + (\varphi(\theta_{n-1}) - \varphi(\theta_n)) \\ &= \varphi(\theta_0) - \varphi(\theta_n) \rightarrow \varphi(\theta_0) - r < \infty, \text{ as } n \rightarrow \infty. \end{aligned}$$

It means that

$$\sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} < \infty.$$

Accordingly, we have

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0. \quad (2)$$

On account of (2), for $\gamma \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \leq \gamma, \quad (3)$$

for all $n \geq n_0$. It yields that

$$d(\theta_{n+1}, \theta_n) \leq \gamma d(\theta_n, \theta_{n-1}), \quad (4)$$

for all $n \geq n_0$. Now using Lemma 1 we obtain that the sequence $\{\theta_n\}$ converges to some $\omega \in X$. We claim that ω is the fixed point of T . Employing assumption (ii) of the theorem, we find that

$$\begin{aligned} d(\omega, T\omega) &\leq s[d(\omega, \theta_{n+1}) + d(\theta_{n+1}, T\omega)] \\ &\leq s[d(\omega, \theta_{n+1}) + (\varphi(\theta_n) - \varphi(\omega))d(\theta_n, \omega)] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Consequently, we obtain $d(\omega, T\omega) = 0$, that is, $T\omega = \omega$. \square

From Theorem 1, we get the corresponding result for complete metric spaces. The following example shows that the Theorem 1 is not a consequence of Banach's contraction principle.

Example 1. Let $X = \{0, 1, 2\}$ endowed with the following metric:

$$d(0, 1) = 1, d(2, 0) = 1, d(1, 2) = \frac{3}{2} \text{ and } d(a, a) = 0, \text{ for all } a \in X, \quad d(a, b) = d(b, a), \text{ for all } a, b \in X.$$

Let $T(0) = 0, T(1) = 2, T(2) = 0$. Define $\varphi : X \rightarrow [0, \infty)$ as $\varphi(2) = 2, \varphi(0) = 0, \varphi(1) = 4$. Thus for all $x \in X$ such that $d(x, Tx) > 0$, (in this example, $x \neq 0$), we have

$$\begin{aligned} d(T1, T2) &\leq (\varphi(1) - \varphi(T(1)))d(2, 1), \\ d(T2, T1) &\leq (\varphi(2) - \varphi(T(2)))d(2, 1), \\ d(T1, T0) &\leq (\varphi(1) - \varphi(T(1)))d(1, 0), \\ d(T2, T0) &\leq (\varphi(2) - \varphi(T(2)))d(2, 0). \end{aligned}$$

Thus the mapping T satisfies our condition and also has a fixed point. Note that $d(T1, T0) = d(1, 0)$. Thus, it does not satisfy the Banach contraction principle.

Remark 1.

1. From Example 1, it follows that Theorem 1 (over metric spaces) is not a consequence of the Banach contraction principle.
2. Question for further study: It is natural to ask if the Banach contraction principle is a consequence of Theorem 1 (over metric spaces).

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