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Advances in the Semilocal Convergence of Newton's Method with Real-World Applications

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Abstract: The aim of this paper is to present a new semi-local convergence analysis for Newton's method in a Banach space setting. The novelty of this paper is that by using more precise Lipschitz constants than in earlier studies and our new idea of restricted convergence domains, we extend the applicability of Newton's method as follows: The convergence domain is extended; the error estimates are tighter and the information on the location of the solution is at least as precise as before. These advantages are obtained using the same information as before, since new Lipschitz constant are tighter and special cases of the ones used before. Numerical examples and applications are used to test favorable the theoretical results to earlier ones.

Keywords: Banach space; Newton's method; semi-local convergence; Kantorovich hypothesis

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution z^* of equation

$$G(x) = 0, \tag{1}$$

where G is a Fréchet-differentiable operator defined on a nonempty, open convex subset D of a Banach space E_1 with values in a Banach space E_2.

Many problems in Computational disciplines such us Applied Mathematics, Optimization, Mathematical Biology, Chemistry, Economics, Medicine, Physics, Engineering and other disciplines can be solved by means of finding the solutions of equations in a form like Equation (1) using Mathematical Modelling [1–7]. The solutions of this kind of equations are rarely found in closed form. That is why most solutions of these equations are given using iterative methods. A very important problem in the study of iterative procedures is the convergence region. In general this convergence region is small. Therefore, it is important to enlarge the convergence region without additional hypotheses.

The study of convergence of iterative algorithms is usually centered into two categories: Semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of theses algorithms while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls.

Newton's method defined for all $n = 0, 1, 2, \dots$ by

$$z_{n+1} = z_n - G'(z_n)^{-1}G(z_n), \tag{2}$$

is undoubtedly the most popular method for generating a sequence $\{z_n\}$ approximating z^*, where z_0 is an initial point. There is a plethora of convergence results for Newton's method [1–4,6,8–14]. We shall increase the convergence region by finding a more precise domain where the iterates $\{z_n\}$ lie leading to smaller Lipschitz constants which in turn lead to a tighter convergence analysis for Newton's method than before. This technique can apply to improve the convergence domain of other iterative methods in an analogous way.

Let us consider the conditions:

There exist $z_0 \in \Omega$ and $\eta \geq 0$ such that

$$G'(z_0)^{-1} \in \mathbb{L}(E_2, E_1) \text{ and } \|G'(z_0)^{-1}G(z_0)\| \leq \eta;$$

There exists $T \geq 0$ such that the Lipschitz condition

$$\|G'(z_0)^{-1}(G'(x) - G'(y))\| \leq T\|x - y\|$$

holds for all $x, y \in \Omega$.

Then, the sufficient convergence condition for Newton's method is given by the famous for its simplicity and clarity Kantorovich sufficient convergence criterion for Newton's method

$$h_K = 2T\eta \leq 1. \tag{3}$$

Let us consider a motivational and academic example to show that this condition is not satisfied. Choose $E_1 = E_2 = \mathbb{R}$, $z_0 = 1$, $p \in [0, 0.5)$, $D = S(z_0, 1 - p)$ and define function G on D by

$$G(x) = z^3 - p.$$

Then, we have $T = 2(2 - p)$. Then, the Kantorovich condition is not satisfied, since $h_K > 1$ for all $p \in (0, 0.5)$. We set $I_K = \emptyset$ to be the set of point satisfying Equation (3). Hence, there is no guarantee that Newton's sequence starting at z_0 converges to $z^* = \sqrt[3]{p}$.

The rest of the paper is structured as follows: In Section 2 we present the semi-local convergence analysis of Newton's method Equation (2). The numerical examples and applications are presented in Section 3 and the concluding Section 4.

2. Semi-Local Convergence Analysis

We need an auxiliary result on majorizing sequences for Newton's method.

Lemma 1. *Let $H > 0$, $K > 0$, $L > 0$, $L_0 > 0$ and $\eta > 0$ be parameters. Suppose that:*

$$h_4 = L_4\eta \leq 1, \tag{4}$$

where

$$L_4^{-1} = \begin{cases} \dfrac{1}{L_0 + H}, \text{ if } b = LK + 2\delta L_0(K - 2H) = 0 \\[2ex] 2\dfrac{-\delta(L_0 + H) + \sqrt{\delta^2(L_0 + H)^2 + \delta(LK + 2\delta L_0(K - 2H))}}{LK + 2\delta L_0(K - 2H)}, \text{ if } b > 0 \\[2ex] -2\dfrac{\delta(L_0 + H) + \sqrt{\delta^2(L_0 + H)^2 + \delta(LK + 2\delta L_0(K - 2H))}}{LK + 2\delta L_0(K - 2H)}, \text{ if } b < 0 \end{cases}$$

and

$$\delta = \frac{2L}{L + \sqrt{L^2 + 8L_0 L}}.$$

holds. Then, scalar sequence $\{t_n\}$ *given by*

$$\begin{aligned} & t_0 = 0, \quad t_1 = \eta, \quad t_2 = t_1 + \tfrac{K\,(t_1 - t_0)^2}{2\,(1 - H\,t_1)}, \\ & t_{n+2} = t_{n+1} + \frac{L\,(t_{n+1} - t_n)^2}{2\,(1 - L_0\,t_{n+1})} \quad for\ all \quad n = 1, 2, \cdots, \end{aligned} \tag{5}$$

is well defined, increasing, bounded from above by

$$t^{**} = \eta + \left(1 + \frac{\delta_0}{1 - \delta}\right) \frac{K\,\eta^2}{2\,(1 - H\,\eta)} \tag{6}$$

and converges to its unique least upper bound t^* *which satisfies*

$$t_2 \le t^* \le t^{**}, \tag{7}$$

where $\delta_0 = \dfrac{L(t_2 - t_1)}{2(1 - L_0 t_2)}$. *Moreover, the following estimates hold:*

$$0 < t_{n+2} - t_{n+1} \le \delta_0\,\delta^{n-1}\,\frac{K\,\eta^2}{2\,(1 - H\,\eta)} \quad for\ all \quad n = 1, 2, \cdots \tag{8}$$

and

$$t^* - t_n \le \frac{\delta_0\,(t_2 - t_1)}{1 - \delta}\,\delta^{n-2} \quad for\ all \quad n = 2, 3, \cdots. \tag{9}$$

Proof. By induction, we show that

$$0 < \frac{L\,(t_{k+1} - t_k)}{2\,(1 - L_0\,t_{k+1})} \le \delta \tag{10}$$

holds for all $k = 1, 2, \cdots$. Estimate Equation (10) is true for $k = 1$ by Equation (4). Then, we have by Equation (5)

$$\begin{aligned} 0 < t_3 - t_2 \le \delta_0\,(t_2 - t_1) \quad & \Longrightarrow \quad t_3 \le t_2 + \delta_0\,(t_2 - t_1) \\ & \Longrightarrow \quad t_3 \le t_2 + (1 + \delta_0)\,(t_2 - t_1) - (t_2 - t_1) \\ & \Longrightarrow \quad t_3 \le t_1 + \tfrac{1 - \delta_0^2}{1 - \delta_0}\,(t_2 - t_1) < t^{**} \end{aligned}$$

and for $m = 2, 3, \cdots$

$$\begin{aligned} t_{m+2} & \le t_{m+1} + \delta_0\,\delta^{m-1}\,(t_2 - t_1) \\ & \le t_m + \delta_0\,\delta^{m-2}\,(t_2 - t_1) + \delta_0\,\delta^{m-1}\,(t_2 - t_1) \\ & \le t_1 + (1 + \delta_0\,(1 + \delta + \cdots + \delta^{m-1}))\,(t_2 - t_1) \\ & = t_1 + (1 + \delta_0\,\tfrac{1 - \delta^m}{1 - \delta})\,(t_2 - t_1) \le t^{**}. \end{aligned}$$

Assume that Equation (10) holds for all natural integers $n \leq m$. Then, we get by Equations (5) and (10) that

$$0 < t_{m+2} - t_{m+1} \leq \delta_0\, \delta^{m-1}\, (t_2 - t_1) \leq \delta^m\, (t_2 - t_1)$$

and

$$t_{m+2} \leq t_1 + (1 + \delta_0\, \frac{1 - \delta^m}{1 - \delta})\, (t_2 - t_1) \leq t_1 + \frac{1 - \delta^{m+1}}{1 - \delta}\, (t_2 - t_1) < t^{**}.$$

Evidently estimate Equation (10) is true, if m is replaced by $m + 1$ provided that

$$\frac{L}{2}\, (t_{m+2} - t_{m+1}) \leq \delta\, (1 - L_0\, t_{m+2})$$

or

$$\frac{L}{2}\, (t_{m+2} - t_{m+1}) + \delta\, L_0\, t_{m+2} - \delta \leq 0$$

or

$$\frac{L}{2}\, \delta^m\, (t_2 - t_1) + \delta\, L_0 \left(t_1 + \frac{1 - \delta^{m+1}}{1 - \delta}\, (t_2 - t_1) \right) - \delta \leq 0. \tag{11}$$

Estimate Equation (11) motivates us to define recurrent functions $\{\psi_k\}$ on $[0, 1)$ by

$$\psi_m(s) = \frac{L}{2}\, (t_2 - t_1)\, t^{m+1} + s\, L_0\, (1 + s + t^2 + \cdots + t^m)\, (t_2 - t_1) - (1 - L_0\, t_1)\, s.$$

We need a relationship between two consecutive functions ψ_k. We get that

$$\begin{aligned}
\psi_{m+1}(s) &= \tfrac{L}{2}\, (t_2 - t_1)\, t^{m+2} + s\, L_0\, (1 + s + t^2 + \cdots + t^{m+1})\, (t_2 - t_1) \\
&\quad -(1 - L_0\, t_1)\, s \\
&= \tfrac{L}{2}\, (t_2 - t_1)\, t^{m+2} + s\, L_0\, (1 + s + t^2 + \cdots + t^{m+1})\, (t_2 - t_1) \\
&\quad -(1 - L_0\, t_1)\, s - \tfrac{L}{2}\, (t_2 - t_1)\, t^m \\
&\quad -s\, L_0\, (1 + s + t^2 + \cdots + t^m)\, (t_2 - t_1) + (1 - L_0\, t_1)\, s + \psi_k(s).
\end{aligned}$$

Therefore, we deduce that

$$\psi_{m+1}(s) = \psi_m(s) + \frac{1}{2}\, (2\, L_0\, t^2 + L\, s - L)\, t^m\, (t_2 - t_1). \tag{12}$$

Estimate Equation (11) is satisfied, if

$$\psi_m(\delta) \leq 0 \quad \text{holds for all} \quad m = 1, 2, \cdots. \tag{13}$$

Using Equation (12) we obtain that

$$\psi_{m+1}(\delta) = \psi_m(\delta) \quad \text{for all} \quad m = 1, 2, \cdots.$$

Let us now define function ψ_∞ on $[0, 1)$ by

$$\psi_\infty(s) = \lim_{m \to \infty} \psi_m(s). \tag{14}$$

Then, we have by Equation (14) and the choice of δ that

$$\psi_\infty(\delta) = \psi_m(\delta) \quad \text{for all} \quad m = 1, 2, \cdots.$$

Hence, Equation (13) is satisfied, if

$$\psi_\infty(\delta) \leq 0. \tag{15}$$

Using Equation (11) we get that

$$\psi_\infty(\delta) = \left(\frac{L_0}{1-\delta} (t_2 - t_1) + L_0 t_1 - 1 \right) \delta. \tag{16}$$

It then, follows from Equations (2.1) and (2.13) that Equation (15) is satisfied. The induction is now completed. Hence, sequence $\{t_n\}$ is increasing, bounded from above by t^{**} given by Equation (6), and as such it converges to its unique least upper bound t^* which satisfies Equation (7). $\square$

Let $S(z, \varrho)$, $\bar{S}(z, \varrho)$ stand, respectively for the open and closed ball in E_1 with center $z \in E_1$ and of radius $\varrho > 0$.

The conditions (A) for the semi-local convergence are:

(A_1) $G : D \subset E_1 \to E_2$ is Fréchet differentiable and there exist $z_0 \in D$, $\eta \geq 0$ such that $G'(z_0)^{-1} \in$ Ł(E_2, E_1) and
$$\|G'(z_0)^{-1} G(z_0)\| \leq \eta.$$

(A_2) There exists $L_0 > 0$ such that for all $x \in D$
$$\|G'(z_0)^{-1}(G'(x) - G'(z_0))\| \leq L_0 \|x - z_0\|.$$

(A_3) $L_0 \eta < 1$ and there exists $L > 0$ such that
$$\|G'(z_0)^{-1}(G'(x) - G'(y))\| \leq L \|x - y\|.$$
for all $x, y \in D_0 := S(z_1, \frac{1}{L_0} - \|G'(z_0)^{-1} G(z_0)\|) \cap D$.

(A_4) There exists $H > 0$ such that
$$\|G'(z_0)^{-1}(G'(z_1) - G'(z_0))\| \leq H \|z_1 - z_0\|,$$
where $z_1 = z_0 - G'(z_0)^{-1} G(z_0)$.

(A_5) There exists $K > 0$ such that for all $\theta \in [0, 1]$
$$\|G'(z_0)^{-1}(G'(z_0 + \theta(z_1 - z_0)) - G'(z_0))\| \leq K\theta \|z_1 - z_0\|.$$

Notice that $(A_2) \implies (A_3) \implies (A_5) \implies (A_4)$. Clearly, we have that

$$H \leq K \leq L_0 \tag{17}$$

and $\frac{L}{L_0}$ can be arbitrarily large [9]. It is worth noticing that $(A_3) - (A_5)$ are not additional to (A_2) hypotheses, since in practice the computation of Lipschitz constant T requires the computation of the other constants as special cases.

Next, first we present a semi-local convergence result relating majorizing sequence $\{t_n\}$ with Newton's method and hypotheses (A).

Theorem 1. *Suppose that hypotheses* (A)*, hypotheses of Lemma 1 and* $\overline{S}(z_0, t^*) \subseteq D$ *hold, where* t^* *is given in Lemma 1. Then, sequence* $\{z_n\}$ *generated by Newton's method is well defined, remains in* $\overline{S}(z_0, t^*)$ *and converges to a solution* $z^* \in \overline{S}(z_0, t^*)$ *of equation* $G(x) = 0$*. Moreover, the following estimates hold*

$$\|z_{n+1} - z_n\| \le t_{n+1} - t_n \tag{18}$$

and

$$\|z_n - z^*\| \le t^* - t_n \quad for\ all \quad n = 0, 1, 2, \cdots, \tag{19}$$

where sequence $\{t_n\}$ *is given in Lemma 1. Furthermore, if there exists* $R \ge t^*$ *such that*

$$\overline{S}(z_0, R) \subseteq D \quad \text{and} \quad L_0\,(t^* + R) < 2,$$

then, the solution z^* *of equation* $G(x) = 0$ *is unique in* $\overline{S}(z_0, R)$*.*

Proof. We use mathematical induction to prove that

$$\|z_{k+1} - x_k\| \le t_{k+1} - t_k \tag{20}$$

and

$$\overline{S}(z_{k+1}, t^* - t_{k+1}) \subseteq \overline{S}(z_k, t^* - t_k) \quad \text{for all} \quad k = 1, 2, \cdots. \tag{21}$$

Let $z \in \overline{S}(z_1, t^* - t_1)$.

Then, we obtain that

$$\|z - z_0\| \le \|z - z_1\| + \|z_1 - z_0\| \le t^* - t_1 + t_1 - t_0 = t^* - t_0,$$

which implies $z \in \overline{S}(z_0, t^* - t_0)$. Note also that

$$\|z_1 - z_0\| = \|G'(z_0)^{-1}\, G(z_0)\| \le \eta = t_1 - t_0.$$

Hence, estimates Equations (20) and (21) hold for $k = 0$. Suppose these estimates hold for $n \le k$. Then, we have that

$$\|z_{k+1} - z_0\| \le \sum_{i=1}^{k+1} \|z_i - z_{i-1}\| \le \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1}$$

and

$$\|z_k + \theta\,(z_{k+1} - z_k) - z_0\| \le t_k + \theta\,(t_{k+1} - t_k) \le t^*$$

for all $\theta \in (0, 1)$. Using Lemma 1 and the induction hypotheses, we get in turn that

$$\|G'(z_0)^{-1}(G'(z_{k+1}) - G'(z_0))\| \le M\|x_{k+1} - z_0\| \le M(t_{k+1} - t_0) \le M t_{k+1} < 1, \tag{22}$$

where

$$M = \begin{cases} H & \text{if} \quad k = 0 \\ L_0 & \text{if} \quad k = 1, 2, \cdots. \end{cases}$$

It follows from Equation (22) and the Banach lemma on invertible operators that $G'(z_{m+1})^{-1}$ exists and

$$\|G'(z_{k+1})^{-1}\, G'(z_0)\| \le (1 - M\,\|z_{k+1} - z_0\|)^{-1} \le (1 - M\,t_{k+1})^{-1}. \tag{23}$$

Using iteration of Newton's method, we obtain the approximation

$$\begin{aligned} G(z_{k+1}) &= G(z_{k+1}) - G(z_k) - G'(z_k)\,(z_{k+1}-z_k) \\ &= \int_0^1 (G'(z_k+\theta\,(z_{k+1}-z_k)) - G'(z_m))\,(z_{k+1}-z_k)\,d\theta. \end{aligned} \tag{24}$$

Then, by Equation (24) we get in turn

$$\begin{aligned} &\|G'(z_0)^{-1}\,G(z_{k+1})\| \\ &\leq \int_0^1 \|G'(z_0)^{-1}\,(G'(z_k+\theta\,(z_{k+1}-z_k)) - G'(z_k))\|\,\|z_{k+1}-z_k\|\,d\theta \\ &\leq M_1 \int_0^1 \|\theta\,(z_{k+1}-z_k)\|\,\|z_{k+1}-z_k\|\,d\theta \leq \frac{M_1}{2}\,(t_{k+1}-t_k))^2, \end{aligned} \tag{25}$$

where

$$M_1 = \begin{cases} K & \text{if} \quad k=0 \\ L & \text{if} \quad k=1,2,\cdots. \end{cases}$$

Moreover, by iteration of Newton's method, Equations (23) and (25) and the induction hypotheses we get that

$$\begin{aligned} \|z_{k+2}-z_{k+1}\| &= \|(G'(z_{k+1})^{-1}\,G'(z_0))\,(G'(z_0)^{-1}\,G(z_{k+1}))\| \\ &\leq \|G'(z_{k+1})^{-1}\,G'(z_0)\|\,\|G'(z_0)^{-1}\,G(z_{k+1})\| \\ &\leq \frac{\frac{M_1}{2}\,(t_{k+1}-t_k)^2}{1-M\,t_{k+1}} = t_{k+2}-t_{k+1}. \end{aligned}$$

That is, we showed Equation (20) holds for all $k \geq 0$. Furthermore, let $z \in \overline{S}(z_{k+2}, t^* - t_{k+2})$. Then, we have that

$$\begin{aligned} \|z - x_{k+1}\| &\leq \|z - z_{k+2}\| + \|z_{k+2} - z_{k+1}\| \\ &\leq t^* - t_{k+2} + t_{k+2} - t_{k+1} = t^* - t_{k+1}. \end{aligned}$$

That is, $z \in \overline{S}(z_{k+1}, t^* - t_{k+1})$. The induction for Equations (20) and (21) is now completed. Lemma 1 implies that sequence $\{s_n\}$ is a complete sequence. It follows from Equations (20) and (21) that $\{z_n\}$ is also a complete sequence in a Banach space E_1 and as such it converges to some $z^* \in \overline{S}(z_0, t^*)$ (since $\overline{S}(z_0, t^*)$ is a closed set). By letting $k \longrightarrow \infty$ in Equation (25) we get $G(^*) = 0$. Estimate Equation (19) is obtained from Equation (18) (cf. [4,6,12]) by using standard majorization techniques. The proof for the uniqueness part has been given in [9]. □

The sufficient convergence criteria for Newton's method using the conditions (A), constants L, L_0 and η given in affine invariant form are:

- Kantorovich [6]

$$h_K = 2T\eta \leq 1. \tag{26}$$

- Argyros [9]

$$h_1 = (L_0 + T)\eta \leq 1. \tag{27}$$

- Argyros [3]

$$h_2 = \frac{1}{4}\left(T + 4L_0 + \sqrt{T^2 + 8L_0T}\right)\eta \leq 1 \tag{28}$$

- Argyros [11]

$$h_3 = \frac{1}{4}\left(4L_0 + \sqrt{L_0 T + 8L_0^2} + \sqrt{L_0 T}\right)\eta \leq 1 \tag{29}$$

- Argyros [12]

$$\begin{aligned} h_4 &= \tilde{L}_4 \eta \leq 1, \\ \tilde{L}_4 &= L_4(T), \quad \delta = \delta(T). \end{aligned} \tag{30}$$

If $H = K = L_0 = L$, then Equations (27)–(30) coincide with Equations (26). If $L_0 < T$, then $L < T$

$$h_K \leq 1 \Rightarrow h_1 \leq 1 \Rightarrow h_2 \leq 1 \Rightarrow h_3 \leq 1 \Rightarrow h_4 \leq 1 \Rightarrow h_5 \leq 1,$$

but not vice versa. We also have that for $\frac{L_0}{T} \to 0$:

$$\begin{aligned} &\frac{h_1}{h_K} \to \frac{1}{2}, \quad \frac{h_2}{h_K} \to \frac{1}{4}, \quad \frac{h_2}{h_1} \to \frac{1}{2} \\ &\frac{h_3}{h_K} \to 0, \quad \frac{h_3}{h_1} \to 0, \quad \frac{h_3}{h_2} \to 0 \end{aligned} \tag{31}$$

Conditions Equations (31) show by how many times (at most) the better condition improves the less better condition.

Remark 1. **(a)** *The majorizing sequence $\{t_n\}$, t^*, t^{**} given in [12] under conditions (A) and Equation (29) is defined by*

$$\begin{aligned} &t_0 = 0, \quad t_1 = \eta, \quad t_2 = t_1 + \frac{L_0(t_1 - t_0)^2}{2(1 - L_0 t_1)} \\ &t_{n+2} = t_{n+1} + \frac{T(t_{n+1} - t_n)^2}{2(1 - L_0 t_{n+1})}, \quad n = 1, 2, \ldots \\ &t^* = \lim_{n\to\infty} t_n \leq t^{**} = \eta + \frac{L_0 \eta^2}{2(1-\delta)(1 - L_0\eta)}. \end{aligned} \tag{32}$$

Using a simple inductive argument and Equation (32) we get for $L_1 < L$ that

$$t_n < t_{n-1}, n = 3, 4, \ldots, \tag{33}$$

$$t_{n+1} - t_n < t_n - t_{n-1}, n = 2, 3, \ldots, \tag{34}$$

and

$$t^* \leq t^{**} \tag{35}$$

Estimates for Equations (5)–(7) show the new error bounds are more precise than the old ones and the information on the location of the solution z^* is at least as precise as already claimed in the abstract of this study (see also the numerical examples). Clearly the new majorizing sequence $\{t_n\}$ is more precise than the corresponding ones associated with other conditions.

(b) *Condition $\bar{S}(z_0, t^*) \subseteq D$ can be replaced by $S(z_0, \frac{1}{L_0})$ (or D_0). In this case condition $(A_2)'$ holds for all $x, y \in S(z_0, \frac{1}{L_0})$ (or D_0).*

(c) *If $L_0\eta \leq 1$, then, we have that $z_0 \in \bar{S}(z_1, \frac{1}{L_0} - \|G'(z_0)^{-1}G(z_0)\|)$, since $\bar{S}(z_1, \frac{1}{L_0} - \|G'(z_0)^{-1}G(z_0)\|) \subseteq S(z_0, \frac{1}{L_0})$.*

3. Numerical Examples

Example 1. *Returning back to the motivational example, we have* $L_0 = 3 - p$.

Conditions Equations (27)–(29) are satisfied, respectively for

$$p \in I_1 := [0.494816242, 0.5),$$

$$p \in I_2 := [0.450339002, 0.5)$$

and

$$p \in I_3 := [0.4271907643, 0.5).$$

We are now going to consider such an initial point which previous conditions cannot be satisfied but our new criteria are satisfied. That is, the improvement that we get with our new weaker criteria.

We get that

$$H = \frac{5+p}{3},$$

$$K = 2,$$

$$L = \frac{2}{3(3-p)}(-2p^2 + 5p + 6).$$

Using this values we obtain that condition Equation (4) is satisfied for $p \in [0.0984119, 0.5)$*, However, must also have that*

$$L_0 \eta < 1$$

which is satisfied for $p \in I_4 := (0, 0.5]$*. That is, we must have* $p \in I_4$*, so there exist numerous values of* p *for which the previous conditions cannot guarantee the convergence but our new ones can. Notice that we have*

$$I_K \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4$$

Hence, the interval of convergence cannot be improved further under these conditions. Notice that the convergence criterion is even weaker than the corresponding one for the modified Newton's method given in [11] by $L_0(\eta) < 0.5$.

For example, we choose different values of p *and we see in Table 1.*

Table 1. Convergence of Newton's method choosing $z_0 = 1$, for different values of p.

p	**0.41**	**0.43**	**0.45**
z_1	0.803333	0.810000	0.816667
z_2	0.747329	0.758463	0.769351
z_3	0.742922	0.754802	0.766321
z_4	0.742896	0.754784	0.766309
z_5	0.742896	0.754784	0.766309

Example 2. *Consider* $E_1 = E_2 = \mathcal{A}[0,1]$*. Let* $D^* = \{x \in \mathcal{A}[0,1]; \|x\| \le R\}$*, such that* $R > 0$ *and* G *defined on* D^* *as*

$$G(x)(u_1) = x(u_1) - f(u_1) - \lambda \int_0^1 \mu(u_1, u_2) x(u_2)^3 \, du_2, \quad x \in C[0,1],\ u_1 \in [0,1],$$

where $f \in \mathcal{A}[0,1]$ *is a given function,* λ *is a real constant and the kernel* μ *is the Green function. In this case, for all* $x \in D^*$*,* $G'(x)$ *is a linear operator defined on* D^* *by the following expression:*

$$[G'(x)(v)](u_1) = v(u_2) - 3\lambda \int_0^1 \mu(u_1, u_2) x(u_2)^2 v(u_2) \, du_2, \quad v \in C[0,1],\ u_1 \in [0,1].$$

If we choose $z_0(u_1) = f(u_1) = 1$, *it follows*

$$\|I - G'(z_0)\| \leq 3|\lambda|/8.$$

Hence, if

$$|\lambda| < 8/3,$$

$G'(z_0)^{-1}$ *is defined and*

$$\|G'(z_0)^{-1}\| \leq \frac{8}{8 - 3|\lambda|},$$

$$\|G(z_0)\| \leq \frac{|\lambda|}{8},$$

$$\eta = \|G'(z_0)^{-1}G(z_0)\| \leq \frac{|\lambda|}{8 - 3|\lambda|}.$$

Consider $\lambda = 1.00$, *we get*

$$\eta = 0.2,$$

$$T = 3.8,$$

$$L_0 = 2.6,$$

$$K = 2.28,$$

$$H = 1.28$$

and

$$L = 1.38154\ldots.$$

By these values we conclude that conditions (26)–(29) are not satisfied, since

$$h_K = 1.52 > 1,$$

$$h_1 = 1.28 > 1,$$

$$h_2 = 1.19343\ldots > 1,$$

$$h_3 > 1.07704\ldots > 1,$$

but condition (2.27) and condition (4) are satisfied, since

$$h_4 = 0.985779\ldots < 1$$

and

$$h_5 = 0.97017\ldots < 1.$$

Hence, Newton's method converges by Theorem 1.

4. Application: Planck's Radiation Law Problem

We consider the following problem [15] :

$$\varphi(\lambda) = \frac{8\pi c P \lambda^{-5}}{e^{\frac{cP}{\lambda BT}} - 1} \tag{36}$$

which calculates the energy density within an isothermal blackbody. The maxima for φ occurs when density $\varphi(\lambda)$. From (36), we get

$$\varphi'(\lambda) = \left(\frac{8\pi c P \lambda^{-6}}{e^{\frac{cP}{\lambda BT}-1}}\right)\left(\frac{(\frac{cP}{\lambda BT})e^{\frac{cP}{\lambda BT}-1}}{e^{\frac{cP}{\lambda kT}-1}} - 5\right) = 0, \tag{37}$$

that is when

$$\frac{(\frac{cP}{\lambda BT})e^{\frac{cP}{\lambda BT}-1}}{e^{\frac{cP}{\lambda BT}-1}} = 5. \tag{38}$$

After using the change of variable $x = \frac{cP}{\lambda BT}$ and reordering terms, we obtain

$$f(x) = e^{-x} - 1 + \frac{x}{5}. \tag{39}$$

As a consequence, we need to find the roots of Equation (39).

We consider $\Omega = \overline{E(5,1)} \subset \mathbb{R}$ and we obtain

$$\eta = 0.0348643\ldots,$$

$$L_0 = 0.0599067\ldots,$$

$$K = 0.0354792\ldots,$$

$$H = 0.0354792\ldots$$

and

$$L = 0.094771\ldots.$$

So (A) are satisfied. Moreover, as $b = 0.000906015 > 0$, then

$$L_4 = 10.0672\ldots,$$

which satisfies

$$L_4\eta = 0.350988\ldots < 1$$

and that means that conditions of Lemmal 1 are also satisfied. Finally, we obtain that

$$t^* = 0.0348859\ldots.$$

Hence, Newton's method converges to the solution $x^* = 4.965114231744276\ldots$ by Theorem 1.

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