mathematics

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Spreading Speed in A Nonmonotone Equation with Dispersal and Delay

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Received: 27 February 2019; Accepted: 18 March 2019; Published: 21 March 2019

Abstract: This paper is concerned with the estimation of spreading speed of a nonmonotone equation, which involves time delay and nonlocal dispersal. Due to the time delay, this equation does not generate monotone semiflows when the positive initial value is given. By constructing an auxiliary monotone equation, we obtain the spreading speed when the initial value admits nonempty compact support. Moreover, by passing to a limit function, we confirm the existence of traveling wave solutions if the wave speed equals to the spreading speed, which states the minimal wave speed of traveling wave solutions and improves the known results.

Keywords: asymptotic spreading; auxiliary equation; minimal wave speed

MSC: 35K57; 92D40

1. Introduction

Spreading speed is a threshold formulating the long time behavior of functions with spatial and temporal variables, which was proposed by Aronson and Weinberger [1]. When the spatial variable is $x \in \mathbb{R}$, its definition is given as follows.

Definition 1. *Let $u(x,t)$ be nonnegative for $x \in \mathbb{R}, t > 0$. Then c' is called the spreading speed of $u(x,t)$ when*

$$\lim_{t\to\infty} \sup_{|x|>(c'+\epsilon)t} u(x,t) = 0, \ \liminf_{t\to\infty} \inf_{|x|<(c'-\epsilon)t} u(x,t) > 0$$

for any given $\epsilon \in (0, c')$.

Here we give the definition that both directions have the same spreading speed, and different directions may have different spreading speeds due to diverse reasons including spatial advection and so on. Let $u(x,t)$ denote the population density in population dynamics, then such a threshold characterizes the invasion speed of invaders or spreading speed of pathogene [2], and the above two limit states the observed phenomena if an observer were moved at a fixed speed [3]. In some evolutionary systems, it has been proven that the threshold may also be the minimal wave speed of traveling wave solutions. Since [1], there have established some important results of these thresholds of monotone semiflows, see a survey paper by Zhao [4] for monotone semiflows, Bao et al. [5] and references cited therein for dispersal models. When the nonmonotone condition is concerned, a few results were also given, see, for example, Wu and Liu [6], Zhang and Ma [7] for a dispersal model, Ducrot [8] and Pan [9] for predator-prey type systems.

In this paper, we study the following nonmonotone equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d[J*u](x,t) + ru(x,t)[1-u(x,t)-au(x,t-\tau)], \\ u(x,s) = \phi(x,s), s \in [-\tau, 0], \end{cases} \tag{1}$$

in which $x \in \mathbb{R}, t > 0, u \in \mathbb{R}, d, a, \tau$ are positive constants, $\phi(x,s)$ is a uniform continuous and bounded positive function, $[J*u](x,t)$ means

$$[J*u](x,t) = \int_{\mathbb{R}} J(x-y)[u(y,t)-u(x,t)]dy,$$

where J is the probability kernel function satisfying the following assumptions:

(J) J is nonnegative and continuous; there exists $\lambda' > 0$ such that $\int_{\mathbb{R}} J(y)e^{\lambda y}dy < \infty$ for any $|\lambda| < \lambda'$, $\int_{\mathbb{R}} J(y)dy = 1$, $J(y) = J(-y), y \in \mathbb{R}$.

The above model admits nonlocal dispersal and time delays [10,11], and it does not generate monotone semiflows when the positive solution is concerned. We shall focus on the long term behavior of $u(x,t)$ defined by (1) when the initial value satisfies

(I1) $\phi(x,s)$ is uniformly continuous and bounded for $x \in \mathbb{R}, s \in [-\tau, 0]$, and $\phi(x,0)$ is a nonnegative function with nonempty support;
(I2) $\phi(x,0)$ has nonempty compact support.

If (I1) and (I2) hold in population dynamics, then the initial habitat size is finite and it may describe the invasion process in many natural phenomena [2]. To estimate the invasion speed is the main purpose of this paper.

In this paper, we shall construct a proper auxiliary equation and estimate the spreading speed of $u(x,t)$, which is motivated by Lin [12] for a delayed reaction-diffusion equation and depends on the conclusions in Jin and Zhao [13] for a monotone dispersal model. The spreading speed in this paper equals the minimal wave speed of traveling wave solutions in Li et al. [14] when the kernel function J has nonempty compact support. By passing to a limit function, we also present the existence of traveling wave solutions when the wave speed is the spreading speed, during which (J) holds. That is, even for minimal wave speed, we improve the known conclusions from the viewpoint of kernel function since a kernel function satisfying (J) may have unbounded support.

2. Preliminaries

We first recall the following initial value problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d[J*u](x,t) + ru(x,t)\left[1-u(x,t)/K\right], x \in \mathbb{R}, t > 0, \\ u(x,0) = \chi(x), x \in \mathbb{R}, \end{cases} \tag{2}$$

where J satisfies (J), $d > 0, K > 0$ and $r > 0$ are constants, and $\chi(x)$ is uniformly continuous and bounded. By [13], we have the following comparison principle of (2).

Lemma 1. *Under the above assumptions, (2) has a solution defined for all $x \in \mathbb{R}, t > 0$. If $w(x,0)$ is uniformly continuous and bounded for $x \in \mathbb{R}$, and $w(x,t)$ satisfies*

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} \geq (\leq) d[J*w](x,t) + rw(x,t)\left[1-w(x,t)/K\right], x \in \mathbb{R}, t > 0, \\ w(x,0) \geq (\leq) \chi(x), x \in \mathbb{R}, \end{cases}$$

then $w(x,t) \geq (\leq) u(x,t)$ holds for all $x \in \mathbb{R}, t > 0$.

For $\lambda > 0$, define

$$c^* = \inf_{\lambda>0} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + r}{\lambda}.$$

Then $c^* > 0$ holds. Moreover, it also admits the following property [13].

Lemma 2. *Assume that $\chi(x) > 0$. Then for any $c < c^*$, we have*

$$\liminf_{t\to\infty} \inf_{|x|<ct} u(x,t) = \limsup_{t\to\infty} \sup_{|x|<ct} u(x,t) = K.$$

Further suppose that $\chi(x)$ has nonempty compact support, then $\lim_{t\to\infty} \sup_{|x|>ct} u(x,t) = 0$ for any $c > c^*$.

Moreover, from Pan [15] (Section 5), we have the following result.

Lemma 3. *Assume that $a \in [0,1)$. If $c > c^*$, then*

$$\frac{\partial u(x,t)}{\partial t} = d[J * u](x,t) + ru(x,t)[1 - u(x,t) - au(x,t-\tau)]$$

has a traveling wave solution defined by

$$u(x,t) = \psi_c(\xi), \xi = x + ct \in \mathbb{R}$$

and so

$$c\psi_c'(\xi) = d\int_{\mathbb{R}} J(y)[\psi_c(\xi - y) - \psi_c(\xi)]dy + r\psi_c(\xi)[1 - \psi_c(\xi) - a\psi_c(\xi - c\tau)], \quad (3)$$

which also satisfies

$$\lim_{\xi\to-\infty} \psi_c(\xi) = 0, \lim_{\xi\to\infty} \psi_c(\xi) = \frac{1}{1+a}. \quad (4)$$

and $0 < \psi_c(\xi) < 1, \xi \in \mathbb{R}$. Moreover, for any $c \in (c^*, c^*+1]$, $\{\psi_c(\xi)\}_{c\in(c^*,c^*+1]}$ is equicontinuous in $\xi \in \mathbb{R}$.

3. Main Results

We now show the main conclusion of this paper.

Theorem 1. *Assume that $u(x,t)$ is defined by (1). If (I1) holds, then for any fixed $c < c^*$, we have $\liminf_{t\to\infty} \inf_{|x|<ct} u(x,t) > 0$. If (I1) and (I2) are true, then $\lim_{t\to\infty} \sup_{|x|>ct} u(x,t) = 0$ for any given $c > c^*$. Moreover, if $a \in [0,1)$, then (3) with $c = c^*$ admits a positive solution satisfying (4).*

Before proving Theorem 1, we make the following remark.

Remark 1. If (I1) and (I2) are true, then c^* is the spreading speed of $u(x,t)$ defined by (1). Moreover, our result holds for any given $\tau > 0, a > 0$. This is different from that in Lin [12], which studied the case of $a \in [0,1)$ for a delayed reaction-diffusion equation.

In the space X of uniformly continuous and bounded functions, we see that $d[J * u](x) : X \to X$ is a bounded linear operator, so it is a sector operator. Thus, it generates an analytic positive semigroup $T(t) : X \to X, t > 0$. By the positivity of $T(t)$, we have the following comparison principle.

Lemma 4. *Assume that $\underline{u}(x,t)$ or $\underline{u}(t) \in X$ satisfies*

$$\underline{u}(t) \geq (\leq) T(t-\theta)\underline{u}(\theta) + \int_\theta^t T(t-s)\left[r\underline{u}(s)(1 - \underline{u}(s)\right], t \geq 0,$$

in which $\theta \in [0, t)$ *if* $t > 0$. *Then* $\underline{u}(x,t) \geq (\leq) u(x,t)$, *where* $u(x,t)$ *is defined by* (2).

By Martin and Smith [16], the mild solution of (1) is defined by

$$u(t) = T(t-\theta)u(\theta) + \int_{\theta}^{t} T(t-s)\left[ru(s)(1-u(s)-au(s-\tau))\right] ds, t > 0, \tag{5}$$

where $\theta \in [0,t), u(\cdot) \in X$. Moreover, from the quasipositivity of $ru(x,t)[1-u(x,t)-au(x,t-\tau)]$, we also have the following conclusion [16].

Lemma 5. (1) *admits a mild solution* $u(x,t)$ *defined by* (5) *for all* $x \in \mathbb{R}, t > 0$. *It is a classical solution satisfying* (1) *if* $t > \tau$ *and*

$$0 < u(x,t) \leq \max\left\{1, \sup_{x\in\mathbb{R}} \phi(x,0)\right\} := M, x \in \mathbb{R}, t > 0.$$

When $\phi(x,0) > 0$ *for some* $x \in \mathbb{R}$, *we have* $u(x,t) > 0, x \in \mathbb{R}, t > 0$. *Moreover, if* $t > \tau + 1$, *then* $u(x,t)$ *is uniformly continuous in* $x \in \mathbb{R}$.

Lemma 6. *If (I1) is true and* $c \in (0, c^*)$ *is fixed, then*

$$\liminf_{t\to\infty} \inf_{|x|<ct} u(x,t) > 0.$$

Proof. Since it involves long time behavior, it suffices to consider $t \geq 2\tau + 2$ such that we can investigate the differential equation, and $u(x,t)$ is uniformly continuous in both x, t. Let $\epsilon > 0$ be a constant such that

$$\inf_{\lambda>0} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + r(1-4\epsilon)}{\lambda} > c.$$

If $au(x,t-\tau) < \epsilon$, then

$$\frac{\partial u(x,t)}{\partial t} \geq d[J*u](x,t) + ru(x,t)[1-\epsilon-u(x,t)].$$

When $au(x,t-\tau) \geq \epsilon$ (and so $a > 0$), from the uniform continuity, we have $\delta > 0$ such that

$$au(y,t-\tau) > \frac{\epsilon}{4}, |x-y| < 2\delta. \tag{6}$$

Consider the initial value problem

$$\begin{cases} \dfrac{\partial \underline{v}(x,t)}{\partial t} = d[J*\underline{v}](x,t) + r\underline{v}(x,t)\left[1-aM-\underline{v}(x,t)\right], \\ \underline{v}(x,0) = \underline{v}(x), \end{cases} \tag{7}$$

where the continuous function $\underline{v}(x)$ satisfies

(v1) $\underline{v}(x) = \frac{\epsilon}{4a}, |x| \leq \delta$;
(v2) $\underline{v}(x) = 0, |x| \geq 2\delta$;
(v3) if $x \in [\delta, 2\delta]$ $(x \in [-2\delta, -\delta])$, then $v(x)$ is decreasing (increasing).

By Lemma 4, we have $\underline{v}(x,t) > 0, x \in \mathbb{R}, t > 0$. Let $\varepsilon = \inf_{t\in[0,\tau]} \underline{v}(0,t)$, then $\varepsilon > 0$. It should be noted that ε is fixed for all x, t such that $au(x,t-\tau) \geq \epsilon, t > 2 + 2\tau$. Since

$$\frac{\partial u(x,t)}{\partial t} \geq d[J*u](x,t) + ru(x,t)\left[1-aM-u(x,t)\right],$$

then (6) indicates $u(x,t) > \varepsilon$ and

$$au(x,t-\tau) \le aM < \frac{aM}{\varepsilon} u(x,t) := ku(x,t),$$

in which k is also independent on x,t satisfying $au(x,t-\tau) \ge \epsilon, t > 2+2\tau$ due to the selection of ε.

Summarizing what we have done, we obtain

$$\frac{\partial u(x,t)}{\partial t} \ge d[J*u](x,t) + ru(x,t)\left[1-\epsilon-(1+k)\,u(x,t)\right], x \in \mathbb{R}, t > 2\tau+1$$

and

$$u(x, 2\tau+1) > 0, x \in \mathbb{R}.$$

Due to Lemmas 1 and 2, we have

$$\liminf_{t\to\infty} \inf_{|x|<c''t} u(x,t) \ge \frac{1-\epsilon}{1+k} > 0,$$

where c'' is defined by

$$\inf_{\lambda>0} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + r(1-2\epsilon)}{\lambda},$$

which completes the proof. □

Lemma 7. *If (I2) is true, then* $\limsup_{t\to\infty} \sup_{|x|>ct} u(x,t) = 0, c > c^*$.

Proof. Since $u(x,t) \ge 0$ for all $x \in \mathbb{R}, t > 0$, we have

$$u(t) \le T(t-\theta)u(\theta) + \int_\theta^t T(t-s)\left[ru(s)(1-u(s))\right]ds$$

for all $0 \le \theta < t$, then the conclusion is clear by Lemmas 2 and 4. The proof is complete. □

Lemma 8. *If* $c = c^*$ *and* $a \in [0,1)$, *then* (3) *has a positive solution satisfying* (4).

Proof. We prove the result by the idea in Brown and Carr [17] for the existence and Lin and Ruan [18] for the limit behavior. Let $\{c_m\}_{m\in\mathbb{N}}$ be a strictly decreasing sequence satisfying $c_m < c^*+1, \lim_{m\to\infty} c_m = c^*$. Then for each $m \in \mathbb{N}$, (3) with $c = c_m$ admits a solution $\psi_{c_m}(\xi)$ satisfying

$$\lim_{\xi\to-\infty} \psi_{c_m}(\xi) = 0, \lim_{\xi\to\infty} \psi_{c_m}(\xi) = \frac{1}{1+a}.$$

Since a traveling wave solution is invariant in the sense of phase shift (that is, if $\psi_{c_m}(\xi)$ is a solution of (3) with $c = c_m$, then for any fixed $e \in \mathbb{R}$, $\psi_{c_m}(\xi+e)$ also satisfies (3) with $c = c_m$), we assume that

$$\psi_{c_m}(0) = \frac{1-a}{4}, \psi_{c_m}(\xi) < \frac{1-a}{4}, \xi < 0$$

after making proper phase shift. By Lemma 3, $\{\psi_{c_m}(\xi)\}_{m\in\mathbb{N}}$ is equicontinuous. In particular, $\psi_{c_m}(\xi)$ also satisfies

$$\psi_{c_m}(\xi) = \frac{1}{c_m}\int_{-\infty}^{\xi} e^{\frac{d(s-\xi)}{c_m}} \left\{d\left(J*\psi_{c_m}\right)(s) + r\psi_{c_m}(s)[1-\psi_{c_m}(s) - a\psi_{c_m}(s-c_m\tau)]\right\} ds$$

with $\left(J*\psi_{c_m}\right)(s) = \int J(y)\psi_{c_m}(s-y)dy$. In particular, any bounded solution of the above integral equation also satisfies (3) with $c = c_m$.

Using Ascoli-Arzela lemma, $\{\psi_{c_m}(\xi)\}_{m\in\mathbb{N}}$ has a subsequence, still denoted by $\{\psi_{c_m}(\xi)\}_{m\in\mathbb{N}}$, and there exists a uniform continuous function $\psi_{c^*}(\xi)$ such that

$$\lim_{m\to\infty}\psi_{c_m}(\xi)=\psi_{c^*}(\xi)\in[0,1],$$

which is uniform in ξ on any compact subset of $\mathbb{R}$, and the convergence is also pointwise in $\xi\in\mathbb{R}$. Letting $m\to+\infty$ and utilizing the dominated convergence theorem, we see that $\psi_{c^*}(\xi)$ satisfies

$$\psi_{c^*}(\xi)=\frac{1}{c^*}\int_{-\infty}^{\xi}e^{\frac{d(s-\xi)}{c^*}}\left\{d\left(J*\psi_{c^*}\right)(s)+r\psi_{c^*}(s)[1-\psi_{c^*}(s)-a\psi_{c^*}(s-c^*\tau)]\right\}ds, \tag{8}$$

which implies the existence of positive solution of (3) with $c=c^*$. In particular, they also satisfy

$$\psi_{c^*}(0)=\frac{1-a}{4},\psi_{c^*}(\xi)\le\frac{1-a}{4},\xi<0.$$

Moreover, from the above integral equation, we see that $\psi_{c^*}(\xi)$ is uniformly continuous.

By the definition of traveling wave solutions, $u(x,t)=\psi_{c^*}(x+c^*t)$ satisfies

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=d[J*u](x,t)+ru(x,t)[1-u(x,t)-au(x,t-\tau)],x\in\mathbb{R},t>0,\\ u(x,s)=\psi_{c^*}(x+c^*s),x\in\mathbb{R},s\in[-\tau,0].\end{cases}$$

So, we have

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}\ge d[J*u](x,t)+ru(x,t)[1-a-u(x,t)],\\ u(x,0)=\psi_{c^*}(x)\end{cases}$$

for $x\in\mathbb{R},t>0$. From Lemma 2, we obtain

$$\liminf_{t\to\infty}u(0,t)=\liminf_{t\to\infty}\psi_{c^*}(c^*t)=\liminf_{\xi\to\infty}\psi_{c^*}(\xi)\ge1-a$$

and

$$\limsup_{\xi\to\infty}\psi_{c^*}(\xi)\le1.$$

By the dominated convergence theorem in (8), we have

$$1-\limsup_{\xi\to\infty}\psi_{c^*}(\xi)-a\liminf_{\xi\to\infty}\psi_{c^*}(\xi)\ge0$$

and

$$1-\liminf_{\xi\to\infty}\psi_{c^*}(\xi)-a\limsup_{\xi\to\infty}\psi_{c^*}(\xi)\le0$$

from the monotonicity. Therefore, we obtain

$$\liminf_{\xi\to\infty}\psi_{c^*}(\xi)=\limsup_{\xi\to\infty}\psi_{c^*}(\xi)=\frac{1}{1+a}.$$

If $\limsup_{\xi\to-\infty}\psi_{c^*}(\xi)>0$, then the definition of lim sup and the uniform continuity imply that there exist $\{\xi_m\}_{m\in\mathbb{N}},\sigma>0$ and $\rho>0$ such that

(p1) $\xi_m\to-\infty,m\to\infty,$
(p2) $\psi_{c^*}(\xi_m)>\sigma/2,$
(p3) $\psi_{c^*}(\xi_m+x)>\sigma/2,|x|<\rho.$

Also consider

$$\frac{\partial w(x,t)}{\partial t} = d[J * w](x,t) + rw(x,t)[1-a-w(x,t)], x \in \mathbb{R}, t > 0$$

with the initial value satisfying

(w1) $w(x,0) = w(-x,0), x \in \mathbb{R}$;
(w2) $w(x,0) = 0, |x| \geq \rho$;
(w3) $w(x,0) = \sigma/2, x \in [-\rho/2, \rho/2]$;
(w4) $w(x,0)$ is decreasing and continuous for $x \in [\rho/2, \rho]$.

Then there exists $T > 0$ such that

$$w(0,t) > \frac{1-a}{2}, t \geq T$$

by Lemmas 1 and 2. Since

$$\frac{\partial u(x,t)}{\partial t} \geq d[J * u](x,t) + ru(x,t)[1-a-u(x,t)], x \in \mathbb{R}, t > 0,$$

we obtain

$$u(\xi_m, c^* T) > \frac{1-a}{2}$$

by the property of $\{\xi_m\}_{m\in\mathbb{N}}$. According to the definition of traveling wave solutions, we see that

$$\limsup_{\xi \to -\infty} \psi_{c^*}(\xi) > \frac{1-a}{2},$$

and a contradiction occurs. The proof is complete. □

4. Conclusion Remarks

Nonmonotone condition may lead to plentiful dynamical behavior in dynamical systems, such as the chaos in simple logistic type difference equations [19]. The nonmonotonicity may have originated from time delay in reaction-diffusion equations, see their complex dynamics by Wu [20]. When the spatial propagation of nonmonotone models is concerned, much attention has been paid to the study of traveling wave solutions, see several recent papers [21,22] and references cited therein. Besides the traveling wave solutions, the entire solutions of some nonmonotone models were also studied [23–26].

In the study of traveling wave solutions and entire solutions, some admissible speed parameters may be larger/smaller than a threshold. The threshold may also be characterized by asymptotic spreading, and is called the spreading speed. For monotone systems, we have mentioned some important works. In this paper, we obtained a threshold of asymptotic spreading for nonmonotone Equation (1), which is also the minimal wave speed in known results. For the existence of traveling wave solutions, we further presented a result under weaker conditions than that in [14].

However, the propagation dynamics of different evolutionary models is more complex than that we have mentioned, which can be formulated by traveling wave solutions, asymptotic spreading and entire solutions. For example, if the initial value does not have compact support, can we characterize the asymptotic spreading of (1) by a constant like that in this paper? Under some conditions, we conjecture that the solution may expand in a finite speed, which likes the stability of traveling wave solutions of monotone dispersal models [27,28]. With different conditions, very likely special initial values lead to non-constant speed of spatial expansion, which is similar to that of the Fisher equation in Hamel and Roques [29]. We shall further investigate these questions in the future.

Author Contributions: X.-L.L. analyzed and prepared/edited the manuscript, S.P. analyzed and prepared/edited the manuscript. Both authors read the final manuscript.

Funding: This research was funded by the NSF of China grant number 10461040.

Acknowledgments: The authors are grateful to the referees and handling editor for their comments.

Conflicts of Interest: The authors declare no conflict of interest.

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