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Asymptotic Profiles and Convergence Rates of the Linearized Compressible Navier–Stokes–Korteweg System

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Abstract: In this paper, we consider the initial value problem for the linearized compressible Navier–Stokes–Korteweg system. Asymptotic profiles and convergence rates are established by Fourier splitting frequency technique. Moreover, some applications of asymptotic profile and convergence rates are exhibited.

Keywords: linearized compressible Navier–Stokes–Korteweg system; asymptotic profiles; convergence rates

1. Introduction

The compressible Navier–Stokes–Korteweg system takes the following form

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla P(\rho) - \mu_1\Delta u - (\mu_1+\mu_2)\nabla(\nabla\cdot u) = \alpha\rho\nabla\Delta\rho, \end{cases} \tag{1}$$

The variables are the density ρ and the velocity u. Furthermore, $p = p(\rho)$ is the pressure function satisfying $P'(\rho) > 0$ for $\rho > 0$. The viscosity coefficients satisfy $\mu_1 > 0, 2\mu_1 + n\mu_2 > 0$, $\alpha > 0$, while $\bar{\rho} > 0$ denotes the background doping profile, and in this paper is taken as a positive constant for simplicity.

The compressible Navier–Stokes–Korteweg systems have strong physical backgrounds, which can be used to describe the dynamics of a liquid–vapor mixture in the setting of the diffuse interface approach: between the two phases lies a thin region of continuous transition and the phase changes are described through the variations of the density, for example a Van der Waals pressure. The system was derived rigorously by Dunn and Serrin [1], see also [2,3]. Equation (1) has attracted interests of lots of mathematicians and physicists and some important results were made, we may refer to [4–16].

Let $\sigma = \rho - 1$ and $P'(1) = 1$. The linearized compressible Navier–Stokes–Korteweg system (1) is

$$\begin{cases} \partial_t\sigma + \nabla\cdot u = 0, \\ \partial_t u + \nabla\sigma - \mu_1\Delta u - (\mu_1+\mu_2)\nabla(\nabla\cdot u) - \alpha\nabla\Delta\sigma = 0, \end{cases} \tag{2}$$

We consider the asymptotic profiles of solutions to Equation (2) with the initial value

$$t = 0 : \sigma = \sigma_0(x), \quad u = u_0(x). \tag{3}$$

As far as we know, there are few results about asymptotic profiles of solutions to the linearized compressible Navier–Stokes–Korteweg system (2). In this paper, our main aim is to establish the asymptotic profiles of solutions to the problems (2) and (3) in the spirit of [17–21]. More precisely, we show that the asymptotic profile of solutions is given by the convolution of the fundamental solutions of diffusion and free wave equations. For the detail, we refer to Theorem 1. Moreover, on one hand, the decay estimate of solutions to Equations (2) and (3) immediately follows from this asymptotic profile result. On the other hand, the decay estimate of solutions is also optimal under suitable conditions. The study of asymptotic profiles of solutions to the problems (2) and (3) may provide some useful ideas and methods of studying the nonlinear problem in future.

To establish the asymptotic profiles of solutions to the problems (2) and (3), it is necessary to derive the solution formula to the problems (2) and (3). We find that solutions operator is related to a fourth order wave equation with strong damping. This wave equation is called as Boussinesq equation with damping. For the Boussinesq equation with damping, the author of this paper has obtained some results. For the details, we may refer to [22].

The following are some notations which are used in this paper. Let $\mathcal{F}[u]$ denote the Fourier transform of u defined by

$$\hat{u}(\xi) = \mathcal{F}[u](\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx.$$

We denote its inverse transform by $\mathcal{F}^{-1}$. For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. For $\gamma \in \mathbb{R}$, let $L^{1,\gamma}(\mathbb{R}^n)$ denote the weighted L^1 space with the norm

$$\|f\|_{L^{1,\gamma}} = \int_{\mathbb{R}^n} (1+|x|)^\gamma |f(x)|\, dx.$$

The paper is organized as follows. Section 2 is devoted to derive solutions formula to the problems (2) and (3). The decay properties of the solutions operator is established in Section 3. While, Section 4 is devoted to establish the asymptotic profiles of solutions to the problems (2) and (3) in the low frequency region. The high frequency case is discussed in Section 5. Finally, the asymptotic profiles of solutions and applications are stated in Section 6.

2. Solution Formula

This section is devoted to deriving the solutions formula to problems (2) and (3). For simplicity, we set $\mu = 2\mu_1 + \mu_2$. It is not difficult to obtain the solutions formula to problems (2) and (3).

$$\begin{pmatrix} \sigma \\ u \end{pmatrix} = G(x,t) * \begin{pmatrix} \sigma_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} * \begin{pmatrix} \sigma_0 \\ u_0 \end{pmatrix}, \tag{4}$$

where

$$G(x,t) = \mathcal{F}^{-1}[\hat{G}(\xi,t)] = \begin{pmatrix} \mathcal{F}^{-1}[\hat{G}_{11}] & \mathcal{F}^{-1}[\hat{G}_{12}] \\ \mathcal{F}^{-1}[\hat{G}_{21}] & \mathcal{F}^{-1}[\hat{G}_{22}] \end{pmatrix}, \tag{5}$$

with

$$\begin{cases} \hat{G}_{11} = \hat{\mathcal{H}}, \ \ \hat{G}_{12} = -i\xi^\tau \hat{\mathcal{G}}, \\ \hat{G}_{21} = -i(1+\alpha|\xi|^2)\xi\hat{\mathcal{G}}, \ \ \hat{G}_{22} = (-\mu|\xi|^2\hat{\mathcal{G}} + \hat{\mathcal{H}})\frac{\xi\xi\cdot}{|\xi|^2}, \end{cases} \tag{6}$$

$$\hat{\mathcal{G}}(\xi,t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \ \ \hat{\mathcal{H}}(\xi,t) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \tag{7}$$

and

$$\lambda_\pm(\xi) = \frac{-\mu|\xi|^2 \pm \sqrt{\mu^2|\xi|^4 - 4(|\xi|^2 + \alpha|\xi|^4)}}{2}. \tag{8}$$

In fact, taking the time derivative for the first equation of (2) and by using the first and second equations of (2), it follows that

$$\sigma_{tt} - \Delta\sigma - \mu\Delta\sigma_t + \alpha\Delta^2\sigma = 0. \tag{9}$$

Notice

$$t = 0: \ \ \sigma = \sigma_0, \ \ \sigma_t = -\nabla\cdot u_0. \tag{10}$$

We apply the Fourier transform to Equations (9) and (10)

$$\begin{cases} \partial_{tt}\hat{\sigma} + \mu|\xi|^2\hat{\sigma}_t + (|\xi|^2 + \alpha|\xi|^4)\hat{\sigma} = 0, \\ t = 0: \quad \hat{\sigma} = \hat{\sigma}_0, \quad \hat{\sigma}_t = -i\xi \cdot \hat{u}_0. \end{cases} \tag{11}$$

Solving the problem (11), we arrive at

$$\hat{\sigma}(\xi, t) = \hat{\mathcal{G}}(\xi, t)(-i\xi \cdot \hat{u}_0)(\xi) + \hat{\mathcal{H}}(\xi, t)\hat{\sigma}_0(\xi). \tag{12}$$

Taking divergence and time derivative for the second equation of (2) and using (2) and (3), it follows that

$$(\nabla \cdot u)_{tt} - \Delta(\nabla \cdot u) - \mu\Delta(\nabla \cdot u)_t + \alpha\Delta^2(\nabla \cdot u) = 0. \tag{13}$$

Notice

$$t = 0: \quad \nabla \cdot u = \nabla \cdot u_0, \quad (\nabla \cdot u)_t = -\Delta\sigma_0 + \mu\Delta(\nabla \cdot u_0) + \alpha\Delta^2\sigma_0. \tag{14}$$

Applying the Fourier transform to Equations (13) and (14), it yields that

$$\begin{cases} \partial_{tt}(\xi \cdot \hat{u}) + \mu|\xi|^2(\xi \cdot \hat{u})_t + (|\xi|^2 + \alpha|\xi|^4)(\xi \cdot \hat{u}) = 0, \\ t = 0: \quad \xi \cdot \hat{u} = \xi \cdot \hat{u}_0, \\ (\xi \cdot \hat{u})_t = -i(|\xi|^2 + \alpha|\xi|^4)\hat{\sigma}_0 - \mu|\xi|^2(\xi \cdot \hat{u}_0). \end{cases} \tag{15}$$

Then we get from (15)

$$\begin{aligned} \xi \cdot \hat{u}(\xi, t) = {} & \hat{\mathcal{G}}(\xi, t)\Big(-i(|\xi|^2 + \alpha|\xi|^4)\hat{\sigma}_0 - \mu|\xi|^2(\xi \cdot \hat{u}_0)\Big)(\xi) \\ & + \hat{\mathcal{H}}(\xi, t)(\xi \cdot \hat{u}_0)(\xi). \end{aligned} \tag{16}$$

Owing to Equations (8) and (16) and the inverse Fourier transform, we obtain the solution Formula (4) to the problems (2) and (3).

Let

$$d(x, t) = \Lambda^{-1}\nabla \cdot u, \quad v(x, t) = \Lambda^{-1} curlu, \tag{17}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$. Owing to the identity

$$\Delta = \nabla\nabla \cdot - curlcurl,$$

then we have the decomposition

$$u(x, t) = -\Lambda^{-1}\nabla d + \Lambda^{-1} curlv. \tag{18}$$

Then from Equations (2) and (3), we deduce that

$$\begin{cases} \partial_t d - \mu\Delta d - \Lambda\sigma - \alpha\Lambda^3 d = 0, \\ \partial_t v - \mu_1\Delta v = 0, \\ t = 0: d = \Lambda^{-1}\nabla \cdot u_0, \quad v = \Lambda^{-1} curlu_0. \end{cases} \tag{19}$$

and

$$\begin{cases} \partial_t v - \mu_1\Delta v = 0, \\ t = 0: v = \Lambda^{-1} curlu_0. \end{cases} \tag{20}$$

3. Decay Properties

In this section, our aim is to derive the decay properties of solution operators G. Since the solution operator G is given in term of $\mathcal{G}$ and $\mathcal{H}$, therefore, we only study the decay properties of the solution operators $\mathcal{G}$ and $\mathcal{H}$. The following estimate has been derived by applying the energy method in the Fourier space to the first equation in (11) (see [22–24]).

Lemma 1. *Let σ be the solution to the problems (9) and (10). Then its Fourier image $\hat{\sigma}$ verifies the pointwise estimate*

$$|\hat{\sigma}_t(\xi,t)|^2 + |\xi|^2(1+|\xi|^2)|\hat{\sigma}(\xi,t)|^2 \le Ce^{-c|\xi|^2 t}(|\xi|^2|\hat{u}_0|^2 + |\xi|^2(1+|\xi|^2)|\hat{\sigma}_0|^2), \tag{21}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$.

It follows from Equation (19) that

$$\begin{cases} d_{tt} - \Delta d - \mu\Delta d_t + \alpha\Delta^2 d = 0, \\ t = 0: \quad d = \Lambda^{-1}\nabla\cdot u_0, \quad d_t = \Lambda\sigma_0 - \mu\Lambda(\nabla\cdot u_0) + \alpha\Lambda^3\sigma_0. \end{cases} \tag{22}$$

From Lemma 1, we directly deduce that

Lemma 2. *Let d be the solution to the problem (22). Then its Fourier image $\hat{d}$ verifies the pointwise estimate*

$$\begin{aligned} |\hat{d}_t(\xi,t)|^2 + |\xi|^2(1+|\xi|^2)|\hat{d}(\xi,t)|^2 \quad \le \quad & Ce^{-c|\xi|^2 t}\Big(\Big||\xi|\hat{\sigma}_0 - \mu|\xi|(\xi\cdot u_0) + \alpha|\xi|^3\hat{\sigma}_0\Big|^2 \\ & + |\xi|^2(1+|\xi|^2)\Big||\xi|^{-1}(\xi\cdot u_0)\Big|^2\Big), \end{aligned} \tag{23}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$.

The pointwise estimate (21) together with the solution formula (12) gives the corresponding pointwise estimates for $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$. The result is stated as follows.

Lemma 3. *Let $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$ be the fundamental solutions of Equation (9) in the Fourier space, which are given explicitly in Equation (7). Then we have the pointwise estimates*

$$\begin{aligned} &|\hat{\mathcal{G}}(\xi,t)| \le C|\xi|^{-1}(1+|\xi|^2)^{-\frac{1}{2}}e^{-c|\xi|^2 t}, \\ &|\hat{\mathcal{H}}(\xi,t)| \le Ce^{-c|\xi|^2 t}, \\ &|\hat{\mathcal{G}}_t(\xi,t)| \le Ce^{-c|\xi|^2 t}, \\ &|\hat{\mathcal{H}}_t(\xi,t)| \le C|\xi|(1+|\xi|^2)^{\frac{1}{2}}e^{-c|\xi|^2 t} \end{aligned} \tag{24}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$.

4. Low Frequency Case

Motivated by Ikehata [18], the purpose of this section is to prove asymptotic profiles of solutions to problems (2) and (3) in the low frequency region. When $|\xi| \le \delta_0$,

$$\hat{\mathcal{G}}(\xi,t) = e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin\big(A(\xi)|\xi|t\big)}{A(\xi)|\xi|} \tag{25}$$

and

$$\hat{\mathcal{H}}(\xi,t) = e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\mu|\xi|}{2A(\xi)}\sin\big(A(\xi)|\xi|t\big) + e^{-\frac{\mu}{2}|\xi|^2 t}\cos\big(A(\xi)|\xi|t\big), \tag{26}$$

where

$$A(\xi) = \sqrt{1 + \frac{4\alpha - \mu^2}{4}|\xi|^2}.$$

Let

$$B(s) = \sin\left(|\xi| t\sqrt{1+\frac{4\alpha-\mu^2}{4}|\xi|^2 s}\ \right).$$

The mean value theorem entails that

$$B(1) - B(0) = B'(\theta_1), \quad \theta_1 \in (0,1).$$

That is,

$$\begin{aligned}\sin\left(A(\xi)|\xi|t\right) &= \sin\left(|\xi|t\right) + \frac{(4\alpha-\mu^2)|\xi|^3 t}{8\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}}\times \\ &\quad \cos\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4)}\theta_1|\xi|^2}\ \right) \\ &= \sin\left(|\xi|t\right) + \cos\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}\ \right)O(|\xi|^3)t.\end{aligned} \tag{27}$$

It follows from the Taylor formula that

$$\frac{1}{A(\xi)} = 1 + O(|\xi|^2). \tag{28}$$

Inserting Equations (27) and (28) into Equation (25) yields

$$\begin{aligned}\hat{\mathcal{G}}(\xi,t) &= \frac{e^{-\frac{\mu}{2}|\xi|^2 t}}{|\xi|}\left[1+O(|\xi|^2)\right]\left[\sin\left(|\xi|t\right)+\right. \\ &\quad \left.\cos\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}\ \right)O(|\xi|^3)t\right] \\ &= e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin\left(|\xi|t\right)}{|\xi|} + e^{-\frac{\mu}{2}|\xi|^2 t}\sin\left(|\xi|t\right)O\left(|\xi|\right) + \\ &\quad e^{-\frac{\mu}{2}|\xi|^2 t}\cos\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}\ \right)O\left(|\xi|^2\right)t.\end{aligned} \tag{29}$$

Let

$$C(s) = \cos\left(|\xi| t\sqrt{1+\frac{4\alpha-\mu^2}{4}|\xi|^2 s}\ \right).$$

Thanks to Mean value theorem, we arrive at

$$C(1) - C(0) = C'(\theta_2), \quad \theta_2 \in (0,1).$$

That is,

$$\begin{aligned}\cos\left(A(\xi)|\xi|t\right) &= \cos\left(|\xi|t\right) - \frac{(4\alpha-\mu^2)|\xi|^3 t}{8\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}}\times \\ &\quad \sin\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\ \right) \\ &= \cos\left(|\xi|t\right) - \sin\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\ \right)O(|\xi|^3)t.\end{aligned} \tag{30}$$

We substitute Equations (28) and (30) into Equation (26) and obtain

$$\begin{aligned}
\hat{\mathcal{H}}(\xi,t) &= e^{-\frac{\mu}{2}|\xi|^2 t}\Big[\cos\left(|\xi|t\right)-\sin\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\ \right)O(|\xi|^3)t\Big]+\\
&\quad e^{-\frac{\mu}{2}|\xi|^2 t}\sin\left(A(\xi)|\xi|t\right)\frac{\mu|\xi|}{2}(1+O(|\xi|^2))\\
&= e^{-\frac{\mu}{2}|\xi|^2 t}\cos\left(|\xi|t\right)-e^{-\frac{\mu}{2}|\xi|^2 t}\sin\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\ \right)O(|\xi|^3)t+\\
&\quad e^{-\frac{\mu}{2}|\xi|^2 t}\sin\left(A(\xi)|\xi|t\right)O(|\xi|).
\end{aligned} \tag{31}$$

Due to Equations (12), (29) and (31), we arrive at

$$\begin{aligned}
\hat{\sigma}(\xi,t) = \quad & e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin\left(|\xi|t\right)}{|\xi|}(-i\xi\cdot\hat{u}_0(\xi))+e^{-\frac{\mu}{2}|\xi|^2 t}\cos\left(|\xi|t\right)\hat{\sigma}_0(\xi)\\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin\left(|\xi|t\right)O\big(|\xi|\big)(-i\xi\cdot\hat{u}_0(\xi))+\\
& e^{-\frac{\mu}{2}|\xi|^2 t}\cos\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}\ \right)O\big(|\xi|^2\big)t(-i\xi\cdot\hat{u}_0(\xi))-\\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin\left(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\ \right)O(|\xi|^3)t\hat{\sigma}_0(\xi)+\\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin\left(A(\xi)|\xi|t\right)O(|\xi|)\hat{\sigma}_0(\xi).
\end{aligned} \tag{32}$$

Owing to the definition of Fourier transform and Euler formula, we deduce that

$$\hat{u}_0(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u_0(x)\,dx=P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0}, \tag{33}$$

where

$$P_{u_0}=(P^1_{u_0},\cdots,P^n_{u_0}),Q_{u_0}=(Q^1_{u_0},\cdots,Q^n_{u_0}),R_{u_0}=(R^1_{u_0},\cdots,R^n_{u_0}),$$

with

$$\begin{aligned}
&P^j_{u_0}(\xi)=\int_{\mathbb{R}^n}\Big(\cos(x\cdot\xi)-1\Big)u^j_0(x)\,dx(j=1,\cdots,n),\\
&Q^j_{u_0}(\xi)=\int_{\mathbb{R}^n}\sin(x\cdot\xi)u^j_0(x)\,dx(j=1,\cdots,n),\\
&R^j_{u_0}=\int_{\mathbb{R}^n}u^j_0(x)\,dx(j=1,\cdots,n).
\end{aligned} \tag{34}$$

Similarly,

$$\hat{\sigma}_0(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\sigma_0(x)\,dx=P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0}, \tag{35}$$

where

$$\begin{aligned}
&P_{\sigma_0}(\xi)=\int_{\mathbb{R}^n}\Big(\cos(x\cdot\xi)-1\Big)\sigma_0(x)\,dx(j=1,\cdots,n),\\
&Q_{\sigma_0}(\xi)=\int_{\mathbb{R}^n}\sin(x\cdot\xi)\sigma_0(x)\,dx(j=1,\cdots,n),\\
&R_{\sigma_0}=\int_{\mathbb{R}^n}\sigma^j_0(x)\,dx(j=1,\cdots,n).
\end{aligned} \tag{36}$$

Combining Equations (32), (33) and (35) yields

$$
\begin{aligned}
\hat{\sigma}(\xi,t) = \quad & e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0}) + e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t)R_{\sigma_0} + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin(|\xi|t)}{|\xi|}\Big(-i\xi\cdot\big(P_{u_0}(\xi)-iQ_{u_0}(\xi)\big)\Big) + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t)\Big(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)\Big) + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin(|\xi|t)O(|\xi|)\Big(-i\xi\cdot\big(P_{u_0}(\xi)-iQ_{u_0}(\xi)\big)\Big) + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin(|\xi|t)O(|\xi|)\Big(-i\xi\cdot R_{u_0}\Big) + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\cos\Big(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4P'(1)}\theta_1|\xi|^2}\ \Big)O(|\xi|^2)t\Big(-i\xi\cdot\big(P_{u_0}(\xi)-iQ_{u_0}(\xi)\big)\Big) - \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\cos\Big(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}\ \Big)O(|\xi|^2)t(-i\xi\cdot R_{u_0}) - \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin\Big(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\ \Big)O(|\xi|^3)t\Big(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)\Big) + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin\Big(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\ \Big)O(|\xi|^3)tR_{\sigma_0} + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin\big(A(\xi)|\xi|t\big)O(|\xi|)\Big(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)\Big) + \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin\big(A(\xi)|\xi|t\big)O(|\xi|)R_{\sigma_0} \\
=: \quad & e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0}) + e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t)R_{\sigma_0} \\
& + \sum_{j=1}^{10} I_j.
\end{aligned}
\tag{37}
$$

Thus, we arrive at

$$
\begin{aligned}
& \int_{|\xi|\le\delta_0}\Big|\hat{\sigma}(\xi,t) - e^{-\frac{\mu}{2}|\xi|^2 t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0}) + \cos(|\xi|t)R_{\sigma_0}\Big)\Big|^2 d\xi \\
& \qquad \le \quad C\sum_{j=1}^{10}\int_{|\xi|\le\delta_0}|I_j|^2 d\xi.
\end{aligned}
\tag{38}
$$

To establish the estimates in low frequency region, we need the following Lemma that comes from [18,25].

Lemma 4. *For $\xi\in\mathbb{R}^n$, then we have*

$$
\begin{aligned}
|P_{\sigma_0}(\xi)| &\le \mathfrak{K}|\xi|\|\sigma_0\|_{L^{1,1}}, \\
|Q_{\sigma_0}(\xi)| &\le \mathfrak{L}|\xi|\|\sigma_0\|_{L^{1,1}}, \\
|P_{u_0}(\xi)| &\le \mathfrak{K}|\xi|\|u_0\|_{L^{1,1}}, \\
|Q_{u_0}(\xi)| &\le \mathfrak{L}|\xi|\|u_0\|_{L^{1,1}},
\end{aligned}
\tag{39}
$$

where

$$\mathfrak{K} \triangleq: \sup_{\tau \neq 0} \frac{|1-\cos\tau|}{|\tau|} < +\infty, \qquad \mathfrak{L} \triangleq: \sup_{\tau \neq 0} \frac{|\sin\tau|}{|\tau|} < +\infty.$$

Lemma 5. *Assume that $\sigma_0, u_0 \in L^{1,1}(\mathbb{R}^n)$. Let (σ, u) be the solution to problems (2) and (3). Then we have*

$$\begin{aligned}&\int_{|\xi|\leq\delta_0} \left|\hat{\sigma}(\xi,t) - e^{-\frac{\mu}{2}|\xi|^2 t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0}) + \cos(|\xi|t)R_{\sigma_0}\Big)\right|^2 d\xi \\ \leq\ & C\Big(\|\sigma_0\|^2_{L^{1,1}} + |R_{\sigma_0}|^2 + \|u_0\|^2_{L^{1,1}} + |R_{u_0}|^2\Big)(1+t)^{-(n/2+1)}.\end{aligned} \tag{40}$$

Proof. It follows from Lemma 4 that

$$\int_{|\xi|\leq\delta_0} |I_1|^2 d\xi \leq C\|u_0\|^2_{L^{1,1}} \int_{|\xi|\leq\delta_0} |\xi|^2 e^{-\mu|\xi|^2 t} d\xi \leq C\|u_0\|^2_{L^{1,1}}(1+t)^{-(n/2+1)}. \tag{41}$$

$$\int_{|\xi|\leq\delta_0} |I_2|^2 d\xi \leq C\|\sigma_0\|^2_{L^{1,1}} \int_{|\xi|\leq\delta_0} |\xi|^2 e^{-\mu|\xi|^2 t} d\xi \leq C\|\sigma_0\|^2_{L^{1,1}}(1+t)^{-(n/2+1)}. \tag{42}$$

$$\int_{|\xi|\leq\delta_0} |I_3|^2 d\xi \leq C\|u_0\|^2_{L^{1,1}} \int_{|\xi|\leq\delta_0} |\xi|^6 e^{-\mu|\xi|^2 t} d\xi \leq C\|u_0\|^2_{L^{1,1}}(1+t)^{-(n/2+3)}. \tag{43}$$

$$\int_{|\xi|\leq\delta_0} |I_4|^2 d\xi \leq C|R_{u_0}|^2 \int_{|\xi|\leq\delta_0} |\xi|^2 e^{-\mu|\xi|^2 t} d\xi \leq C|R_{u_0}|^2(1+t)^{-(n/2+1)}. \tag{44}$$

$$\int_{|\xi|\leq\delta_0} |I_5|^2 d\xi \leq C\|u_0\|^2_{L^{1,1}} t^2 \int_{|\xi|\leq\delta_0} |\xi|^8 e^{-\mu|\xi|^2 t} d\xi \leq C\|u_0\|^2_{L^{1,1}}(1+t)^{-(n/2+2)}. \tag{45}$$

$$\int_{|\xi|\leq\delta_0} |I_6|^2 d\xi \leq C|R_{u_0}|^2 t^2 \int_{|\xi|\leq\delta_0} |\xi|^6 e^{-\mu|\xi|^2 t} d\xi \leq C|R_{u_0}|^2(1+t)^{-(n/2+1)}. \tag{46}$$

$$\int_{|\xi|\leq\delta_0} |I_7|^2 d\xi \leq C\|\sigma_0\|^2_{L^{1,1}} t^2 \int_{|\xi|\leq\delta_0} |\xi|^8 e^{-\mu|\xi|^2 t} d\xi \leq C\|\sigma_0\|^2_{L^{1,1}}(1+t)^{-(n/2+2)}. \tag{47}$$

$$\int_{|\xi|\leq\delta_0} |I_8|^2 d\xi \leq C|R_{\sigma_0}|^2 t^2 \int_{|\xi|\leq\delta_0} |\xi|^6 e^{-\mu|\xi|^2 t} d\xi \leq C|R_{\sigma_0}|^2(1+t)^{-(n/2+1)}. \tag{48}$$

$$\int_{|\xi|\leq\delta_0} |I_9|^2 d\xi \leq C\|\sigma_0\|^2_{L^{1,1}} \int_{|\xi|\leq\delta_0} |\xi|^4 e^{-\mu|\xi|^2 t} d\xi \leq C\|\sigma_0\|^2_{L^{1,1}}(1+t)^{-(n/2+2)}. \tag{49}$$

and

$$\int_{|\xi|\leq\delta_0} |I_{10}|^2 d\xi \leq C|R_{\sigma_0}|^2 \int_{|\xi|\leq\delta_0} |\xi|^2 e^{-\mu|\xi|^2 t} d\xi \leq C|R_{\sigma_0}|^2(1+t)^{-(n/2+1)}. \tag{50}$$

We substitute Equations (41)–(50) into Equation (38) and obtain Equation (40). The Lemma is proved. □

Noting that

$$\begin{aligned}
\hat{u}(\xi,t) &= -i(1+\alpha|\xi|^2)\xi\hat{\mathcal{G}}\hat{\sigma}_0 + (-\mu|\xi|^2\hat{\mathcal{G}}+\hat{\mathcal{H}})\frac{\xi(\xi\cdot\hat{u}_0)}{|\xi|^2} \\
&= e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin(|\xi|t)}{|\xi|}(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0}))+ \\
&\quad e^{-\frac{\mu}{2}|\xi|^2 t}\sin(|\xi|t)O(|\xi|)(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0}))+ \\
&\quad e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}\,)O(|\xi|^2)t(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0})) \\
&\quad \alpha|\xi|^2\hat{\mathcal{G}}(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0}))- \\
&\quad \mu\hat{\mathcal{G}}(\xi(\xi\cdot(P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0})))+ \\
&\quad e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t)(\frac{\xi\xi\cdot}{|\xi|^2}(P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0}))- \\
&\quad {}^{-\frac{\mu}{2}|\xi|^2 t}\sin(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\,)O(|\xi|^3)t(\frac{\xi\xi\cdot}{|\xi|^2}(P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0}))+ \\
&\quad e^{-\frac{\mu}{2}|\xi|^2 t}\sin(A(\xi)|\xi|t)O(|\xi|)(\frac{\xi\xi\cdot}{|\xi|^2}(P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0})).
\end{aligned} \tag{51}$$

From (51), we obtain

$$\begin{aligned}
&\hat{u}(\xi,t) - e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin(|\xi|t)}{|\xi|}(-i\xi R_{\sigma_0}) - e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t)\frac{\xi(\xi\cdot R_{u_0})}{|\xi|^2} \\
= \;& e^{-\frac{\mu}{2}|\xi|^2 t}\frac{\sin(|\xi|t)}{|\xi|}(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)))+ \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin(|\xi|t)O(|\xi|)(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0}))+ \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_1|\xi|^2}\,)O(|\xi|^2)t(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0})) \\
& \alpha|\xi|^2\hat{\mathcal{G}}(-i\xi(P_{\sigma_0}(\xi)-iQ_{\sigma_0}(\xi)+R_{\sigma_0}))- \\
& \mu\hat{\mathcal{G}}(\xi(\xi\cdot(P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0})))+ \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\cos(|\xi|t)(\frac{\xi\xi\cdot}{|\xi|^2}(P_{u_0}(\xi)-iQ_{u_0}(\xi)))- \\
& {}^{-\frac{\mu}{2}|\xi|^2 t}\sin(|\xi|t\sqrt{1+\frac{4\alpha-\mu^2}{4}\theta_2|\xi|^2}\,)O(|\xi|^3)t(\frac{\xi\xi\cdot}{|\xi|^2}(P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0}))+ \\
& e^{-\frac{\mu}{2}|\xi|^2 t}\sin(A(\xi)|\xi|t)O(|\xi|)(\frac{\xi\xi\cdot}{|\xi|^2}(P_{u_0}(\xi)-iQ_{u_0}(\xi)+R_{u_0})) \\
=: \;& \sum_{j=1}^{8} J_j.
\end{aligned} \tag{52}$$

Lemma 6. *Assume that σ_0, $u_0 \in L^{1,1}(\mathbb{R}^n)$. Let (σ,u) be the solution to problems (2) and (3). Then we have*

$$\begin{aligned}
&\int_{|\xi|\le\delta_0}\Big|\hat{u}(\xi,t) - e^{-\frac{\mu}{2}|\xi|^2 t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{\sigma_0}) + \cos(|\xi|t)\frac{\xi(\xi\cdot R_{u_0})}{|\xi|^2}\Big)\Big|^2 d\xi \\
\le\;& C\Big(\|\sigma_0\|^2_{L^{1,1}} + |R_{\sigma_0}|^2 + \|u_0\|^2_{L^{1,1}} + |R_{u_0}|^2\Big)(1+t)^{-(n/2+1)}.
\end{aligned} \tag{53}$$

Proof. Lemma 4 entails that

$$\int_{|\xi|\le\delta_0}|J_1|^2d\xi \le C\|\sigma_0\|^2_{L^{1,1}}\int_{|\xi|\le\delta_0}|\xi|^2e^{-\mu|\xi|^2t}d\xi \le C\|\sigma_0\|^2_{L^{1,1}}(1+t)^{-(n/2+1)},$$

$$\begin{aligned}\int_{|\xi|\le\delta_0}|J_2|^2d\xi &\le C(\|\sigma_0\|^2_{L^{1,1}}+|R_{\sigma_0}|^2)\int_{|\xi|\le\delta_0}|\xi|^4e^{-\mu|\xi|^2t}d\xi\\ &\le C(\|\sigma_0\|^2_{L^{1,1}}+|R_{\sigma_0}|^2)(1+t)^{-(n/2+2)},\\ \int_{|\xi|\le\delta_0}|J_3|^2d\xi &\le C(\|\sigma_0\|^2_{L^{1,1}}+|R_{\sigma_0}|^2)t^2\int_{|\xi|\le\delta_0}|\xi|^6e^{-\mu|\xi|^2t}d\xi\\ &\le C(\|\sigma_0\|^2_{L^{1,1}}+|R_{\sigma_0}|^2)(1+t)^{-(n/2+1)},\\ \int_{|\xi|\le\delta_0}|J_4|^2d\xi &\le C(\|\sigma_0\|^2_{L^{1,1}}+|R_{\sigma_0}|^2)\int_{|\xi|\le\delta_0}|\xi|^4e^{-\mu|\xi|^2t}d\xi\\ &\le C(\|\sigma_0\|^2_{L^{1,1}}+|R_{\sigma_0}|^2)(1+t)^{-(n/2+2)},\\ \int_{|\xi|\le\delta_0}|J_5|^2d\xi &\le C(\|u_0\|^2_{L^{1,1}}+|R_{u_0}|^2)\int_{|\xi|\le\delta_0}|\xi|^2e^{-c|\xi|^2t}d\xi\\ &\le C(\|u_0\|^2_{L^{1,1}}+|R_{u_0}|^2)(1+t)^{-(n/2+1)},\end{aligned}$$

$$\int_{|\xi|\le\delta_0}|J_6|^2d\xi \le C\|u_0\|^2_{L^{1,1}}\int_{|\xi|\le\delta_0}|\xi|^2e^{-\mu|\xi|^2t}d\xi \le C\|u_0\|^2_{L^{1,1}}(1+t)^{-(n/2+1)},$$

$$\begin{aligned}\int_{|\xi|\le\delta_0}|J_7|^2d\xi &\le C(\|u_0\|^2_{L^{1,1}}+|R_{u_0}|^2)t^2\int_{|\xi|\le\delta_0}|\xi|^6e^{-\mu|\xi|^2t}d\xi\\ &\le C(\|u_0\|^2_{L^{1,1}}+|R_{u_0}|^2)(1+t)^{-(n/2+1)}\end{aligned}$$

and

$$\begin{aligned}\int_{|\xi|\le\delta_0}|J_8|^2d\xi &\le C(\|u_0\|^2_{L^{1,1}}+|R_{u_0}|^2)\int_{|\xi|\le\delta_0}|\xi|^2e^{-\mu|\xi|^2t}d\xi\\ &\le C(\|u_0\|^2_{L^{1,1}}+|R_{u_0}|^2)(1+t)^{-(n/2+1)}.\end{aligned}$$

Inserting the above estimates into Equation (52) yields Equation (53). We have completed the proof of Lemma 6. □

5. High Frequency Case

Lemma 7. *Assume that $\sigma_0\in H^1(\mathbb{R}^n)\bigcap L^{1,1}(\mathbb{R}^n)$ and $u_0\in L^2(\mathbb{R}^n)\bigcap L^{1,1}(\mathbb{R}^n)$. Let (σ,u) be the solution to problems (2) and (3). Then for $t\gg 1$ we have*

$$\begin{aligned}&\int_{|\xi|\ge\delta_0}\Big|\hat\sigma(\xi,t)-e^{-\frac{\mu}{2}|\xi|^2t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0})+\cos(|\xi|t)R_{\sigma_0}\Big)\Big|^2d\xi\\ \le\ & C\Big(\|\sigma_0\|^2_{L^2}+\|u_0\|^2_{L^2}+|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)e^{-ct}\end{aligned}\tag{54}$$

and

$$\begin{aligned}&\int_{|\xi|\ge\delta_0}\Big|\hat u(\xi,t)-e^{-\frac{\mu}{2}|\xi|^2t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{\sigma_0})+\cos(|\xi|t)\frac{\xi(\xi\cdot R_{u_0})}{|\xi|^2}\Big)\Big|^2d\xi\\ \le\ & C\Big(\|\sigma_0\|^2_{H^1}+\|u_0\|^2_{L^2}+|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)e^{-ct}.\end{aligned}\tag{55}$$

Proof. Owing to the Minkowski inequality and Lemma 1, we deduce that

$$\begin{aligned}
&\int_{|\xi|\geq\delta_0}\Big|\hat{\sigma}(\xi,t)-e^{-\frac{\mu}{2}|\xi|^2t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0})+\cos(|\xi|t)R_{\sigma_0}\Big)\Big|^2d\xi\\
\leq\;& 2\int_{|\xi|\geq\delta_0}\big|\hat{\sigma}(\xi,t)|^2d\xi+C(|R_{\sigma_0}|^2+|R_{u_0}|^2)\int_{|\xi|\geq\delta_0}e^{-\mu|\xi|^2t}d\xi\\
\leq\;& C\int_{|\xi|\geq\delta_0}e^{-c|\xi|^2t}((1+|\xi|^2)^{-1}|\hat{u}_0|^2+|\hat{\sigma}_0|^2)d\xi+C(|R_{\sigma_0}|^2+|R_{u_0}|^2)e^{-\frac{\mu}{2}\delta_0^2t}\int_{|\xi|\geq\delta_0}e^{-\frac{\mu}{2}|\xi|^2t}d\xi\\
\leq\;& C\Big(\|\sigma_0\|_{L^2}^2+\|u_0\|_{L^2}^2\Big)e^{-ct}+C\Big(|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)e^{-ct}t^{-n/2}\\
\leq\;& C\Big(\|\sigma_0\|_{L^2}^2+\|u_0\|_{L^2}^2+|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)e^{-ct},
\end{aligned}$$

where we have used $t\gg 1$ in the third inequality. By using Minkowski inequality, (18), Lemma 2, and Equation (20), we arrive at

$$\begin{aligned}
&\int_{|\xi|\geq\delta_0}\Big|\hat{u}(\xi,t)-e^{-\frac{\mu}{2}|\xi|^2t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{\sigma_0})+\cos(|\xi|t)\frac{\xi(\xi\cdot R_{u_0})}{|\xi|^2}\Big)\Big|^2d\xi\\
\leq\;& 2\int_{|\xi|\geq\delta_0}\big|\hat{u}(\xi,t)|^2d\xi+C(|R_{\sigma_0}|^2+|R_{u_0}|^2)\int_{|\xi|\geq\delta_0}e^{-\mu|\xi|^2t}d\xi\\
\leq\;& C\int_{|\xi|\geq\delta_0}(|\hat{d}(\xi,t)|^2+|\hat{\sigma}(\xi,t)|^2)d\xi+C(|R_{\sigma_0}|^2+|R_{u_0}|^2)e^{-\frac{\mu}{2}\delta_0^2t}\int_{|\xi|\geq\delta_0}e^{-\frac{\mu}{2}|\xi|^2t}d\xi\\
\leq\;& C\Big(\|\sigma_0\|_{H^1}^2+\|u_0\|_{L^2}^2\Big)e^{-ct}+C\int_{|\xi|\geq\delta_0}e^{-2\mu_1|\xi|^2t}|\widehat{\Lambda^{-1}\mathrm{curl}u_0}|^2d\xi+\\
&C\Big(|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)e^{-ct}t^{-n/2}\\
\leq\;& C\Big(\|\sigma_0\|_{H^1}^2+\|u_0\|_{L^2}^2+|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)e^{-ct}.
\end{aligned}$$

Thus, Lemma 7 is proved. □

6. Asymptotic Profiles and Applications

In this section, our aim is to establish the asymptotic profiles of solutions to problems (2) and (3) and give two applications.

6.1. Asymptotic Profiles

In this section, we state the asymptotic profiles of solutions to problems (2) and (3).

Theorem 1. *Assume that $\sigma_0\in H^1\bigcap\in L^{1,1}$ and $u_0\in L^2\bigcap\in L^{1,1}$. Let (σ,u) be the solutions to problems (2) and (3). Then for $t\gg 1$ we have*

$$\begin{aligned}
&\int_{\mathbb{R}^n}\Big|\hat{\sigma}(\xi,t)-e^{-\frac{\mu}{2}|\xi|^2t}\Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0})+\cos(|\xi|t)R_{\sigma_0}\Big)\Big|^2d\xi\\
\leq\;& C\Big(\|\sigma_0\|_{L^{1,1}}^2+\|u_0\|_{L^{1,1}}^2+|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)t^{-(n/2+1)}+\\
&C\Big(\|\sigma_0\|_{L^2}^2+\|u_0\|_{L^2}^2+|R_{\sigma_0}|^2+|R_{u_0}|^2\Big)e^{-ct}
\end{aligned}\tag{56}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^n} \Big| \hat{u}(\xi,t) - e^{-\frac{\mu}{2}|\xi|^2 t} \Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{\sigma_0}) + \cos(|\xi|t)\frac{\xi(\xi\cdot R_{u_0})}{|\xi|^2} \Big) \Big|^2 d\xi \\ \leq \;& C\Big(\|\sigma_0\|^2_{L^{1,1}} + \|u_0\|^2_{L^{1,1}} + |R_{\sigma_0}|^2 + |R_{u_0}|^2 \Big) t^{-(n/2+1)} + \\ & C\Big(\|\sigma_0\|^2_{H^1} + \|u_0\|^2_{L^2} + |R_{\sigma_0}|^2 + |R_{u_0}|^2 \Big) e^{-ct}, \end{aligned} \tag{57}$$

where R_{σ_0} and R_{u_0} are given by Equations (34) and (36), respectively.

Proof. Lemmas 5–7 imply that Equations (56) and (57) hold immediately. The proof of Theorem 1 is completed. □

6.2. Application I

By the above asymptotic profile of solutions, it is not difficult to find that solution (σ, u) satisfies the following decay estimate, which may be found in [10].

Corollary 1. *Assume that $\sigma_0 \in H^1 \bigcap \in L^{1,1}$ and $u_0 \in L^2 \bigcap \in L^{1,1}$. Let (σ, u) be the solution to problems (2) and (3). Then for $t \gg 1$ we have*

$$\|\sigma(t)\|^2_{L^2} \leq C(\|\sigma_0\|^2_{L^{1,1}} + \|u_0\|^2_{L^{1,1}} + |R_{\sigma_0}|^2 + |R_{u_0}|^2 + \|\sigma_0\|^2_{L^2} + \|u_0\|^2_{L^2}) t^{-n/2} \tag{58}$$

and

$$\|u(t)\|^2_{L^2} \leq C(\|\sigma_0\|^2_{L^{1,1}} + \|u_0\|^2_{L^{1,1}} + |R_{\sigma_0}|^2 + |R_{u_0}|^2 + \|\sigma_0\|^2_{H^1} + \|u_0\|^2_{L^2}) t^{-n/2}. \tag{59}$$

Proof. The proof of Equations (58) and (59) is similar. We only prove (58). Due to, Equation (56), it holds that

$$\begin{aligned} \|\sigma(t)\|^2_{L^2} &= \int_{\mathbb{R}^n} |\hat{\sigma}(\xi,t)|^2 d\xi \\ &\leq 2\int_{\mathbb{R}^n} \Big| \hat{\sigma}(\xi,t) - e^{-\frac{\mu}{2}|\xi|^2 t} \Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0}) + \cos(|\xi|t) R_{\sigma_0} \Big) \Big|^2 d\xi \\ &\quad + 2\int_{\mathbb{R}^n} \Big| e^{-\frac{\mu}{2}|\xi|^2 t} \Big(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi\cdot R_{u_0}) + \cos(|\xi|t) R_{\sigma_0} \Big) \Big|^2 d\xi \\ &\leq C\Big(\|\sigma_0\|^2_{L^{1,1}} + \|u_0\|^2_{L^{1,1}} + |R_{\sigma_0}|^2 + |R_{u_0}|^2 + \|\sigma_0\|^2_{L^2} + \|u_0\|^2_{L^2} \Big) t^{-(n/2+1)} + \\ &\quad C(|R_{\sigma_0}|^2 + |R_{u_0}|^2) t^{-n/2} \\ &\leq C\Big(\|\sigma_0\|^2_{L^{1,1}} + \|u_0\|^2_{L^{1,1}} + |R_{\sigma_0}|^2 + |R_{u_0}|^2 + \|\sigma_0\|^2_{L^2} + \|u_0\|^2_{L^2} \Big) t^{-n/2}. \end{aligned}$$

Hence Corollary 1 is proved. □

6.3. Application II

By the above asymptotic profile of solutions, we may prove that solution u has optimal decay estimate under suitable conditions. We may refer to [17] for the wave equation with strong damping.

Corollary 2. *Let $n \geq 2$. Under the same conditions of Theorem 1. Assume that $|R_{\sigma_0}| \neq 0$ and $|R_{u_0}|/|R_{\sigma_0}| \ll 1$. Then for $t \gg 1$ we have*

$$ct^{-\frac{n}{4}} \leq \|u(t)\|_{L^2} \leq Ct^{-\frac{n}{4}}. \tag{60}$$

To complete the proof of Corollary 2, we need the following estimates, which have been established in [17].

Lemma 8. *Let $n \geq 1$. There exist constants $C > 0$ depending on n and μ such that*

$$Ct^{-n/2} \leq \int_{\mathbb{R}^n} \left| e^{-\frac{\mu}{2}|\xi|^2 t} \frac{\sin(|\xi|t)}{|\xi|}(i\xi) \right|^2 d\xi \leq Ct^{-n/2} \tag{61}$$

and

$$Ct^{-n/2}|R_{u_0}|^2 \leq \int_{\mathbb{R}^n} \left| e^{-\frac{\mu}{2}|\xi|^2 t} \cos(|\xi|t) \frac{\xi(\xi \cdot R_{u_0})}{|\xi|^2} \right|^2 d\xi \leq C|R_{u_0}|^2 t^{-n/2}. \tag{62}$$

In what follows, we give the details of Corollary 2.

Proof. The right inequality immediately follows from (59). Now, we prove that the left inequality holds. It follows from Equations (57), (61) and (62) that

$$\begin{aligned}
& \|u(t)\|_{L^2} = \|\hat{u}(t)\|_{L^2} \\
\geq \; & \left\| e^{-\frac{\mu}{2}|\xi|^2 t} \left(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi \cdot R_{\sigma_0}) + \cos(|\xi|t) \frac{\xi(\xi \cdot R_{u_0})}{|\xi|^2} \right) \right\|_{L^2} - \\
& \left\| \hat{u}(t) - e^{-\frac{\mu}{2}|\xi|^2 t} \left(\frac{\sin(|\xi|t)}{|\xi|}(-i\xi \cdot R_{\sigma_0}) + \cos(|\xi|t) \frac{\xi(\xi \cdot R_{u_0})}{|\xi|^2} \right) \right\|_{L^2} \\
\geq \; & \left\| e^{-\frac{\mu}{2}|\xi|^2 t} \frac{\sin(|\xi|t)}{|\xi|}(-i\xi \cdot R_{\sigma_0}) \right\|_{L^2} \\
& - \left\| e^{-\frac{\mu}{2}|\xi|^2 t} \cos(|\xi|t) \frac{\xi(\xi \cdot R_{u_0})}{|\xi|^2} \right\|_{L^2} + O(t^{-n/4-1/2}) \\
\geq \; & C_*|R_{\sigma_0}|t^{-n/4} - \tilde{C}_*|R_{u_0}|t^{-n/4} + o(t^{-n/4}).
\end{aligned} \tag{63}$$

Noting that $|R_{\sigma_0}| \neq 0$ and $|R_{u_0}|/|R_{\sigma_0}| \ll 1$, by Equation (63), when $t \to \infty$, we arrive at

$$\|u(t)\|_{L^2} \geq \frac{C_*}{4}(|R_{\sigma_0}| + |R_{u_0}|)t^{-n/4}. \tag{64}$$

Thus we complete the proof of Corollary 2. □

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