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The Decomposition Theorems of AG-Neutrosophic Extended Triplet Loops and Strong AG-(l, l)-Loops

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Abstract: In this paper, some new properties of Abel Grassmann's Neutrosophic Extended Triplet Loop (AG-NET-Loop) were further studied. The following important results were proved: (1) an AG-NET-Loop is weakly commutative if, and only if, it is a commutative neutrosophic extended triplet (NETG); (2) every AG-NET-Loop is the disjoint union of its maximal subgroups. At the same time, the new notion of Abel Grassmann's (l, l)-Loop (AG-(l, l)-Loop), which is the Abel-Grassmann's groupoid with the local left identity and local left inverse, were introduced. The strong AG-(l, l)-Loops were systematically analyzed, and the following decomposition theorem was proved: every strong AG-(l, l)-Loop is the disjoint union of its maximal sub-AG-groups.

Keywords: neutrosophic extended triplet; Abel Grassmann's groupoid; AG-NET-Loop; decomposition theorem; AG-(l, l)-Loop

1. Introduction

The theory of group and semigroup [1–11] are the basic abstract algebraic structure and they all have an associative binary relation. As a generalization of a commutative semigroup, the notion of an Abel Grassmann's groupoid was introduced by Kazim and Naseeruddin [12] in 1972 and this structure is known as the left almost semigroup (LA-semigroup). An AG-groupoid is a non-associative algebraic structure and many features of the AG-groupoid can be studied in [13]. In [14–21], some properties and connections of AG-groupoid, with some classes of algebraic structures, have been investigated. An AG-groupoid is called an AG-group if the left identity and inverse exists, while further research on the AG-group can be found in [22].

As a mathematical tool for dealing with uncertain information, the fuzzy set and the intuitionistic fuzzy set theories are widely used in many fields, such as engineering technology and management science. For example, fuzzy sets can be applied in multi-criteria decision-making (MCDM), and the characteristic objects method (COMET) was developed to solve the problem of MCDM (more information about this topic can be found here: www.comet.edu.pl). As an extension of fuzzy set and intuitionistic fuzzy set, the new concept of neutrosophic logic and neutrosophic set was first proposed by Smarandache in [23], and some new theoretical studies were developed [24–26]. Moreover, the theory of neutrosophic set has been applied in many domains, such as [27], which applies neutrosophic set to the decision-making, proposed a new model for the selection of transport service providers, and the model was tested on a hypothetical example of the evaluation of five transport service providers.

For a neutrosophic set over the universe, let T, I, F be real functions from U to [0,1], an element x from U is noted with respect to ($T(x)$, $I(x)$, $F(x)$). Then $T(x)$, $I(x)$, $F(x)$ are called neutrosophic components. Recently, the new concepts of the neutrosophic triplet group (NTG) and neutrosophic extended triplet group (NETG) were proposed by Smarandache and Ali in [28,29] as an application for neutrosophic sets. For a neutrosophic triplet group (N, *), for any a in N, having its own neutral element (denoted by $neut(a)$) and at least one opposite element (denoted by $anti(a)$) in N relative to $neut(a)$ satisfying the condition:

$$a*neut(a) = neut(a)*a = a,$$

$$a*anti(a) = anti(a)*a = neut(a).$$

The contrast between the neutrosophic set and the neutrosophic triplet group are as shown in Figure 1.

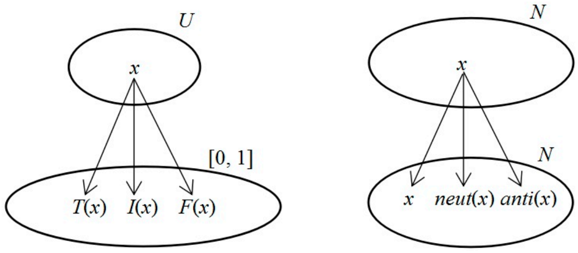

Figure 1. The contrast between neutrosophic sets and neutrosophic triplet groups.

And in [30], sorts of general neutrosophic triplet structures were pointed out, and their basic properties were investigated. For the structure of NETG, some research papers are published with a series of results [31–35].

In [35], the concept of Abel Grassmann's neutrosophic extended triplet loop (AG-NET-loop) was introduced, which is both an AG-groupoid and a neutrosophic extended triplet loop (NET-loop). In this paper, we investigated the Abel Grassmann's neutrosophic extended triplet loop (AG-NET-Loop) further, and introduced the new concept of AG-(l, l)-Loop, which is defined as an Abel Grassmann's groupoid with the local left identity and local left inverse. We analyzed the decomposition theorems of AG-NET-Loop and AG-(l, l)-Loop. The differences between the contents and results of several related papers are described intuitively in Table 1. We also describe the development of groupoids and the relationship with AG-(l, l)-Loop, AG-NET-Loop, and NETG in Figure 2, where the symbol "A→B" means that "A includes B."

Table 1. The differences between the contents and results of several related papers.

Papers	The Algebra Structures Involved	The Associative Law Is Satisfied or Not	Whether to Study Decomposition Theorem
Ref. [30]	Quasi NTL/NETL	√	×
Ref. [32]	NTG/NETG	√	×
Ref. [35]	NETG	√	√
	AG-groupoid	×	×
This paper	AG-NET-Loop	×	√
	AG-(l, l)-Loop	×	√

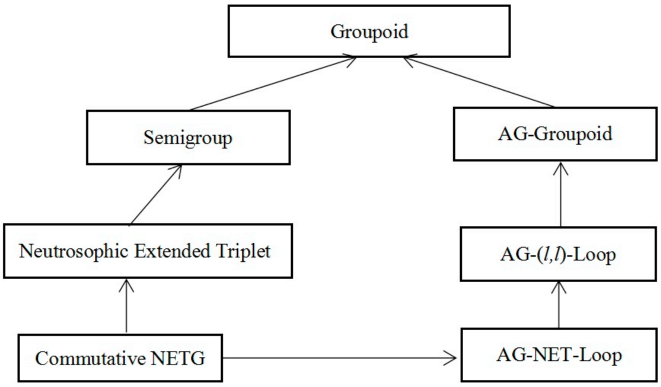

Figure 2. The relationships among various special groupoids.

2. Preliminaries

A groupoid $(S, *)$ is called an Abel Grassmann's groupoid (AG-groupoid) [18,19] if it holds the left invertive law, that is, for all $a, b, c \in S$, $(a*b)*c = (c*b)*a$. In an AG-groupoid the medial law holds, for all $a, b, c, d \in S$, $(a*b)*(c*d) = (a*c)*(b*d)$. An AG-groupoid $(S, *)$ is called locally associative if it satisfies $(a*a)*a = a*(a*a)$, for all a in S. If an AG-groupoid$(S, *)$ with left identity, then it holds $a*(b*c) = b*(a*c)$ and $(a*b)*(c*d) = (d*b)*(c*a)$, for all a, b, c, d in S. If an AG-groupoid $(S, *)$ contains left identity e, then $SS = S$ and $Se = S = eS$. An AG-groupoid $(S, *)$ is called a (left) AG-group, if there exists left identity $e \in S$, for all a $\in$ S there exists $\boldsymbol{a^{-1}} \in \mathbf{S}$ such that $\boldsymbol{a^{-1}}*a = a*\boldsymbol{a^{-1}} = e$.

Proposition 1. *([17]) Let (S, *) be an AG-groupoid with a left identity e. Then the following conditions are equivalent,*

(1) S is an AG-group,
(2) Every element of S has a right inverse,
(3) Every element a of A has a unique inverse $\boldsymbol{a^{-1}}$*,*
*(4) The equation x*a = b has a unique solution for all a, b ∈ S.*

An AG-groupoid is a non-associative algebraic structure midway between a groupoid and a commutative semigroup, because if an AG-groupoid contains a right identity then it becomes a commutative semigroup. An AG-group is a generalization of the abelian group and a special case of quasigroup, which is not commutative or associative in general. But if one of them is allowed, an AG-group becomes an abelian group.

Theorem 1. *([18]) Let (S, *) be an AG-group with local associativity. Then S is an abelian group.*

Theorem 2. *([19]) Let (S, *) be an anti-commutative AG-groupoid. Then the following are equivalent:*

(1) S is a left distributive AG-groupoid,
(2) S is a right distributive AG-groupoid,
(3) S is a distributive AG-groupoid.

Theorem 3. *([19]) Let (N, *) be an AG-group right identity e. Then N is an abelian group.*

Theorem 4. *([19]) Let (N, *) be an AG-group. Then N has exactly one idempotent, which is the left identity.*

Definition 1. *([23]) Let T, I, F be the real standard or non-standard subsets of* $]^{-}0, 1^{+}[$*, with sup T = t_sup, inf T = t_int, sup I = i_sup, inf I = i_inf, sup F = f_sup, inf F = f_inf, and n_sup = t_sup + i_sup + f_sup, n_inf*

= t_inf + i_inf + f_inf. T, I, F are called neutrosophic components. Let U be a universe of discourse, and M a set included in U. An element x from U is noted with respect to the set M as x(T,I,F).

Definition 2. *([28,29]) Assume that N is a non-empty set, and * is a binary operation on N. If for any a ∈ N, there exist neut(a) ∈ N and anti(a) ∈ N such that*

$$a*neut(a) = neut(a)*a = a,$$

$$a*anti(a) = anti(a)*a = neut(a).$$

Then, we call N a neutrosophic extended triplet set. Thus, a neutrosophic extend triplet is (a, neut(a), anti(a)), where neut(a) is extend neutral of "a"(not necessarily the identity element), and anti(a) is the opposite of "a."

In the following, we use the notations $\{neut(a)\}$ and $\{anti(a)\}$ to represent the sets of $neut(a)$ and $anti(a)$; we also use $neut(a)$ and $anti(a)$ to represent any certain one of $neut(a)$ and $anti(a)$.

Definition 3. *([31]) Assume that (N, *) is a neutrosophic extended triplet set (NETS). When (N,*) is a semigroup, N is said to be a neutrosophic extended triplet group. Moreover, when (N,*) is a commutative semigroup, N is said to be a commutative neutrosophic extended triplet group.*

Definition 4. *([32]) Assume that (N, *) is a NETG. When for all a, b ∈ N, neut(a) ∗ b = b ∗ neut(a), N is said to be a weak commutative neutrosophic extended triplet group (WCNETG).*

Proposition 2. *([32]) Let (N, *) be a weak commutative NETG with respect to * and for any a, b ∈ N,*

*(1) neut(a)*neut(b) = neut(b*a),*
*(2) anti(a)*anti(b) ∈ {anti(b*a)}.*

Definition 5. *([32]) Assume that (N, *) is a neutrosophic extended triplet set. When * is well-defined (i.e., ∀a, b ∈ N, a*b ∈ N), (N, *) is said to be a neutrosophic extended triplet loop (NET-loop).*

Remark 1. *In [30,32], the name of neutrosophic triplet loop is used. To be more rigorous with and echo the neutrosophic extended triplet group (NETG), the name of the neutrosophic extended triplet loop (NET-loop) is used in this paper.*

Definition 6. *([35]) Assume that (N, *) is a neutrosophic extended triplet loop (NET-loop). N is called an AG-NET-loop if (N, *) is an AG-groupoid.*

Theorem 5. *([35]) Assume that (N, *) is an AG-NET-loop. Then,*

(1) For all a in N, neut(a) is unique,
*(2) For all a in N, neut(a) = neut(a) * neut(a).*

Theorem 6. *([35]) Let (N, *) be an AG-NET-loop. Then for any a in N,*

(1) neut(a) = neut(neut(a)),
*(2) ∀p ∈ {anti(a)}, neut(a)*p ∈ {anti(a)} and p*neut(a) ∈ {anti(a)}.*

3. AG-NET-Loop

Definition 7. *Assume that (N, *) is an AG-NET-Loop. N is said to be a weak commutative Abel Grassmann's neutrosophic extended triplet loop (AG-NET-Loop), if for all a, b ∈ N, neut(a)∗ b = b∗ neut(a).*

Theorem 7. *Let (N, *) be a groupoid. Then N is a weak commutative Abel Grassmann's neutrosophic extended triplet loop (AG-NET-Loop) if and only if it is a commutative neutrosophic extended triplet group (NETG).*

Proof. Assume that N is a weak commutative AG-NET-Loop. Applying medial law, then for any $a, b \in N$,

$$\begin{aligned} a*b &= (neut(a)*a)*(neut(b)*b) \\ &= (neut(a)*neut(b))*(a*b) \\ &= (neut(b)*neut(a))*(a*b) \\ &= (neut(b)*a)*(neut(a)*b) \\ &= (a*neut(b))*(neut(a)*b) \\ &= (neut(b)*a)*(b*neut(a)) \\ &= (neut(b)*b)*(a*neut(a)) \\ &= b*a \end{aligned}$$

Then N is a commutative AG-NET-Loop, and for any $a, b \in N$,

$$(a*b)*c = (c*b)*a = a*(c*b) = a*(b*c)$$

Therefore, N is a commutative neutrosophic extended triplet group (*NETG*).
Conversely, it is obvious. □

Theorem 8. *Assume that (N, *) is an AG-NET-Loop. Then,*

(1) *For all a, b in N, neut(a*b) = neut(a)*neut(b).*
(2) *For all a, b in N, anti(a)*anti(b) ∈ {anti(a*b)}.*

Proof. For any $a, b \in N$, by the left invertive law and Definition 2, we have

$$\begin{aligned} a*b &= (neut(a)*a)*b \\ &= (b*a)*neut(a) \\ &= [(neut(b)*b)*a]*neut(a) \\ &= [(a*b)*neut(b)]*neut(a) \\ &= [neut(a)*neut(b)]*(a*b) \end{aligned}$$

Similarly, we have

$$a*b = (a*b)*[neut(a)*neut(b)]. \tag{1}$$

Besides, $\forall anti(a) \in \{anti(a)\}$ and $\forall anti(b) \in \{anti(b)\}$, we have

$$\begin{aligned} neut(a)*neut(b) &= (anti(a)*a)*neut(b) \\ &= (neut(b)*a)*anti(a) \\ &= [(anti(b)*b)*a]*anti(a) \\ &= [(a*b)*anti(b)]*anti(a) \\ &= [anti(a)*anti(b)]*(a*b) \end{aligned}$$

Similarly, we have

$$(a*b)*[anti(a)*anti(b)] = neut(a)*neut(b). \tag{2}$$

Through (1) and (2) and by Theorem 8, we get $neut(a)*neut(b) = neut(a*b)$. Hence, using (2), $anti(a)*anti(b) \in \{anti(a*b)\}$. □

Example 1. *Let* $X = \{(a, b) \mid a \in R, b = 1, -1, i \text{ or } -i\}$, *definition* $(a, b)*(c, d) = (ac, b/d)$. *Then*

$$[(a, b)*(c, d)]*(e, f) = (ac, b/d)*(e,f) = (ace, b/df)$$

$$[(e, f)*(c, d)]*(a, b) = (ec, f/d)*(a, b) = (ace, f/bd)$$

Because $b, f \in \{1, -1, i, -i\}$*, hence* $b^2 = f^2$*, and b/f = f/b. We can get b/df = f/bd. Therefore [(a, b)*(c, d)]*(e, f) = [(e, f)*(c, d)]*(a, b), satisfying left invertive law. (1,* -1*) is the neutral of (a, b) and (1/a,* $-i$*) is the opposite of (a, b). when b =* $\pm i$*, we have*

$$(1, -1)*(a, \pm i) = (a, \pm i) \text{ and } (a, \pm i)*(1, -1) = (a, \pm i)$$

$$(1/a, -i)*(a, i) = (1, -1) \text{ and } (a, i)* (1/a, -i) = (1, -1)$$

Example 2. *Denote N = {a, b, c, d, e}, define operations * on N as shown in Table 2. We can verify that (N, *) is an AG-NET-Loop, and*

$$neut\ (a) = a,\ anti\ (a) = a\ neut\ (b) = a\ anti\ (b) = b;$$

$$neut\ (c) = a\ anti\ (c) = d\ neut\ (d) = a\ anti\ (d) = c$$

*It is easy to verify that (N, *) is an AG-NET-Loop.*

Table 2. The operation * on *N*.

*	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*d*
b	*b*	*a*	*d*	*c*
c	*c*	*d*	*b*	*a*
d	*d*	*c*	*a*	*b*

Theorem 9. *Let (N, *) be an AG-NET-Loop. Define a binary* $\approx$ *on N as follows,*

$$\forall x,y \in N,\ x \approx y \Longleftrightarrow neut(x) = neut(y)$$

Then

(1) The binary $\approx$ *is a congruence relation on N, and we denote the equivalent class contained x by* $[x]_\approx$*,*
(2) $\forall a \in N$*,* $[x]_\approx$ *is a subgroup,*
(3) $\forall a \in N$*,* $[x]_\approx$ *is a maximal subgroup, that is, if M is a subgroup of N and* $[x]_\approx \subseteq M$*, then* $[x]_\approx = M$*,*
(4) $N = \bigcup_{x \in N} [x]_\approx$*, that is, every AG-NET-Loop is the disjoint union of its maximal subgroups.*

Proof. (1) Obviously, $\forall x \in N$, *neut(x)* = *neut(x)* $\in N$. Thus $x \approx x$.

Assume $x \approx y$, then *neut(x)* = *neut(y)*, and we know *neut(y)* = *neut(x)*. Thus $y \approx x$.

If $x \approx y$ and $y \approx z$, then we have *neut(x)* = *neut(y)* and *neut(y)* = *neut(z)*, it is obvious that *neut(x)* = *neut(z)*. Thus $x \approx z$.

(2) For any $a \in [x]_\approx$, let *neut(a)* = e_x. By Definition 1, we have $a*e_x = e_x*a = a$.

For any $a, b \in [x]_\approx$, Suppose *neut(a)* = e_x and *neut(b)* = e_x. By Theorem 8, we get *neut(a*b)* = *neut(a)* *neut(b)* = $e_x \in [x]_\approx$.

For any $a, b, c \in [x]_\approx$, let $b = e_x$, then

$$(a * b) * c = (a * e_x) * c = a * c \text{ and } (c * b) * a = (c * e_x) * a = c * a$$

By the left invertive law, *(a*b)*c* = *(c*b)*a*. Thus *a*c* = *c*a*, that is, $[x]_\approx$ satisfies the commutative law. And

$$(a*b)*c = (c*b)*a = a*(c*b) = a*(b*c)$$

Thus $[x]_\approx$ satisfies the associative law.

Suppose $p \in \{anti(a)\}$, by Theorem 6(2), we get $neut(a)*p \in \{anti(a)\}$. Then $\forall a \in [x]_{\approx}$,

$$\begin{aligned} neut(neut(a) * p) &= neut(neut(a)) * neut(p) \\ &= neut(a) * neut(p) \text{ (by Theorem 6(1))} \\ &= neut(a * p) \\ &= neut(neut(a)) \\ &= neut(a) \end{aligned}$$

Therefore, $\forall x \in N$, $[x]_{\approx}$ is the subgroup of N.

(3) For any $a \in M$, because M is the subgroup of N, then $a \in N$. By definition and theorem, every element has a unique neutral element, then it is obvious that $[x]_{\approx} \supseteq$ M. Hence $[x]_{\approx} = M$.

(4) By Theorem 5, for all a in N, *neut(a)* is unique. Then we can know that $N = \bigcup_{x \in N} [x]_{\approx}$. □

Example 3. *Denote N = {a, b, c, d, e}, define operations * on N as shown in Table 3. We can verify that (N, *) is the disjoint union of its maximal subgroups, and*

neut (a) = a, anti (a) = {a,b,c}; neut (b) = b, anti (b) = b;

neut (c) = b, anti (c) = c; neut (d) = a, anti (d) = d;

neut (e) = a, anti (e) = d.

Let S is the set of neutral element "a" and H is the set of neutral element "b". Then S = {a,d,e} and H = {b,c}.

Table 3. The operation * on *N*.

*	a	b	c	d	e
a	a	a	a	d	e
b	a	b	c	d	e
c	a	c	b	d	e
d	d	d	d	e	a
e	e	e	e	a	d

It is easy to verify that N = S∪H, both S and H are subgroups of N.

4. AG-(*l*, *l*)-Loop

Definition 8. *Let (N, *) be an AG-groupoid. Then, N is called an AG-(l, l)-Loop, if for any $a \in N$, exist two elements b and c in N satisfy the condition: b * a = a, and c * a = b.*

Example 4. *Denote N = {a, b, c, d, e}, define operations * on N as shown in Table 4. We can verify that (N, *) is an AG-(l, l)-Loop, and,*

$neut_{(l, l)}(a) = a$, $anti_{(l, l)}(a) = a$; $neut_{(l, l)}(b) = a$, $anti_{(l, l)}(b) = b$; $neut_{(l, l)}(c) = a$, $anti_{(l, l)}(c) = d$;

$neut_{(l, l)}(d) = a$, $anti_{(l, l)}(d) = c$; $neut_{(l, l)}(e) = e$, $anti_{(l, l)}(e) = e$.

*It is easy to verify that (N, *) is an AG-(l, l)-Loop.*

Table 4. The operation * on *N*.

*	a	b	c	d	e
a	a	b	c	d	a
b	b	a	d	c	b
c	d	c	b	a	d
d	c	d	a	b	c
e	a	b	c	d	e

Example 5. *Let X = {(a, b) | a,b ∈ R − {0}, }, definition (a, b)*(c, d) = (a + c − 2ac, d/b). Then*

$$[(a, b)*(c, d)]*(e, f) = (a + c - 2ac, d/b)*(e,f) = (a + c + e - 2ac - 2ea - 2ec - 4ace, fb/d)$$

$$[(e, f)*(c,d)]*(a, b) = (e + c - 2ec, d/f)*(a, b) = (a + c + e - 2ac - 2ea - 2ec - 4ace, fb/d)$$

Therefore [(a, b)(c, d)]*(e, f) = [(e, f)*(c, d)]*(a, b), satisfies the left invertive law. (a, 1) is the neutral of (c, d) and (e, d) is the opposite of (c, d).*

Case 1: when c = 1/2, a = 1/2 and e = 1/2, we have (1/2, 1)(1/2, d) = (1/2, d) and (1/2, d)*(1/2, d) = (1/2, 1)*
Case 2: when c ≠ 1/2, a = 0, we have (0, 1)(c, d) = (c, d) and (e, d)*(c, d) = (0, 1) (when a + c − 2ac = 0).*

Definition 9. *Let (N, ∗) be an AG-(l, l)-Loop. Then N is called a weak commutative AG-(l, l)-Loop, when neut(a) ∗ b = b ∗ neut(a), ∀ a, b ∈ N.*

Theorem 10. *Assume that (N, *) is a weak commutative AG-(l, l)-Loop. Then,*

(1) *For any a, b in N,* $neut_{(l,\,l)}(b) * neut_{(l,\,l)}(a) = neut_{(l,\,l)}(b*a)$.
(2) *For any a, b in N,* $anti_{(l,\,l)}(b) * anti_{(l,\,l)}(a) \in \{anti_{(l,\,l)}(b*a)\}$.

Proof. For any $a, b \in N$, by the left invertive law and Definition 2,

$$\begin{aligned}[neut(l,\ l)(b) *\ neut(l,\ l)(a)] * (b * a) &= [(b * a) *\ neut(l,\ l)\ (a)] *\ neut(l,\ l)\ (b)\\ &= [(neut(l,\ l)(a) * a) * b] *\ neut(l,\ l)(b)\\ &= (a * b) *\ neut(l,\ l)(b)\\ &= (neut(l,\ l)\ (b) * b) * a\\ &= b * a\end{aligned}$$

In addition, we can get

$$(b*a)*[neut(b)*neut(a)] = [neut(a)*neut(b)]*(b*a) = b*a$$

Besides, $\forall anti(a) \in \{anti(a)\}$ and $\forall anti(b) \in \{anti(b)\}$,

$$\begin{aligned}[anti(b) * anti(a)] * (b * a) &= [(b * a) * anti(a)] * anti(b)\\ &= [(anti(a) * a) * b] * anti(b)\\ &= (neut(a) * b) * anti(b)\\ &= (anti(b) * b) * neut(a)\\ &= neut(b) * neut(a)\end{aligned}$$

Similarly, we have $(b*a)*[anti(b)*anti(a)] = neut(b)*neut(a)$. That is

$$(b*a)*[anti(b)*anti(a)] = [anti(b)*anti(a)]*(b*a) = neut(b)*neut(a).$$

From the above, for any a, b in N, by Definition 2, we can get $neut_{(l,\,l)}(b) * neut_{(l,\,l)}(a) = neut_{(l,\,l)}(b*a)$ and $anti_{(l,\,l)}(b) * anti_{(l,\,l)}(a) \in \{anti_{(l,\,l)}(b*a)\}$. □

Theorem 11. *Let (N, ∗) be an AG-(l, l)-Loop. Then N is a weak commutative AG-(l, l)-Loop if and only if it is a commutative neutrosophic extended triplet group (NETG).*

Proof. Assume that N is a weak commutative *AG-(l, l)-Loop*, then for any $a, b \in N$, if $a = b$, we have $a*neut(a) = neut(a)*a = a$. Therefore it is similar to Theorem 7. □

Definition 10. *Let (N, *) be an AG-groupoid. Then, N is called an AG-(r, r)-Loop, if for any $a \in N$, exist two elements b and c in N such that $a * b = b$ and $a * c = b$.*

Example 6. *Let $X = \{(a, b) \mid a,b \in R-\{0\}\}$, definition $(a, b)*(c, d) = (ac, d/b)$. Then*

$$[(a, b)*(c, d)]*(e, f) = (ac, d/b)*(e, f) = (ace, d/bf)$$

$$[(e, f)*(c, d)]*(a, b) = (ec, d/f)*(a, b) = (ace, d/bf)$$

Therefore $[(a, b)*(c, d)]*(e, f) = [(e, f)*(c, d)]*(a, b)$, satisfies the left invertive law. $(1, b^2)$ is the right neutral of (a, b) and $(1/a, b^3)$ is the right opposite of (a, b).

$$(a, b)*(1, b^2) = (a, b)$$

$$(a, b)*(1/a, b^3) = (1, b^2)$$

Theorem 12. *Let (N, ∗) be an AG-(r, r)-Loop. Then N is a weak commutative AG-(r, r)-Loop if, and only if, it is a commutative NETG.*

Proof. It is similar to Theorem 11. □

Definition 11. *Let (N, ∗) be an AG-(l, l)-Loop. Then N is a strong AG-(l, l)-Loop if*

$$neut_{(l,\,l)}(a) * neut_{(l,\,l)}(a) = neut_{(l,\,l)}(a),\ \forall\, a \in N.$$

Example 7. *Denote $N = \{a, b, c, d, e\}$, define operations * on N as shown in Table 5. We can verify that (N, *) is the strong AG-(l, l)-Loop, and*

$$neut_{(l,\,l)}(a) = a,\ anti_{(l,\,l)}(a) = \{d,e,f\};\ neut_{(l,\,l)}(b) = a,\ anti_{(l,\,l)}(b) = b;$$

$$neut_{(l,\,l)}(c) = a,\ anti_{(l,\,l)}(c) = c;\ neut_{(l,\,l)}(d) = d,\ anti_{(l,\,l)}(d) = \{d,e,f\};$$

$$neut_{(l,\,l)}(e) = e,\ anti_{(l,\,l)}(e) = e;\ neut_{(l,\,l)}(f) = e,\ anti_{(l,\,l)}(f) = f.$$

It is easy to verify that N is a strong AG-(l, l)-Loop. Example 5 is also a strong AG-(l, l)-Loop.

Table 5. The operation * on *N*.

*	**a**	**b**	**c**	**d**	**e**	**f**
a	a	b	c	a	a	a
b	c	a	b	c	c	c
c	b	c	a	b	b	b
d	a	b	c	d	d	d
e	a	b	c	d	e	f
f	a	b	c	d	f	e

Theorem 13. *Assume that (N, *) is a strong AG-(l, l)-Loop. Then for all a in N, $neut_{(l,\,l)}(a)$ is unique.*

Proof. Suppose that there exists $x, y \in \{neut_{(l,\, l)}\ (a)\}$. By Definition 2 and 8, $x*a = a$, $y*a = a$, and there exists $p, q \in N$ which satisfy $p*a = x$, $q*a = y$. Applying the invertive law, we have

(i) $x*x = (p*a)*(p*a) = ((p*a)*a)*p = (s*a)*p = a*p$.
(ii) $y*y = (q*a)*(q*a) = ((q*a)*a)*q = (y*a)*q = a*q$.
(iii) $x*x = x$ and $y*y = y$. (by Definition 11)
(iv) $x*y = (p*a)*y = (y*a)*p = a*p = x$.
(v) $y*x = (q*a)*x = (x*a)*q = a*q = y$.
(vi) $x = x*y = (x*x)*y = (y*x)*x = y*x = y$.

Therefore, $neut_{(l,\, l)}\ (a)$ is unique. Moreover, by (i) and (iii) we can get that $p*a = x$ implies $a*p=x$. □

Theorem 14. *Let $(N, *)$ be a strong AG-(l, l)-Loop. Then*

(1) For all a in N, $neut_{(l,\, l)}\ (neut_{(l,\, l)}\ (a)) = neut_{(l,\, l)}\ (a)$;
(2) For any a in N, and for any $q \in \{anti(a)\}$, $neut(a)*q \in \{anti(a)\}$.

Proof. (1) It is similar to Theorem 7(3) in [35].
(2) Suppose $q \in \{anti(a)\}$, then

$$\begin{aligned} a*(neut(a)*q) &= (neut(a)*a)*(neut(a)*q) \\ &= (neut(a)*neut(a))*(a*q) \text{ (by } \textbf{medial law}) \\ &= neut(a)*neut(a) \text{ (by Definition 9)} \\ &= neut(a) \end{aligned}$$

And, $(neut(a)*q)*a = (a*q)*(neut(a)) = neut(a)*neut(a) = neut(a)$.
Therefore we can get $neut(a)*q \in \{anti(a)\}$. □

Theorem 15. *Let $(N, *)$ be a strong AG-(l, l)-Loop. Define a binary $\approx$ on N as follows,*

$$\forall x,y \in N,\ x \approx y \Longleftrightarrow neut(x) = neut(y)$$

Then

(1) The binary $\approx$ is a congruence relation on N, and we denote the equivalent class contained x by $[x]_{\approx}$,
(2) $\forall a \in N$, $[x]_{\approx}$ is a sub-AG-group,
(3) $\forall a \in N$, $[x]_{\approx}$ is maximal sub-AG-group, that is, if M is a sub-AG-group of N and $[x]_{\approx} \subseteq M$, then $[x]_{\approx} = M$,
(4) $N = \bigcup_{x \in N} [x]_{\approx}$, that is, every strong AG-(l, l)-Loop is the disjoint union of its maximal sub-AG-groups.

Proof. (1) It is similar to Theorem 8.
(2) For any $a \in [x]_{\approx}$, let $neut(a) = e_x$, By Definition 8, we have $e_x*a = a$.
For any $a, b, c \in [x]_{\approx}$, by the left invertive law, we have $(a*b)*c = (c*b)*a$.
For any $a, b \in [x]_{\approx}$, Suppose $neut_{(l,\, l)}\ (a) = e_x$ and $neut_{(l,\, l)}\ (b) = e_x$. By Theorem 10, we have $(neut_{(l,\, l)}\ (b)*\ neut_{(l,\, l)}\ (a))*(b*a) = (b*a)$ and $anti_{(l,\, l)}\ (b)*\ anti_{(l,\, l)}\ (a))*(b*a) = neut_{(l,\, l)}\ (b)*\ neut_{(l,\, l)}\ (a)$, therefore $neut_{(l,\, l)}\ (b)*\ neut_{(l,\, l)}\ (a) = neut_{(l,\, l)}\ (b*a) \in [x]_{\approx}$.
Suppose $p \in \{anti(a)\}$, by Theorem 14(2), we have $neut(a)*p \in \{anti(a)\}$.

$$\begin{aligned} neut(neut(a)*p) &= neut(neut(a))*neut(p) \\ &= neut(a)*neut(p) \text{ (by Theorem14(1))} \\ &= neut(a*p) \\ &= neut(neut(a)) \\ &= neut(a) \end{aligned}$$

Thus, $\forall x \in N$, $[x]_{\approx}$ is the sub-AG-group of N.

Therefore, $\forall x \in N$, $[x]_{\approx}$ is the sub-AG-group of N.

(3) It is similar to Theorem 9.

(4) By Theorem 13, for all a in N, *neut(a)* is unique. Then we can know that $N = \bigcup;_{x\in N}[x]_{\approx}$. □

Example 8. *Assume N = {a, b, c, d, e}, define operations * on N as following Table 6. We can verify that (N, *) is the disjoint union of its maximal sub-AG-groups, and*

$$neut(l, l)(a) = a, anti(l, l)\ (a) = a; neut(l, l)(b) = a, anti(l, l)(b) = b;$$

$$neut_{(l,\ l)}(c) = a, anti_{(l,\ l)}(c) = d; neut_{(l,\ l)}(d) = a, anti_{(l,\ l)}(d) = c;$$

$$neut_{(l,\ l)}(e) = e, anti_{(l,\ l)}(e) = e; neut_{(l,\ l)}(f) = e, anti_{(l,\ l)}(f) = f.$$

Let S be the set of neutral element "a" and H the set of neutral element "e". Then S = {a, b, c, d} and H = {e, f}. It is easy to verify that N = S ∪H, both S and H are sub-AG-groups of N.

Table 6. The operation * on *N*.

*	***a***	***b***	***c***	***d***	***e***	***f***
a	*a*	*b*	*c*	*d*	*a*	*b*
b	*b*	*a*	*d*	*c*	*b*	*a*
c	*d*	*c*	*b*	*a*	*d*	*c*
d	*c*	*d*	*a*	*b*	*c*	*d*
e	*a*	*b*	*c*	*d*	*e*	*f*
f	*b*	*a*	*d*	*c*	*f*	*e*

5. Conclusions

We have investigated the structure and properties of NETG and AG-NET-Loop in [35]. In the paper, we studied the structure of the AG-NET-Loop further and introduced the AG-(l, l)-Loop (which is the Abel Grassmann's groupoid with local left identity and local left inverse), gave some infinite examples of them, and obtained some important results. We proved that every weak commutative AG-NET-Loop (or weak commutative AG-(l, l)-Loop) is commutative NETG (CNETG) and every AG-NET-loop is the disjoint union of its maximal subgroups. Moreover, we introduced the new notion of an AG-(l, l)-Loop and investigated the decomposition theorem of a strong AG-(l, l)-Loop. The main results of this paper are described in Figure 3. As the next research direction, we will explore the structure of the combination of the neutrosophic set, fuzzy set, soft set, and algebra systems (see [36–38]).

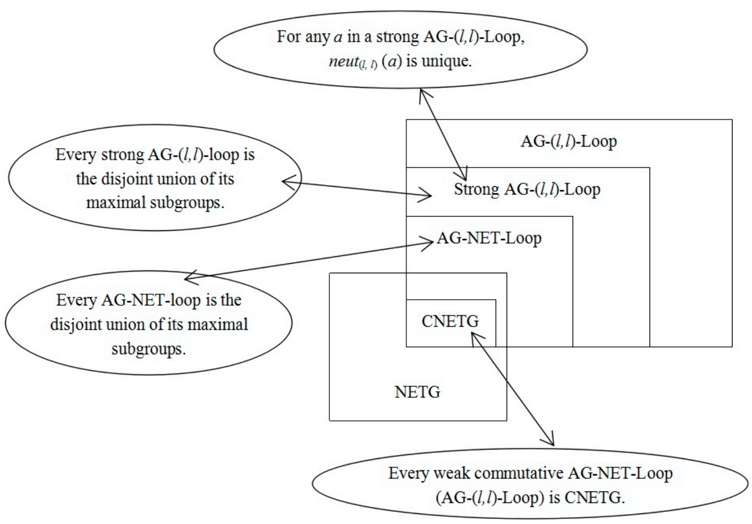

Figure 3. The main results of this paper.

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