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Fractional Calculus of Extended Mittag-Leffler Function and Its Applications to Statistical Distribution

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Abstract: Several fractional calculus operators have been introduced and investigated. In this sequence, we aim to establish the Marichev-Saigo-Maeda (MSM) fractional calculus operators and Caputo-type MSM fractional differential operators of extended Mittag-Leffler function (EMLF). We also investigate the statistical distribution associated with the EMLF. Finally, we derive some of the particular cases of the main results.

Keywords: extended Mittag-Leffler function; Wright-type hypergeometric functions; extended Wright-type hypergeometric functions; statistical distribution

MSC: 33E12; 33B15; 26A33

1. Introduction and Preliminaries

Fractional calculus (FC) is a discipline of mathematics that derives from the conventional definitions of integral and derivative operators by considering fractional values. The reason for attracting the scientist towards FC is that fractional derivatives have been recognized as powerful modeling and simulation tools for engineering problems. Many physical laws are expressed more accurately in terms of differential equations of arbitrary order. The fractal calculus can efficiently deal with kinetics, which is termed the fractal kinetics [1–3]. The Mittag-Leffler (M-L) function and its generalizations are widely used in the field of fractals. The generalized M-L law with fractal calculus appears in [4]. The use of M-L function in the medical field with fractals is given in [5]. In [6], authors defined the M-L function on fractal sets. For more details about the use of the Mittag-Leffler function in the field of fractal calculus and applications, interested readers can refer to [7–9]. FC has potential applications in the variational iteration method (VIM). In [10], authors used the local fractional operators to investigate the application of local fractional VIM for solving the local fractional Laplace equations. A new VIM for a class of fractional convection-diffusion equations is given in [11]. Numerous papers on VIM and its various applications are found in many research articles [12–14].

The Mittag-Leffler (M-L) function introduced in [15] as

$$E_{\rho}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\rho n + 1)} \quad (x \in \mathbb{C};\ \Re(\rho) > 0). \tag{1}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, non-positive integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Many generalizations of the M-L function (1) and the following Wiman's generalization [16]

$$E_{\rho,\sigma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\rho n + \sigma)} \quad (x,\ \sigma \in \mathbb{C};\ \Re(\rho) > 0) \tag{2}$$

have been presented and applied to a variety of research subjects (see, e.g., [17–21]).

Prabhakar [22] introduced the following generalized M-L function

$$E_{\rho,\sigma}^{\gamma}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\rho n + \sigma)} \frac{x^n}{n!} \quad (x, \sigma, \gamma \in \mathbb{C};\ \Re(\rho) > 0), \tag{3}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda,\ \nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ (see, e.g., Section 1.1 of [23]), by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{4}$$

Ozarslan and Yilmaz [24] introduced and investigated the following extended M-L function

$$\begin{aligned} E_{\theta,\vartheta}^{\gamma;c}(x;p) &= \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)(c)_n}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\theta n + \vartheta)} \frac{x^n}{n!} \\ &(x,\ \vartheta \in \mathbb{C};\ p \geq 0;\ \Re(c) > \Re(\gamma) > 0,\ \Re(\theta) > 0). \end{aligned} \tag{5}$$

Here $\mathbf{B}_p(x,y)$ is the extended beta function defined by (see [25])

$$\mathbf{B}_p(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} e^{-\frac{p}{u(1-u)}}\, du \quad (\min\{\Re(p),\ \Re(x),\ \Re(y)\} > 0), \tag{6}$$

where $\mathbf{B}_0(x,y) = \mathbf{B}(x,y)$ is the familiar beta function given by (see, e.g., Section 1.1 of [23])

$$\mathbf{B}(x,\ y) = \begin{cases} \displaystyle\int_0^1 u^{x-1}(1-u)^{y-1}\, du & (\min\{\Re(x),\ \Re(y)\} > 0) \\ \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} & (x,\ y \in \mathbb{C} \setminus \mathbb{Z}_0^-)\,. \end{cases} \tag{7}$$

The familiar generalized hypergeometric series ${}_rF_s$ is defined by (see, e.g., Section 1.5 of [23])

$$\begin{aligned} {}_rF_s\left[\begin{matrix} \alpha_1,\ \ldots,\ \alpha_r\,; \\ \beta_1,\ \ldots,\ \beta_s\,; \end{matrix}\ x\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{x^n}{n!} \\ &= {}_rF_s(\alpha_1,\ \ldots,\ \alpha_r;\ \beta_1,\ \ldots,\ \beta_s;\ x). \end{aligned}$$

Sharma and Devi [26] introduced and investigated the following extended Wright generalized hypergeometric function

$$
\begin{aligned}
&{}_{r+1}\Psi_{s+1}\left[\begin{matrix}(a_i, A_i)_{1,r}\,,\ (\gamma,1); \\ (b_j, B_j)_{1,s}\,,\ (c,1)\,;\end{matrix}\ x; p\right] \\
&= \frac{1}{\Gamma(c-\gamma)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma\left(a_i + kA_i\right)}{\prod_{j=1}^{s} \Gamma\left(b_j + kB_j\right)} \frac{B_p(\gamma+k, c-\gamma)\, x^k}{k!} \\
&\Big(\Re(p) > 0,\ \Re(c) > \Re(\gamma) > 0;\ r,\ s \in \mathbb{N}_0; \\
&a_i,\ b_j \in \mathbb{C},\ A_i,\ B_j \in \mathbb{R}^+,\ i = 1, \ldots, r,\ j = 1, \ldots, s\Big),
\end{aligned}
\tag{8}
$$

where the empty product is understood to be 1 and when the summation is assumed to be convergent.

We recall the fractional integral operators with the Appell function F_3 (see, e.g., [27], p. 53, Equation (6)) as a kernel (see [28,29]): The generalized fractional integral operators involving the Appell functions F_3 are defined for ν, ν', ξ, ξ', $\vartheta \in \mathbb{C}$ with $\Re(\vartheta) > 0$ and $x \in \mathbb{R}^+$as follows:

$$
\left(I_{0+}^{\nu,\nu',\xi,\xi',\vartheta} f\right)(x) = \frac{x^{-\nu}}{\Gamma(\vartheta)} \int_0^x (x-t)^{\vartheta-1} t^{-\nu'} F_3\left(\nu, \nu', \xi, \xi'; \vartheta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)\, \mathrm{d}t \tag{9}
$$

and

$$
\left(I_{-}^{\nu,\nu',\xi,\xi',\vartheta} f\right)(x) = \frac{x^{-\nu'}}{\Gamma(\vartheta)} \int_x^{\infty} (t-x)^{\vartheta-1} t^{-\nu} F_3\left(\nu, \nu', \xi, \xi'; \vartheta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)\, \mathrm{d}t. \tag{10}
$$

The integral operators of the types (9) and (10) have been introduced by Marichev [28] and later extended and studied by Saigo and Maeda [29]. Recently, many researchers (see [30–36]) have studied the image formulas for MSM fractional integral operators involving various special functions.

The corresponding fractional differential operators have their respective forms:

$$
\left(D_{0+}^{\nu,\nu',\xi,\xi',\vartheta} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\vartheta)]+1} \left(I_{0+}^{-\nu',-\nu,-\xi'+[\Re(\vartheta)]+1,-\xi,-\vartheta+[\Re(\vartheta)]+1} f\right)(x) \tag{11}
$$

and

$$
\left(D_{-}^{\nu,\nu',\xi,\xi',\vartheta} f\right)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\vartheta)]+1} \left(I_{-}^{-\nu',-\nu,-\xi',-\xi+[\Re(\vartheta)]+1,-\vartheta+[\Re(\vartheta)]+1} f\right)(x). \tag{12}
$$

Here, we recall the following lemmas (see [29,37]).

Lemma 1. *Let $\nu, \nu', \xi, \xi', \vartheta, \rho \in \mathbb{C}$ be such that $\Re(\vartheta) > 0$ and*

$$
\Re(\tau) > \max\{0, \Re(\nu - \nu' - \xi - \vartheta), \Re(\nu' - \xi')\}.
$$

then there exists the relation

$$
\left(I_{0+}^{\nu,\nu',\xi,\xi',\vartheta}\, t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\,\Gamma(\rho+\vartheta-\nu-\nu'-\xi)\,\Gamma(\tau+\xi'-\nu')}{\Gamma(\rho+\xi')\,\Gamma(\rho+\vartheta-\nu-\nu')\,\Gamma(\rho+\vartheta-\nu'-\xi)} x^{\rho-\nu-\nu'+\vartheta-1}. \tag{13}
$$

Lemma 2. *Let ν, ν', ξ, ξ', ϑ, $\rho \in \mathbb{C}$ such that $\Re(\vartheta) > 0$ and*

$$
\Re(\rho) > \max\{\Re(\xi),\ \Re(-\nu - \nu' + \vartheta),\ \Re(-\nu - \xi' + \vartheta)\}.
$$

then

$$\left(I_{-}^{\nu,\nu',\xi,\xi',\vartheta}t^{-\rho}\right)(x)=\frac{\Gamma\left(-\xi+\rho\right)\Gamma\left(\nu+\nu'-\vartheta+\rho\right)\Gamma\left(\nu+\xi'-\vartheta+\rho\right)}{\Gamma\left(\rho\right)\Gamma\left(\nu-\xi+\rho\right)\Gamma\left(\nu+\nu'+\xi'-\vartheta+\rho\right)}x^{-\nu-\nu'+\vartheta-\rho}.$$

The left- and right-sided generalized integral transforms defined for $x>0$ and $\nu,\xi,\vartheta\in\mathbb{C},\Re(\nu)>0$, respectively by (see [38])

$$\left(I_{0+}^{\nu,\xi,\vartheta}f\right)(x)=\frac{x^{-\nu-\xi}}{\Gamma\left(\nu\right)}\int_{0}^{x}(x-t)^{\nu-1}\ {}_{2}F_{1}\left(\nu+\xi,-\vartheta;\nu;1-\frac{t}{x}\right)f\left(t\right)dt \tag{14}$$

and

$$\left(I_{-}^{\nu,\xi,\vartheta}f\right)(x)=\frac{1}{\Gamma\left(\nu\right)}\int_{x}^{\infty}(t-x)^{\nu-1}t^{-\nu-\xi}\ {}_{2}F_{1}\left(\nu+\xi,-\vartheta;\nu;1-\frac{x}{t}\right)f\left(t\right)dt, \tag{15}$$

where ${}_{2}F_{1}\left(.\right)$ is the Gauss hypergeometric series.

The left- and right-hand-sided Riemann-Liouville fractional integrals of order $\nu\in\mathbb{C}$ are defined by

$$\left(I_{0+}^{\nu}f\right)(x)=\frac{1}{\Gamma\left(\nu\right)}\int_{0}^{x}\frac{1}{(x-t)^{1-\nu}}f(t)dt\quad(\nu\in\mathbb{C},\Re(\nu)>0,x>0) \tag{16}$$

and

$$\left(I_{-}^{\nu}f\right)(x)=\frac{1}{\Gamma\left(\nu\right)}\int_{x}^{\infty}\frac{1}{(t-x)^{1-\nu}}f(t)dt\quad(\nu\in\mathbb{C},\Re(\nu)>0,x<0). \tag{17}$$

Also, we need the following lemmas [38].

Lemma 3. *Let $\nu,\xi,\vartheta\in\mathbb{C}$ be such that $\Re(\nu)>0,\Re(\rho)>\max[0,\Re(\xi-\vartheta)]$. Then*

$$\left(I_{0+}^{\nu,\xi,\vartheta}\ t^{\rho-1}\right)(x)=\frac{\Gamma\left(\rho\right)\Gamma\left(\rho+\vartheta-\xi\right)}{\Gamma\left(\rho-\xi\right)\Gamma\left(\rho+\nu+\vartheta\right)}x^{\rho-\xi-1}. \tag{18}$$

In particular,

$$\left(I_{0+}^{\nu}\ t^{\rho-1}\right)(x)=\frac{\Gamma\left(\rho\right)}{\Gamma\left(\rho+\nu\right)}x^{\rho+\nu-1},\Re(\nu)>0,\Re(\rho)>0 \tag{19}$$

and

$$\left(I_{\vartheta,\nu}^{+}\ t^{\rho-1}\right)(x)=\frac{\Gamma\left(\rho+\vartheta\right)}{\Gamma\left(\rho+\nu+\vartheta\right)}x^{\rho-1},\Re(\nu)>0,\Re(\rho)>-\Re(\vartheta). \tag{20}$$

Lemma 4. *Let $\nu,\xi,\vartheta\in\mathbb{C}$ be such that $\Re(\nu)>0,\Re(\rho)<1+\min[\Re(\xi),\Re(\vartheta)]$. Then*

$$\left(I_{-}^{\nu,\xi,\vartheta}\ t^{\rho-1}\right)(x)=\frac{\Gamma\left(\xi-\rho+1\right)\Gamma\left(\vartheta-\rho+1\right)}{\Gamma\left(1-\rho\right)\Gamma\left(\nu+\xi+\vartheta-\rho+1\right)}x^{\rho-\xi-1}. \tag{21}$$

In particular,

$$\left(I_{-}^{\nu}\ t^{\rho-1}\right)(x)=\frac{\Gamma\left(1-\nu-\rho\right)}{\Gamma\left(1-\rho\right)}x^{\rho+\nu-1},0<\Re(\nu)<1-\Re(\rho) \tag{22}$$

and

$$\left(K_{\vartheta,\nu}^{-}\ t^{\rho-1}\right)(x)=\frac{\Gamma\left(\vartheta-\rho+1\right)}{\Gamma\left(\nu+\vartheta-\rho+1\right)}x^{\rho-1},\Re(\rho)<1+\Re(\vartheta). \tag{23}$$

The generalized forms of the M-L function and its properties have appeared in recent papers [39–41]. The objective of this paper is to present generalized fractional integral and differential operators of EMLF and their application to statistical distribution. The presented work is arranged as follows: In Sections 2 and 3, a form of MSM fractional integral and differential representations of (5) is presented alongside its properties. In Section 4, Caputo-type MSM fractional differential operators are discussed.

In Section 5, we also presented some statistical distribution regarding (5) and conclusion drawn in Section 6.

2. MSM Fractional Integral Representations of Extended Mittag-Leffler Function

Here we present generalized EMLF in view of the MSM fractional integral representations and consider some particular cases.

Theorem 1. *Let $\nu, \nu', \eta, \eta', c, \alpha, \beta, \vartheta, \gamma, \varrho \in \mathbb{C}$ with $\Re(\vartheta) > 0$ and $\Re(\varrho) > max\{0, \Re(\nu + \nu' + \eta - \vartheta), \Re(\nu - \eta')\}$ and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then*

$$\begin{aligned}\left(I_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\vartheta,c}(t;p)\right)(x) &= \tfrac{x^{\varrho-\nu-\nu'+\vartheta-1}}{\Gamma(\gamma)} \\ \times \quad {}_5\psi_5 &\left[\begin{array}{ll}(c,1),(\varrho,1),(\varrho+\vartheta-\nu-\nu'-\eta,1),(\varrho+\eta'-\nu',1),(\gamma,1); & \\ & (x;p) \\ (\beta,\alpha),(\varrho+\eta',1),(\varrho+\vartheta-\nu'-\eta-\nu),(\varrho+\vartheta-\nu-\nu',1),(c,1); & \end{array}\right]\end{aligned} \tag{24}$$

Proof. Let S_1 be LHS of (24), then using (5), we have

$$\begin{aligned} S_1 &= \left(I_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\vartheta,c}(t;p)\right)(x) \\ &= \left(I_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{t^n}{n!}\right)(x)\end{aligned}$$

Interchanging summation and integration order which is verified under the condition in this theorem, we get

$$S_1 = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)n!} \left(I_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho+n-1}\right)(x)$$

Applying the Lemma 1, we get

$$\begin{aligned} S_1 &= \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)n!} \\ &\times \frac{\Gamma(\varrho+n)\Gamma(\varrho+\vartheta-\nu-\nu'-\eta+n)\Gamma(\varrho+\eta'-\nu'+n)}{\Gamma(\varrho+\eta'+n)\Gamma(\varrho+\vartheta-\nu-\nu'+n)\Gamma(\varrho+\vartheta-\nu'-\eta+n)} x^{\varrho+n+\vartheta-\nu-\nu'-1} \\ &= \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)n!} \\ &\times \frac{\Gamma(\varrho+n)\Gamma(\varrho+\vartheta-\nu-\nu'-\eta+n)\Gamma(\varrho+\eta'-\nu'+n)}{\Gamma(\varrho+\eta'+n)\Gamma(\varrho+\vartheta-\nu-\nu'+n)\Gamma(\varrho+\vartheta-\nu'-\eta+n)} x^{\varrho+n+\vartheta-\nu-\nu'-1} \\ &= \frac{x^{\varrho+\vartheta-\nu-\nu'-1}}{\Gamma(\gamma)} \sum_{0}^{\infty} \frac{\mathbf{B}_p(\vartheta+n, c-\vartheta)}{\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)} \\ &\times \frac{\Gamma(\varrho+n)\Gamma(\varrho+\vartheta-\nu-\nu'-\eta+n)\Gamma(\varrho+\eta'-\nu'+n)}{\Gamma(\varrho+\eta'+n)\Gamma(\varrho+\vartheta-\nu-\nu'+n)\Gamma(\varrho+\vartheta-\nu'-\eta+n)} \frac{x^n}{n!}.\end{aligned}$$

Thus, by using (8), we get the result. □

Corollary 1. *Let $\nu, \eta, c, \alpha, \beta, \vartheta, \varrho \in \mathbb{C}$ with $\Re(\vartheta) > 0$ and $\Re(\varrho) > max\{0, \Re(\eta - \vartheta)\}$ and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then*

$$\left(I_{0+}^{\nu,\eta,\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) = \frac{x^{\varrho+\vartheta-1}}{\Gamma(\gamma)} \times {}_4\psi_4 \left[\begin{array}{c} (c,1),(\varrho,1),(\varrho+\vartheta-\eta,1),(\gamma,1); \\ \\ (c,1)(\beta,\alpha),(\varrho-\eta,1),(\varrho+\vartheta+\nu,1); \end{array} (x;p) \right] \tag{25}$$

Theorem 2. *Let $\nu, \nu', \eta, \eta', \alpha, \beta, \vartheta, \gamma, \varrho \in \mathbb{C}$ with $\Re(\nu) > 0$ and $\Re(\varrho) > max\{\Re(\eta), \Re(-\nu-\nu'+\vartheta), \Re(-\nu-\eta'+\vartheta)\}$ and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then*

$$\left(I_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) = \frac{x^{-\varrho-\nu-\nu'+\vartheta}}{\Gamma(\gamma)} \times {}_5\psi_5 \left[\begin{array}{c} (c,1),(\varrho-\eta,1),(\nu+\nu'-\vartheta+\varrho,1),(\nu+\eta'-\vartheta+\varrho,1),(\gamma,1); \\ \\ (c,1),(\beta,\alpha),(1,1)(\nu-\eta+\varrho,1),(\nu+\nu'-\eta'-\vartheta+\varrho,1),(\vartheta,1); \end{array} (x;p) \right] \tag{26}$$

Proof. Let S_2 be LHS of (26), then using (5), we have

$$\begin{aligned} S_2 &= \left(I_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) \\ &= \left(I_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} \sum_0^\infty \frac{\mathbf{B}_p(\gamma+n,c-\gamma)}{\mathbf{B}(\gamma,c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{t^n}{n!}\right)(x) \end{aligned}$$

Interchanging the order of summation and integration i.e., verified under the condition in this theorem, we get

$$S_2 = \sum_0^\infty \frac{\mathbf{B}_p(\gamma+n,c-\gamma)}{\mathbf{B}(\gamma,c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)n!} \left(I_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho+n-1}\right)(x)$$

Applying the Lemma 2, we get

$$\begin{aligned} S_2 &= \sum_0^\infty \frac{\mathbf{B}_p(\gamma+n,c-\gamma)}{\mathbf{B}(\gamma,c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)n!} \\ &\quad \frac{\Gamma(-\eta+\varrho+n)\Gamma(\nu+\nu'+\varrho-\vartheta+n)\Gamma(\nu+\varrho+\eta'-\vartheta+n)}{\Gamma(\varrho+n)\Gamma(\varrho-\eta+\nu+n)\Gamma(\varrho+\nu+\nu'-\eta'-\vartheta+n)} x^{-\varrho+n+\vartheta-\nu-\nu'} \\ &= \sum_0^\infty \frac{\mathbf{B}_p(\gamma+n,c-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)n!} \\ &\times \frac{\Gamma(-\eta+\varrho+n)\Gamma(\nu+\nu'+\varrho-\vartheta+n)\Gamma(\nu+\varrho+\eta'-\vartheta+n)}{\Gamma(\varrho+n)\Gamma(\varrho-\eta+\nu+n)\Gamma(\varrho+\nu+\nu'-\eta'-\vartheta+n)} x^{-\varrho+n+\vartheta-\nu-\nu'} \\ &= \frac{x^{-\varrho+\vartheta-\nu-\nu'}}{\Gamma(\gamma)} \sum_0^\infty \frac{\mathbf{B}_p(\gamma+n,c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)} \\ &\times \frac{\Gamma(-\eta+\varrho+n)\Gamma(\nu+\nu'+\varrho-\vartheta+n)\Gamma(\nu+\varrho+\eta'-\vartheta+n)}{\Gamma(\varrho+n)\Gamma(\varrho-\eta+\nu+n)\Gamma(\varrho+\nu+\nu'-\eta'-\vartheta+n)} \frac{x^n}{n!} \end{aligned}$$

Again, by using (8), we arrived the desired result. □

Corollary 2. *Let $\nu, \eta, c, \alpha, \beta, \gamma, \vartheta, \varrho \in \mathbb{C}$ with $\Re(\vartheta) > 0$ and $\Re(\varrho) > max\{\Re(\eta), \Re(-\vartheta)\}$ and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then*

$$\left(I_{-}^{\nu,\eta,\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) = \frac{x^{\varrho-\eta-1}}{\Gamma(\gamma)} \times {}_4\psi_4 \left[\begin{array}{c} (c,1),(\eta-\varrho+1,1),(\vartheta-\varrho+1,1),(\gamma,1); \\ (c,1),(\beta,\alpha),(1-\varrho,1),(\nu+\eta+\vartheta-\varrho+1,1); \end{array} (x;p) \right] \tag{27}$$

3. MSM Fractional Differential Representations of Extended Mittag-Leffler Function

In this part, we present the MSM fractional differentiation of (5). We recall the following lemmas (see [37]).

Lemma 5. *Let ν, ν', ξ, ξ', ϑ, $\rho \in \mathbb{C}$ such that*

$$\Re(\rho) > \max\left\{0, \Re\left(-\nu+\xi'\right), \Re\left(-\nu-\nu'-\xi+\vartheta\right)\right\}.$$

then

$$\left(D_{0+}^{\nu,\nu',\xi,\xi',\vartheta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(-\xi+\nu+\rho)\Gamma\left(\nu+\nu'+\xi'-\vartheta+\rho\right)}{\Gamma(-\xi+\rho)\Gamma\left(\nu+\nu'-\vartheta+\rho\right)\Gamma\left(\nu+\xi'-\vartheta+\rho\right)} x^{\nu+\nu'-\vartheta+\rho-1}. \tag{28}$$

Lemma 6. *Let ν, ν', ξ, ξ', ϑ, $\rho \in \mathbb{C}$ such that*

$$\Re(\rho) > \max\left\{\Re\left(-\xi'\right), \Re\left(\nu'+\xi-\vartheta\right), \Re\left(\nu+\nu'-\vartheta\right) + \left[\Re(\vartheta)\right] + 1\right\}.$$

then

$$\left(D_{-}^{\nu,\nu',\xi,\xi',\vartheta} t^{-\rho}\right)(x) = \frac{\Gamma(\xi'+\rho)\Gamma(-\nu-\nu'+\vartheta+\rho)\Gamma(-\nu'-\xi+\vartheta+\rho)}{\Gamma(\rho)\Gamma(-\nu'+\xi'+\rho)\Gamma(-\nu-\nu'-\xi+\vartheta+\rho)} x^{\nu+\nu'-\vartheta-\rho}. \tag{29}$$

Now, we establish the following theorems.

Theorem 3. *Let $\nu, \nu', \eta, \eta', c, \alpha, \beta, \gamma, \vartheta, \varrho \in \mathbb{C}$ and $\Re(\varrho) > max\{0, \Re(-\nu+\eta), \Re(-\nu-\nu'-\eta'+\vartheta)\}$ and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then*

$$\left(D_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) = \frac{x^{\varrho+\nu+\nu'-\vartheta-1}}{\Gamma(\gamma)} \times {}_5\psi_5 \left[\begin{array}{c} (c,1),(\varrho,1),(\varrho+\nu-\eta,1),(\nu+\nu'+\eta'-\vartheta+\varrho,1),(\gamma,1); \\ (c,1),(\beta,\alpha),(\varrho-\eta,1),(\nu+\nu'-\vartheta+\varrho,1),(\nu+\varrho-\vartheta+\eta',1); \end{array} (x;p) \right]. \tag{30}$$

Proof. Let S_3 be L. H. S. of (30), then using (5), we have

$$\begin{aligned} S_3 &= \left(D_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) \\ &= \left(D_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho-1} \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{t^n}{n!}\right)(x) \end{aligned}$$

Interchanging summation and integration order i.e., verified under the condition in this theorem, we get

$$S_3 = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)n!} \left(D_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho+n-1}\right)(x)$$

Applying the Lemma 5, we get

$$\begin{aligned}
S_3 &= \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta) n!} \\
&\times \frac{\Gamma(\varrho+n)\Gamma(\varrho+\nu-\eta+n)\Gamma(\varrho+\nu+\nu'+\eta'-\vartheta+n)}{\Gamma(\varrho-\eta+n)\Gamma(\varrho+\nu+\nu'-\vartheta+n)\Gamma(\varrho+\nu+\eta'-\vartheta+n)} x^{\varrho+n-\vartheta+\nu+\nu'-1} \\
&= \frac{x^{\varrho-\vartheta+\nu+\nu'-1}}{\Gamma(\gamma)} \sum_{0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)} \\
&\times \frac{\Gamma(\varrho+n)\Gamma(\varrho+\nu-\eta+n)\Gamma(\varrho+\nu+\nu'+\eta'-\vartheta+n)}{\Gamma(\varrho-\eta+n)\Gamma(\varrho+\nu+\nu'-\vartheta+n)\Gamma(\varrho+\nu+\eta'-\vartheta+n)} \frac{x^n}{n!}
\end{aligned}$$

By using (8), we get the desired result. □

Theorem 4. *Let $\nu, \nu', \eta, \eta', c, \alpha, \beta, \gamma, \vartheta, \varrho \in \mathbb{C}$ and*

$$\Re(\varrho) > max\left\{\Re(-\eta'), \Re(\nu'+\eta-\vartheta), \Re(\nu+\nu'-\vartheta) + [\Re(\vartheta)] + 1\right\}$$

and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then

$$\begin{aligned}
&\left(D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{-\varrho} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) = \tfrac{x^{\nu+\nu'-\vartheta-\varrho}}{\Gamma(\vartheta)} \\
&\times {}_5\psi_5 \left[\begin{array}{lc} (c,1), (\eta'+\varrho,1), (\varrho-\nu-\nu'+\vartheta,1), (\varrho+\eta-\nu',1), (\gamma,1); & \\ & (x;p) \\ (c,1), (\beta,\alpha), (\varrho,1), (\varrho-\nu'+\eta',1), (\varrho-\nu-\nu'+\vartheta-\eta',1); & \end{array}\right].
\end{aligned} \tag{31}$$

Proof. Let S_4 be LHS of (31), then using (5), we have

$$\begin{aligned}
S_4 &= \left(D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{-\varrho} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) \\
&= \left(D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{-\varrho} \sum_{0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{t^n}{n!}\right)(x)
\end{aligned}$$

Interchanging summation and integration order i.e., verified under the condition in this theorem, we get

$$S_4 = \sum_{0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta) n!} \left(D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{-\varrho+n}\right)(x)$$

Applying the Lemma 6, we get

$$\begin{aligned}
S_4 &= \sum_{0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}_p(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta) n!} \\
&\times \frac{\Gamma(\eta'+\varrho+n)\Gamma(\varrho-\nu-\nu'+\vartheta+n)\Gamma(\varrho-\nu'+\eta+\vartheta+n)}{\Gamma(\varrho+n)\Gamma(\varrho-\nu'+\eta'+n)\Gamma(\varrho-\nu-\nu'-\eta'+\vartheta+n)} x^{\nu+\nu'-\varrho-\vartheta+n} \\
&= \frac{x^{\nu+\nu'-\varrho-\vartheta}}{\Gamma(\gamma)} \sum_{0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)} \\
&\times \frac{\Gamma(\eta'+\varrho+n)\Gamma(\varrho-\nu-\nu'+\vartheta+n)\Gamma(\varrho-\nu'+\eta+\vartheta+n)}{\Gamma(\varrho+n)\Gamma(\varrho-\nu'+\eta'+n)\Gamma(\varrho-\nu-\nu'-\eta'+\vartheta+n)} \frac{x^n}{n!}
\end{aligned}$$

Again, by using (8), we get the desired result. □

4. Caputo-Type MSM Fractional Differentiation of Extended Mittag-Leffler Function

In this part, we discuss the left- and right-hand-sided Caputo-type fractional derivatives that have the Gauss hypergeometric function in the kernel are given as:

$$({}^cD^{\nu,\xi,\vartheta}_{0,+}f)(x) = (I^{-\nu,+[\Re(\nu)]+1,-\xi,-\vartheta+[\Re(\nu)]+1}_{0,+}f^{[\Re(\nu)]+1})(x). \tag{32}$$

and

$$({}^cD^{\nu,\xi,\vartheta}_{-}f)(x) = (-1)^{[\Re(\nu)]+1}(I^{-\nu,+[\Re(\nu)]+1,-\xi+[\Re(\nu)]+1,\nu+\vartheta}_{-}f^{[\Re(\nu)]+1})(x). \tag{33}$$

where $\nu, \xi, \vartheta, \varrho \in \mathbb{C}$ and $x \in \mathbb{R}^+$.

The left- and right-hand-sided Caputo-type MSM fractional differential operators:

$$({}^cD^{\nu,\nu',\xi,\xi',\vartheta}_{-}f)(x) = (-1)^{[\Re(\nu)]+1}(I^{-\nu,-\nu',-\xi'+[\Re(\vartheta)]+1,-\xi,-\vartheta+[\Re(\vartheta)]+1,-\vartheta+[\Re(\nu)]+1}_{-}f^{[\Re(\nu)]+1})(x), \tag{34}$$

and

$$({}^cD^{\nu,\nu',\xi,\xi',\vartheta}_{-}f)(x) = (-1)^{[\Re(\nu)]+1}(I^{-\nu,-\nu',-\xi,-\xi'+[\Re(\vartheta)]+1,-\vartheta+[\Re(\vartheta)]+1,-\vartheta+[\Re(\nu)]+1}_{-}f^{[\Re(\nu)]+1})(x), \tag{35}$$

where $\nu, \nu', \xi, \xi', \vartheta, \varrho \in \mathbb{C}$ and $x \in \mathbb{R}^+$.

To discuss the Caputo-type MSM fractional differential operator of the extended MLF (5), the following lemmas will be required to prove the proposed result.

Lemma 7. *[37] Let $\nu, \nu', \eta, \eta', \vartheta, \varrho \in \mathbb{C}$ and $[\Re(\nu)] + 1$ with $\Re(\varrho) - m > max\,\{0, \Re(-\nu + \eta), \Re(-\nu - \nu' - \eta' + \vartheta)\}$ and $p \geq 0$. Then*

$$\left(D^{\nu,\nu',\eta,\eta',\vartheta}_{0+}t^{\varrho-1}\right)(x) = \frac{\Gamma(\varrho)\Gamma(\varrho+\nu-\eta-m)\Gamma(\varrho+\nu+\nu'+\eta'-\vartheta-m)}{\Gamma(\varrho-\eta-m)\Gamma(\varrho+\nu+\nu'-\vartheta)\Gamma(\varrho+\nu+\eta'-\vartheta-m)}x^{\varrho-\vartheta+\nu+\nu'-1}.$$

Lemma 8. *[37] Let $\nu, \nu', \eta, \eta', \vartheta, \varrho \in \mathbb{C}$ and $[\Re(\nu)] + 1$ with $\Re(\varrho) + m > max\,\{\Re(-\eta'), \Re(\nu' + \eta - \vartheta), \Re(\nu + \nu' - \vartheta) + [\Re(\vartheta)] + 1\}$. Then*

$$\left(D^{\nu,\nu',\eta,\eta',\vartheta}_{-}t^{-\varrho}\right)(x) = \frac{\Gamma(\varrho+\eta'+m)\Gamma(\varrho-\nu-\nu'+\vartheta)\Gamma(\varrho-\nu'-\eta+\vartheta+m)}{\Gamma(\varrho)\Gamma(\varrho-\nu'+\eta'+m)\Gamma(\varrho-\nu-\nu'-\eta+\vartheta+m)}x^{\nu+\nu'-\vartheta-\varrho}$$

Theorem 5. *Let $\nu, \nu', \eta, \eta', c, \alpha, \beta, \gamma, \vartheta, \varrho \in \mathbb{C}$ and $m = [\Re(\vartheta) + 1]$ $\Re(\varrho) - m > max\,\{0, \Re(-\nu + \eta'), \Re(-\nu - \nu' - \eta' + \vartheta)\}$ and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then*

$$\left({}^cD^{\nu,\nu',\eta,\eta',\vartheta}_{0+}t^{\varrho-1}E^{\gamma,c}_{\alpha,\beta}(t;p)\right)(x) = \frac{x^{\varrho+\nu+\nu'-\vartheta+1}}{\Gamma(\gamma)} \times {}_5\psi_5\left[\begin{array}{c} (c,1),(\varrho,1),(\varrho+\nu-\eta-m,1),(\nu+\nu'+\eta'-\vartheta-m+\varrho,1),(\gamma,1); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x;p) \\ (c,1),(\beta,\alpha),(\varrho-\eta-m,1),(\nu+\nu'-\vartheta+\varrho,1),(\nu+\varrho-\vartheta+\eta'-m,1); \end{array}\right]. \tag{36}$$

Proof. Let S_5 be LHS of (36), then using (5), we have

$$\begin{aligned} S_5 &= \left({}^cD^{\nu,\nu',\eta,\eta',\vartheta}_{0+}t^{\varrho-1}E^{\gamma,c}_{\alpha,\beta}(t;p)\right)(x) \\ &= \left({}^cD^{\nu,\nu',\eta,\eta',\vartheta}_{0+}t^{\varrho-1}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)}{\mathbf{B}(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{t^n}{n!}\right)(x) \end{aligned}$$

Interchanging the order of summation and integration i.e., verified under the condition in this theorem, we get

$$S_5 = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta) n!} \left({}^{c}D_{0+}^{\nu,\nu',\eta,\eta',\vartheta} t^{\varrho+n-1}\right)(x)$$

Applying the Lemma 7, we get

$$\begin{aligned} S_5 &= \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta) n!} \\ &\times \frac{\Gamma(\varrho+n)\Gamma(\varrho+\nu-\eta-m+n)\Gamma(\varrho+\nu+\nu'+\eta'-\vartheta-m+n)}{\Gamma(\varrho-\eta-m+n)\Gamma(\varrho+\nu+\nu'-\vartheta+n)\Gamma(\varrho+\nu+\eta'-\vartheta-m+n)} x^{\varrho+n-\vartheta+\nu+\nu'+1} \\ &= \frac{x^{\varrho-\vartheta+\nu+\nu'+1}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)} \\ &\times \frac{\Gamma(\varrho+n)\Gamma(\varrho+\nu-\eta-m+n)\Gamma(\varrho+\nu+\nu'+\eta'-\vartheta-m+n)}{\Gamma(\varrho-\eta-m+n)\Gamma(\varrho+\nu+\nu'-\vartheta+n)\Gamma(\varrho+\nu+\eta'-\vartheta-m+n)} \frac{x^n}{n!} \end{aligned}$$

In view of (8), we obtain the required result. □

Theorem 6. *Let $\nu, \nu', \eta, \eta', c, \alpha, \beta, \gamma, \vartheta, \varrho \in \mathbb{C}$ and $m = [\Re(\vartheta)+1]$ with*

$$\Re(\varrho) + m > max\left\{\Re(-\eta'), \Re(\nu+\nu'-\vartheta)+m\right\}$$

and $p \geq 0$. Also let $x \in \mathbb{R}^+$, then

$$\begin{aligned} &\left({}^{c}D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{-\varrho} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) = \frac{x^{\varrho+\nu+\nu'-\vartheta}}{\Gamma(\gamma)} \\ &\times {}_5\psi_5 \left[\begin{array}{r} (c,1),(\eta'+\varrho+m,1),(-\nu-\nu'+\vartheta+\varrho,1),(\varrho-\nu'-\eta+\vartheta+m,1),(\gamma,1); \\ (x;p) \\ (c,1),(\beta,\alpha),(\varrho,1),(-\nu'+\eta'+m+\varrho,1),(-\nu-\nu'-\eta+\varrho+\vartheta+m,1); \end{array}\right]. \end{aligned} \tag{37}$$

Proof. Let S_6 be L. H. S. of (37), then using (5), we have

$$\begin{aligned} S_6 &= \left({}^{c}D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{-\varrho} E_{\alpha,\beta}^{\gamma,c}(t;p)\right)(x) \\ &= \left({}^{c}D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{-\varrho} \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{t^n}{n!}\right)(x) \end{aligned}$$

Interchanging summation and integration order, which is verified under the given condition, we get

$$S_6 = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta) n!} \left({}^{c}D_{-}^{\nu,\nu',\eta,\eta',\vartheta} t^{n-\varrho}\right)(x)$$

Applying the Lemma 8, we get

$$\begin{aligned}
S_6 &= \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\mathbf{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta) n!} \\
&\times \frac{\Gamma(\varrho+\eta'+m+n)\Gamma(\varrho-\nu-\nu'+\vartheta+n)\Gamma(\varrho-\nu'-\eta+\vartheta+m+n)}{\Gamma(\varrho+n)\Gamma(\varrho-\nu'+\eta'+m+n)\Gamma(\varrho-\nu-\nu'-\eta+\vartheta+m+n)} x^{-\varrho+n-\vartheta+\nu+\nu'} \\
&= \frac{x^{-\varrho-\vartheta+\nu+\nu'}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)} \\
&\times \frac{\Gamma(\varrho+\eta'+m+n)\Gamma(\varrho-\nu-\nu'+\vartheta+n)\Gamma(\varrho-\nu'-\eta+\vartheta+m+n)}{\Gamma(\varrho+n)\Gamma(\varrho-\nu'+\eta'+m+n)\Gamma(\varrho-\nu-\nu'-\eta+\vartheta+m+n)} \frac{x^n}{n!}.
\end{aligned}$$

By using (8), we get the required result. □

5. Extended Mittag-Leffler Function and Statistical Distribution

For a random variable X, the distribution function is defined by

$$F(x) = P(X \leq x), \tag{38}$$

where x is any real number $-\infty < x < \infty$. The properties of distribution function $F(x)$ as follows

1. $F(x)$ is non-decreasing
2. $\lim_{x\mapsto -\infty} F(x) = 0; \lim_{x\mapsto\infty} F(x) = 1$
3. $F(x)$ is continuous from the right. Many authors studied the distribution function which involves the M-L function [41–44]. In this line, we develop the distribution function involving extended M-L function (5) and deduce particular cases of our result.

Theorem 7. *Let $\mu, \beta, x \in \mathbb{R}^+$ with $0 < \mu \leq 1$ and also let $\gamma, c \in \mathbb{C}$ and $\Re(\beta) > 0$, $\Re(\gamma) > 0, p \geq 0$. Let*

$$F_x(x) = 1 - E_{\mu,\beta}^{\gamma,c}(-x^{\mu}; p) \tag{39}$$

then the density function $f(x)$ of $F_x(x)$ is given as follows:

$$f(x) = \frac{\mu c^2}{\gamma} x^{\mu-1} E_{\mu,\mu+\beta}^{\gamma+1,c+1}(-x^{\mu}; p).$$

Proof. Using (39) and (5),we have

$$\begin{aligned}
F_x(x) &= 1 - E_{\mu,\beta}^{\gamma;c}(-x^{\mu}; p) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\mathbf{B}_p(\gamma+n, c-\gamma)(c)_n}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\mu n+\beta)} \frac{(x^{\mu})^n}{n!}.
\end{aligned} \tag{40}$$

Differentiating each side of (40) with respect to x gives the density function

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\mathbf{B}_p(\gamma+n, c-\gamma)(c)_n}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\mu n+\beta)} \frac{\mu(x^{\mu n-1})}{(n-1)!}.$$

which, upon replacing n by $n+1$, yields

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n}\mathbf{B}_p(\gamma+n+1, c-\gamma)(c)_{n+1}}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\mu n+\mu+\beta)} \frac{\mu(x^{\mu n+\mu-1})}{n!}. \tag{41}$$

Now, by using the following relation in (41), we have

$$\mathbf{B}(b, c-b) = \frac{c}{b}\mathbf{B}(b+1, c-b)$$

$$f(x) \quad = \quad \frac{c}{\gamma}\sum_{n=0}^{\infty}\frac{(-1)^n\mathbf{B}_p(\gamma+1+n, c-\nu)(c)_{n+1}}{\mathbf{B}(\gamma+1, c-\gamma)\Gamma(\mu n+\mu+\beta)}\frac{\mu(x^{\mu n+\mu-1})}{n!}. \tag{42}$$

Using the relation $(c)_{n+1} = c(c+1)_n$ in (42), we have

$$= \quad \frac{\mu c^2}{\gamma}x^{\mu-1}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n+1, c-\gamma)(c+1)_n}{\mathbf{B}(\gamma+1, c-\gamma)\Gamma(\mu n+\mu+\beta)}\frac{(-x^\mu)^n}{n!}$$

Which gives the required result. □

Corollary 3. *Let $\mu, x \in \mathbb{R}^+$ with $0 < \mu \leq 1$ and $\beta = 1$. Also let $\gamma, c \in \mathbb{C}$ and $\Re(\beta) > 0, \Re(\gamma) > 0, p \geq 0$. Let*

$$F_x(x) = 1 - E_{\mu,1}^{\gamma,c}(-x^\mu; p). \tag{43}$$

then the density function $f(x)$ of $F_x(x)$ is given as follows:

$$f(x) \quad = \quad \frac{\mu c^2}{\gamma}x^{\mu-1}E_{\mu,\mu+1}^{\gamma+1,c+1}(-x^\mu; p).$$

6. Concluding Remarks

FC operators have significant applications in the field of science and engineering. Many research papers have been used to solve nonlinear differential equations, VIMs, and fractal related problems with the help of fractional operators. In this lineage, we established generalized fractional formulas to derive numerous results. The operators developed in this paper may have applications in applied mathematics and physics. The significant generality of these results rendered some existing results as particular cases of our result. For instance, if we let $p = 0$, then we obtain MSM fractional integral, MSM fractional differential formulas and Caputo-type MSM fractional differentiation formulas of Mittag-Leffler function defined in (3) (see [34]).

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