mathematics

Article

The "Generator" of Int-Soft Filters on Residuated Lattices

Huarong Zhang * and Minxia Luo *

College of Science, China Jiliang University, Hangzhou 310000, Zhejiang, China
* Correspondence: hrzhang2008@cjlu.edu.cn (H.Z.); 06a0802068@cjlu.edu.cn (M.L.)

Received: 15 January 2019; Accepted: 28 February 2019; Published: 6 March 2019

Abstract: In this paper, we give the "generator" of int-soft filters and propose the notion of t-int-soft filters on residuated lattices. We study the properties of t-int-soft filters and obtain some commonalities (e.g., the extension property, quotient characteristics, and a triple of equivalent characteristics). We also use involution-int-soft filters as an example and show some basic properties of involution-int-soft filters. Finally, we investigate the relations among t-int-soft filters and give a simple method for judging their relations.

Keywords: residuated lattice; soft set; filter; t-filter; t-int-soft filter

---

1. Introduction

Uncertainty widely exists in many practical problems. Compared with probability theory [1], fuzzy sets [2], rough sets [3], intuitionistic fuzzy sets [4], and soft sets [5] have an unparalleled advantage in solving uncertain problems. At present, many scholars are devoted to the study of soft set theory and its applications. The theoretical work mainly focuses on some operations of soft sets [6–8], soft and fuzzy soft relations [9], soft algebraic structures [10], and distance, similarity measures, and equality of soft sets [11–13]. In applications, soft sets are extensively used in decision making problems [14] and forecasting approaches [15].

Residuated lattices [16] originated from mathematical logic without contraction. They combine the fundamental notions of multiplication, order, and residuation. In recent years, many logical algebras have been introduced as the semantic systems of logical systems, for example, Boolean algebras, MV-algebras, BL-algebras, R0-algebras, Heyting algebras, MTL-algebras, and so on. These logical algebras are all special cases of residuated lattices.

Filters correspond to sets of formulae, closed with respect to Modus ponens. So, filters play an important role in investigating the above logical algebras. At present, the filter theories of many fuzzy logical algebras have been extensively studied. On residuated lattices, the relative literature is as follows: [17–26]. In the literature, many concrete types of filters (implicative filters, fantastic filters, Boolean filters, and so on) have been introduced, their equivalent characterizations studied, and the relations among specific filters were investigated on residuated lattices. Notably, Víta [24] proposed the notion of t-filters, in order to cover the great amount of special types of filters, and obtained some basic properties of t-filters.

Recently, some scholars have applied soft sets to the filter theory of logical algebras. In [27,28], Jun et al. proposed the (strong, implicative) intersection-soft (int-soft for short) filters, and established their equivalent characterizations and the extension property of (strong) implicative int-soft filters on R0-algebras. Lin and Ma [29] proved that all int-soft filters formed a bounded distributive lattice and investigated the int-soft congruences, with respect to int-soft filters on residuated lattices. Jun et al. [30] introduced the concepts of int-soft filters, MV-int-soft filters, int-soft G-filters, and regular int-soft filters, explored their properties and characterizations, and provided the conditions for an int-soft filter to be an int-soft G-filter on residuated lattices.

T-filters provide a path toward the unifying of some types of filters. It is natural to ask whether we can give a framework to cover as many int-soft filters as possible and obtain some of their basic features. Based on this, we propose the notion of t-int-soft filters on residuated lattices, and show how particular results about many kinds of int-soft filters are unified by our framework. In addition, research about relations among kinds of int-soft filters is a hot topic. Is there a simple method to find them? In order to answer this question, we particularly investigate the quotient structure of t-int-soft filters, use it to study the relations among t-int-soft filters, and give a simple method for judging their relations. The paper is organized as follows:

In Section 2, some preliminary definitions and theorems are recalled. In Section 3, the notion of t-int-soft filters is proposed. Some characterizations of t-int-soft filters are derived. In Section 4, a specific example of a t-int-soft filter is given. In Section 5, the general principles of investigating the relations among t-int-soft filters are given.

2. Preliminaries

Definition 1. *[16] A residuated lattice is an algebra $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, such that the following conditions hold:*

R1 $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
R2 $(L, \otimes, 1)$ *is a commutative monoid, and*
R3 $x \otimes y \leq z$ *if and only if* $x \leq y \rightarrow z$ *for all* $x, y, z \in L$.

For $x \in L$, we define $x^* = x \rightarrow 0$ and $x^{**} = (x^*)^*$. For a natural number n, we define $x^0 = 1$ and $x^n = x^{n-1} \otimes x$ for $n \geq 1$.

Definition 2. *[17,22,31,32] Let L be a residuated lattice. Then, L is called:*

- *An involutive (or regular) residuated lattice if* $x^{**} = x$ *for* $x \in L$;
- *a Heyting algebra if* $x \otimes y = x \wedge y$ *for all* $x, y \in L$, *which is equivalent to* $x^2 = x$ *for all* $x \in L$;
- *a Rl-monoid if* $x \wedge y = x \otimes (x \rightarrow y)$ *for all* $x, y \in L$ *(axiom of divisibility);*
- *an MTL-algebra if* $(x \rightarrow y) \vee (y \rightarrow x) = 1$ *for all* $x, y \in L$ *(axiom of prelinearity);*
- *a BL-algebra if it satisfies both axioms of prelinearity and divisibility;*
- *an MV-algebra if it is a regular Rl-monoid; and*
- *a Boolean algebra if it is an idempotent MV-algebra.*

In what follows, L denotes a residuated lattice, unless otherwise specified.

We give some rules of calculus on L, which will be needed in the rest of study.

(1) $0^* = 1, 1^* = 0$.
(2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.
(3) $x \otimes y \rightarrow z = x \rightarrow (y \rightarrow z)$.
(4) $x \vee y \rightarrow y = x \rightarrow y$.
(5) $x \leq y$ if and only if $x \rightarrow y = 1$.

Definition 3. *[23] A nonempty subset F of L is called a filter if*

(1) If $x \in F$ *and* $x \leq y$, *then* $y \in F$, *and*
(2) If $x, y \in F$, *then* $x \otimes y \in F$.

Theorem 1. *[23] F is a filter if and only if*

(1) $1 \in F$, *and*

(2) *If $x, x \to y \in F$, then $y \in F$.*

It is clear that $\{1\}$ and L are filters. In addition, given a filter F of L, we can define the relation $\equiv_F$ on L by $x \equiv_F y$ if and only if $x \to y \in F$ and $y \to x \in F$. We can also prove that $\equiv_F$ is a congruence relation. We use L/F to denote the set of the congruence classes of $\equiv_F$ (i.e., $L/F = \{[x]_F | x \in L\}$), where $[x]_F := \{y \in L | y \equiv_F x\}$. If we define the following operations on L/F: $[x]_F \sqcap' [y]_F = [x \wedge y]_F$, $[x]_F \sqcup' [y]_F = [x \vee y]_F$, and $[x]_F \odot' [y]_F = [x \otimes y]_F$, $[x]_F \rightharpoonup' [y]_F = [x \to y]_F$, then, we have the following lemma.

Lemma 1. *[23,26] Let F be a filter of L. Then $(L/F, \sqcap', \sqcup', \odot', \rightharpoonup', [0]_F, [1]_F)$ is a residuated lattice, with respect to F.*

In what follows, we recall some basic concepts of soft sets.

Let U be an initial universe set, and E a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and $A, B, C \cdots \subseteq E$.

Definition 4. *[5] A soft set $(\tilde{f}, A)$ over U is defined to be the set of ordered pairs*

$$(\tilde{f}, A) := \{(x, \tilde{f}(x)) : x \in E, \tilde{f}(x) \in \mathscr{P}(U)\},$$

where $\tilde{f} : E \to \mathscr{P}(U)$, such that $\tilde{f}(x) = \varnothing$ if $x \notin A$.

Now, we take a residuated lattice L as the set of parameters.

Definition 5. *[30] A soft set $(\tilde{f}, L)$ over U is called an int-soft filter of L if it satisfies*

(1) *$\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x \otimes y), \forall x, y \in L$, and*
(2) *If $x \leq y$, then $\tilde{f}(x) \subseteq \tilde{f}(y), \forall x, y \in L$.*

Lemma 2. *[30] Every int-soft filter $(\tilde{f}, L)$ of L satisfies*

(1) *$\tilde{f}(x) \subseteq \tilde{f}(1), \forall x \in L$, and*
(2) *$\tilde{f}(x) \cap \tilde{f}(x \to y) \subseteq \tilde{f}(y), \forall x, y \in L$.*

Theorem 2. *[30] A soft set $(\tilde{f}, L)$ over U is an int-soft filter of L if and only if the set*

$$\tilde{f}_\tau := \{x \in L \mid \tau \subseteq \tilde{f}(x)\}$$

is a filter of L, for all $\tau \in \mathscr{P}(U)$, with $\tilde{f}_\tau \neq \varnothing$.

Lemma 3. *A soft set $(\tilde{T}, L)$ is an int-soft filter of L, where $\tilde{T}$ has the following form*

$$\tilde{T}(x) = \begin{cases} U, & \text{if } x = 1, \\ \varnothing, & \text{if } x \neq 1. \end{cases} \tag{1}$$

Proof. $\forall \tau \in \mathscr{P}(U)$,

$$\tilde{T}_\tau = \begin{cases} L, & \text{if } \tau = \varnothing, \\ \{1\}, & \text{if } \tau \neq \varnothing. \end{cases} \tag{2}$$

By Theorem 2, we know that $(\tilde{T}, L)$ is an int-soft filter. $\square$

Lemma 4. *Let $(\tilde{f}, L)$ be a soft set. If $(\tilde{f}, L)$ is an int-soft filter, then $\tilde{f}_{\tilde{f}(1)}$ is a filter.*

Proof. Let $(\tilde{f}, L)$ be an int-soft filter. Since $1 \in \tilde{f}_{\tilde{f}(1)}$, we have $\tilde{f}_{\tilde{f}(1)} \neq \varnothing$. By Theorem 2, we have $\tilde{f}_{\tilde{f}(1)}$ is a filter. □

3. T-Filters and T-Int-Soft Filters

In this section, we use the symbol $\bar{x}$ to indicate the abbreviation of $x, y, \ldots$; that is, $\bar{x}$ is a formal listing of variables used in a given context. We use the term t to denote a term in the language of residuated lattices. Given a variety $\mathbb{B}$ of residuated lattices, we denote its subvariety satisfying the equation t = 1 by the symbol $\mathbb{B}$[t], and we call this algebra the t-algebra.

Definition 6. *[24] Let t be an arbitrary term on the language of residuated lattices. A filter F of L is a t-filter if $t(\bar{x}) \in F$ for all $\bar{x} \in L$.*

Example 1. *[17,18,21,24–26] Let F be a filter of L. Then F is,*

- *an involution filter (or a regular filter) for $t(\bar{x}) = x^{**} \to x$;*
- *a Heyting filter for $t(\bar{x}) = x \to x^2$;*
- *a Rl-monoid filter (or a divisible filter) for $t(\bar{x}) = (x \wedge y) \to (x \otimes (x \to y))$;*
- *a MTL-filter for $t(\bar{x}) = (x \to y) \vee (y \to x)$;*
- *a BL-filters for $t(\bar{x}) = ((x \to y) \to (x \to z)) \to ((x \to z) \vee (y \to z))$;*
- *a MV-filter for $t(\bar{x}) = ((x \to y) \to y) \to ((y \to x) \to x)$; and*
- *a Boolean filter for $t(\bar{x}) = x \vee x^*$.*

Remark 1. *Many of the filters in Example 1 have different names. To avoid confusion, Buşneag and Piciu [18] proposed a new approach for classifying filters and renamed some filters.*

Theorem 3. *[24] Let $\mathbb{B}$ be a variety of residuated lattices and $L \in \mathbb{B}$. Then, the following statements are equivalent:*

(1) Every filter of L is a t-filter.
(2) $\{1\}$ is a t-filter.
(3) $L \in \mathbb{B}(t)$.

Theorem 4. *[24] Let $\mathbb{B}$ be a variety of residuated lattices, $L \in \mathbb{B}$, and F a filter of L. Then, F is a t-filter if and only if $L/F \in \mathbb{B}(t)$.*

Definition 7. *Let $(\tilde{f}, L)$ be an int-soft filter of L. $(\tilde{f}, L)$ is called a t-int-soft filter if, for all $\tau \in \mathscr{P}(U)$, $\tilde{f}_\tau$ is either empty or a t-filter.*

Remark 2. *According to Definition 7, we can define involution-int-soft filters, MTL-int-soft filters, and BL-int-soft filters.*

Theorem 5. *Let $(\tilde{f}, L)$ be an int-soft filter of L. Then, $(\tilde{f}, L)$ is a t-int-soft filter if and only if $\tilde{f}_{\tilde{f}(1)}$ is a t-filter.*

Proof. Let $(\tilde{f}, L)$ be a t-int-soft filter. Since $1 \in \tilde{f}_{\tilde{f}(1)}$, then $\tilde{f}_{\tilde{f}(1)} \neq \varnothing$. By Definition 7, we have that $\tilde{f}_{\tilde{f}(1)}$ is a t-filter.

Conversely, suppose $(\tilde{f}, L)$ is an int-soft filter of L. Then, by Theorem 2, $\forall \tau \in \mathscr{P}(U)$, we have if $\tilde{f}_\tau \neq \varnothing$, then $\tilde{f}_\tau$ is a filter. Let $\tilde{f}_{\tilde{f}(1)}$ be a t-filter, then $t(\bar{x}) \in \tilde{f}_{\tilde{f}(1)}$. For all $\tau \in \mathscr{P}(U)$, there exist two cases:

(i) $\tau \subseteq \tilde{f}(1)$:

Suppose $x \in \tilde{f}_{\tilde{f}(1)}$, then $\tilde{f}(x) \supseteq \tilde{f}(1) \supseteq \tau$. This shows that $x \in \tilde{f}_\tau$. Thus, $\tilde{f}_{\tilde{f}(1)} \subseteq \tilde{f}_\tau$. Hence, $\mathrm{t}(\bar{x}) \in \tilde{f}_\tau$. By Definition 6, we have that $\tilde{f}_\tau$ is a t-filter.

(ii) $\tau \supset \tilde{f}(1)$:

Since $(\tilde{f}, L)$ is an int-soft filter, then $\forall x \in L$, $\tau \supset \tilde{f}(1) \supseteq \tilde{f}(x)$. Thus, $\tilde{f}_\tau = \varnothing$.

By Definition 7, we know that $(\tilde{f}, L)$ is a t-int-soft filter. □

Theorem 6. *(Extension property) Let $\tilde{f}, \tilde{g}$ be int-soft filters of L, $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in L$, and, moreover, $\tilde{f}(1) \supseteq \tilde{g}(1)$. If $(\tilde{f}, L)$ is a t-int-soft filter, then $(\tilde{g}, L)$ is a t-int-soft filter.*

Proof. Let $(\tilde{f}, L)$ be a t-int-soft filter and $\tilde{f}(1) \supseteq \tilde{g}(1)$. Then, $\tilde{f}_{\tilde{f}(1)}$ is a t-filter. Thus, $\mathrm{t}(\bar{x}) \in \tilde{f}_{\tilde{f}(1)}$. Suppose $x \in \tilde{f}_{\tilde{f}(1)}$, then $\tilde{f}(x) \supseteq \tilde{f}(1)$. Also, $\tilde{g}(x) \supseteq \tilde{f}(x)$, and thus $\tilde{g}(x) \supseteq \tilde{f}(x) \supseteq \tilde{f}(1) \supseteq \tilde{g}(1)$. This shows that $x \in \tilde{g}_{\tilde{g}(1)}$. That is, $\tilde{f}_{\tilde{f}(1)} \subseteq \tilde{g}_{\tilde{g}(1)}$. Thus, $\mathrm{t}(\bar{x}) \in \tilde{g}_{\tilde{g}(1)}$. We have that $\tilde{g}_{\tilde{g}(1)}$ is a t-filter. Hence, $\tilde{g}$ is a t-int-soft filter. □

The next part concerns the quotient structure.

Lemma 5. *If $(\tilde{f}, L)$ is an int-soft filter, then $\tilde{f}(x \to z) \supseteq \tilde{f}(x \to y) \cap \tilde{f}(y \to z)$ for all $x, y, z \in L$.*

Proof. Assume that $(\tilde{f}, L)$ is an int-soft filter. Since $x \to y \leq (y \to z) \to (x \to z)$, it follows, from Definition 5 (2), that $\tilde{f}((y \to z) \to (x \to z)) \supseteq \tilde{f}(x \to y)$. From Lemma 2 (2), we have $\tilde{f}(x \to z) \supseteq \tilde{f}(y \to z) \cap \tilde{f}((y \to z) \to (x \to z))$. Thus, $\tilde{f}(x \to z) \supseteq \tilde{f}(x \to y) \cap \tilde{f}(y \to z)$. □

Lemma 6. *Let $(\tilde{f}, L)$ be an int-soft filter of L and $x, y \in L$. For any $z \in L$, we define $\tilde{f}^x : L \to \mathscr{P}(U)$, $\tilde{f}^x(z) = \tilde{f}(x \to z) \cap \tilde{f}(z \to x)$. Then, $\tilde{f}^x = \tilde{f}^y$ if and only if $\tilde{f}(x \to y) \supseteq \tilde{f}(1)$ and $\tilde{f}(y \to x) \supseteq \tilde{f}(1)$.*

Proof. Suppose $\tilde{f}^x = \tilde{f}^y$, then $\tilde{f}^x(x) = \tilde{f}^y(x)$. Also, $\tilde{f}^x(x) = \tilde{f}(1), \tilde{f}^y(x) = \tilde{f}(y \to x) \cap \tilde{f}(x \to y)$. Thus, $\tilde{f}(x \to y) \supseteq \tilde{f}(1)$ and $\tilde{f}(y \to x) \supseteq \tilde{f}(1)$.

Conversely, suppose $\tilde{f}(x \to y) \supseteq \tilde{f}(1)$ and $\tilde{f}(y \to x) \supseteq \tilde{f}(1)$. For any $z \in L$, $\tilde{f}^x(z) = \tilde{f}(x \to z) \cap \tilde{f}(z \to x)$, $\tilde{f}^y(z) = \tilde{f}(y \to z) \cap \tilde{f}(z \to y)$. By Lemma 5, $\tilde{f}(x \to z) \supseteq \tilde{f}(x \to y) \cap \tilde{f}(y \to z) \supseteq \tilde{f}(1) \cap \tilde{f}(y \to z) \supseteq \tilde{f}(y \to z)$. We have $\tilde{f}(z \to x) \supseteq \tilde{f}(z \to y) \cap \tilde{f}(y \to x) \supseteq \tilde{f}(z \to y) \cap \tilde{f}(1) \supseteq \tilde{f}(z \to y)$. Thus, $\tilde{f}(x \to z) \cap \tilde{f}(z \to x) \supseteq \tilde{f}(z \to y) \cap \tilde{f}(y \to z)$. Similarly, $\tilde{f}(y \to z) \cap \tilde{f}(z \to y) \supseteq \tilde{f}(x \to z) \cap \tilde{f}(z \to x)$. Therefore, $\tilde{f}(x \to z) \cap \tilde{f}(z \to x) = \tilde{f}(y \to z) \cap \tilde{f}(z \to y)$. This shows that $\tilde{f}^x(z) = \tilde{f}^y(z)$. By the arbitrariness of z, we have $\tilde{f}^x = \tilde{f}^y$. □

Theorem 7. *Let $(\tilde{f}, L)$ be an int-soft filter of L. Then, $\tilde{f}^x = \tilde{f}^y$ if and only if $x \to y \in \tilde{f}_{\tilde{f}(1)}$ and $y \to x \in \tilde{f}_{\tilde{f}(1)}$ if and only if $x \equiv_{\tilde{f}_{\tilde{f}(1)}} y$.*

Proof.

$$\begin{aligned} \tilde{f}^x = \tilde{f}^y &\Longleftrightarrow \tilde{f}(x \to y) \supseteq \tilde{f}(1), \tilde{f}(y \to x) \supseteq \tilde{f}(1) \\ &\Longleftrightarrow x \to y \in \tilde{f}_{\tilde{f}(1)}, y \to x \in \tilde{f}_{\tilde{f}(1)} \\ &\Longleftrightarrow x \equiv_{\tilde{f}_{\tilde{f}(1)}} y. \end{aligned}$$

□

Theorem 8. *Let $(\tilde{f}, L)$ be an int-soft filter of L, $L/\tilde{f} := \{\tilde{f}^x | x \in L\}$. For any $\tilde{f}^x, \tilde{f}^y \in L/\tilde{f}$, if we define* $\tilde{f}^x \sqcap \tilde{f}^y = \tilde{f}^{x \wedge y}$, $\tilde{f}^x \sqcup \tilde{f}^y = \tilde{f}^{x \vee y}$, $\tilde{f}^x \odot \tilde{f}^y = \tilde{f}^{x \otimes y}$, $\tilde{f}^x \rightharpoonup \tilde{f}^y = \tilde{f}^{x \to y}$ *, then* $L/\tilde{f} = (L/\tilde{f}, \sqcap, \sqcup, \odot, \rightharpoonup, \tilde{f}^0, \tilde{f}^1)$ *is a residuated lattice.*

Proof. Suppose $\tilde{f}^x = \tilde{f}^s, \tilde{f}^y = \tilde{f}^t$. By Theorem 7, we have $x \equiv_{\tilde{f}_{\tilde{f}(1)}} s, y \equiv_{\tilde{f}_{\tilde{f}(1)}} t$. Since $\equiv_{\tilde{f}_{\tilde{f}(1)}}$ is a congruence relation on L, we have $x \vee y \equiv_{\tilde{f}_{\tilde{f}(1)}} s \vee t$, and so $\tilde{f}^{x \vee y} = \tilde{f}^{s \vee t}$. Similarly, we have $\tilde{f}^{x \wedge y} = \tilde{f}^{s \wedge t}, \tilde{f}^{x \otimes y} = \tilde{f}^{s \otimes t}, \tilde{f}^{x \to y} = \tilde{f}^{s \to t}$. This shows that the operators on $L/\tilde{f}$ are well defined.

Clearly, $L/\tilde{f}$ satisfies (R1) and (R2). We only need to prove that $(\odot, \rightharpoonup)$ is an adjoint pair. We note that the lattice order $\tilde{f}^x \preceq \tilde{f}^y$ if and only if $\tilde{f}^x \sqcup \tilde{f}^y = \tilde{f}^y$ and

$$\begin{aligned} \tilde{f}^x \preceq \tilde{f}^y &\iff \tilde{f}^x \sqcup \tilde{f}^y = \tilde{f}^y \\ &\iff \tilde{f}^{x \vee y} = \tilde{f}^y \\ &\iff \tilde{f}(x \vee y \to y) \supseteq \tilde{f}(1) \\ &\iff \tilde{f}(x \to y) \supseteq \tilde{f}(1). \end{aligned}$$

Suppose $\tilde{f}^x, \tilde{f}^y, \tilde{f}^z \in L/\tilde{f}$, then

$$\begin{aligned} \tilde{f}^x \odot \tilde{f}^y \preceq \tilde{f}^z &\iff \tilde{f}^{x \otimes y} \preceq \tilde{f}^z \\ &\iff \tilde{f}(x \otimes y \to z) \supseteq \tilde{f}(1) \\ &\iff \tilde{f}(x \to (y \to z)) \supseteq \tilde{f}(1) \\ &\iff \tilde{f}^x \preceq \tilde{f}^{y \to z} \\ &\iff \tilde{f}^x \preceq \tilde{f}^y \rightharpoonup \tilde{f}^z. \end{aligned}$$

Therefore, (R3) holds. This shows that $L/\tilde{f}$ is a residuated lattice. □

Theorem 9. *Let $(\tilde{f}, L)$ be an int-soft filter. Then, the residuated lattice $L/\tilde{f} \cong L/\tilde{f}_{\tilde{f}(1)}$.*

Proof. Define a mapping $\varphi : L \longrightarrow L/\tilde{f}$ by $\varphi(x) = \tilde{f}^x$. Then,

$$\begin{aligned} x \in ker(\varphi) &\iff \varphi(x) = \tilde{f}^1 \\ &\iff \tilde{f}^x = \tilde{f}^1 \\ &\iff \tilde{f}(x) \supseteq \tilde{f}(1) \\ &\iff x \in \tilde{f}_{\tilde{f}(1)}. \end{aligned}$$

Therefore, $ker(\varphi) = \tilde{f}_{\tilde{f}(1)}$. Clearly, φ is surjective. It is easy to verify that φ is a homomorphism. Thus, $L/\tilde{f} \cong L/\tilde{f}_{\tilde{f}(1)}$. □

Theorem 10. *(Quotient characteristics) Let $(\tilde{f}, L)$ be* **an** *int-soft filter of L. Then, $(\tilde{f}, L)$ is a t-int-soft filter if and only if $L/\tilde{f} \in \mathbb{B}$(t).*

Proof. $(\tilde{f}, L)$ is a t-int-soft filter if and only if $\tilde{f}_{\tilde{f}(1)}$ is a t-filter if and only if $L/\tilde{f}_{\tilde{f}(1)} \in \mathbb{B}$(t) if and only if $L/\tilde{f} \in \mathbb{B}$(t) . □

Theorem 11. *(Triple of equivalent characteristics) Let L be a residuated lattice. Then, the following statements are equivalent:*

(1) *Every int-soft filter of L is a t-int-soft filter.*
(2) $(\tilde{T}, L)$ *is a t-int-soft filter.*
(3) $L \in \mathbb{B}(t)$.

Proof. (1)$\Longrightarrow$(2)

Since $(\tilde{T}, L)$ is an int-soft filter, the result is obvious.

(2)$\Longrightarrow$(3)

Suppose $(\tilde{T}, L)$ is a t-int-soft filter, then $\tilde{T}_{\tilde{T}(1)}$ is a t-filter. Thus, $L/\tilde{T}_{\tilde{T}(1)} \in \mathbb{B}$(t). Also, $\tilde{T}_{\tilde{T}(1)} = \{1\}$ and $L/\{1\} \approx L$. Thus, $L \in \mathbb{B}$(t).

(3)$\Longrightarrow$(1)

Suppose $(\tilde{f}, L)$ is an int-soft filter of L, then $\tilde{f}_{\tilde{f}(1)}$ is a filter. Also, $L \in \mathbb{B}$(t), and thus $\tilde{f}_{\tilde{f}(1)}$ is a t-filter. Hence, $(\tilde{f}, L)$ is a t-int-soft filter. □

4. A Specific Example

In this section, we use involution-int-soft filters as an example.

Definition 8. *Let* $(\tilde{f}, L)$ *be an int-soft filter of L. Then,* $(\tilde{f}, L)$ *is called an involution-int-soft filter if, for all* $\tau \in \mathscr{P}(U)$, $\tilde{f}_\tau$ *is either empty or an involution filter.*

Lemma 7. *Let* $(\tilde{f}, L)$ *be an int-soft filter of L. Then,* $(\tilde{f}, L)$ *is an involution-int-soft filter if and only if* $\tilde{f}(x^{**} \to x) \supseteq \tilde{f}(1)$, *for all* $x \in L$.

Proof. Suppose $(\tilde{f}, L)$ is an involution-int-soft filter, then, by Theorem 5, we have that $\tilde{f}_{\tilde{f}(1)}$ is an involution filter. By Example 1, we have $x^{**} \to x \in \tilde{f}_{\tilde{f}(1)}$. This shows that $\tilde{f}(x^{**} \to x) \supseteq \tilde{f}(1)$.

Conversely, since $(\tilde{f}, L)$ is an int-soft filter, then $\tilde{f}_{\tilde{f}(1)}$ is a filter. Suppose $\tilde{f}(x^{**} \to x) \supseteq \tilde{f}(1)$, then $x^{**} \to x \in \tilde{f}_{\tilde{f}(1)}$. By Example 1, we know that $\tilde{f}_{\tilde{f}(1)}$ is an involution filter. Thus, $(\tilde{f}, L)$ is an involution-int-soft filter. □

Theorem 12. *(Extension property) Let* $\tilde{f}, \tilde{g}$ *be int-soft filters of L,* $\tilde{f}(x) \subseteq \tilde{g}(x)$ *for all* $x \in L$*, and, moreover,* $\tilde{f}(1) \supseteq \tilde{g}(1)$*. If* $(\tilde{f}, L)$ *is an involution-int-soft filter, then* $(\tilde{g}, L)$ *is an involution-int-soft filter.*

Theorem 13. *(Quotient characteristics) Let* $(\tilde{f}, L)$ *be* **an** *int-soft filter of L. Then,* $(\tilde{f}, L)$ *is an involution-int-soft filter if and only if* $L/\tilde{f}$ *is an involutive residuated lattice.*

Theorem 14. *(Triple of equivalent characteristics) Let L be a residuated lattice. Then, the following statements are equivalent:*

(1) *Every int-soft filter of L is an involution-int-soft filter.*
(2) $(\tilde{T}, L)$ *is an involution-int-soft filter.*
(3) *L is an involutive residuated lattice.*

5. The Relations among T-Int-Soft Filters on Residuated Lattices

Lemma 8. *[21] Let L be a residuated lattice. Every Heyting algebra is a Rl-monoid.*

Lemma 9. *[33] Let L be a residuated lattice. Then, the following statements are equivalent:*

(1) *L is a Boolean algebra.*
(2) *L is involutive and idempotent.*

Lemma 10. *[31,32] Let L be a residuated lattice. Then, L is an MV-algebra if and only if L is an involutive BL-algebra.*

In what follows, let $(\tilde{f}, L)$ be an int-soft filter of L.

Theorem 15. *If $(\tilde{f}, L)$ is a t_1-int-soft filter and $\mathbb{B}(t_1) \subseteq \mathbb{B}(t_2)$, then $(\tilde{f}, L)$ is a t_2-int-soft filter.*

Proof. $(\tilde{f}, L)$ is a t_1-int-soft filter $\Longrightarrow L/\tilde{f} \in \mathbb{B}(\mathrm{t}_1) \Longrightarrow L/\tilde{f} \in \mathbb{B}(\mathrm{t}_2) \Longrightarrow (\tilde{f}, L)$ is a t_2-int-soft filter. □

Theorem 16. *If $\mathbb{B}(t_1) \subseteq \mathbb{B}(t_2)$ and $\mathbb{B}(t_2) \subseteq \mathbb{B}(t_1)$, then $(\tilde{f}, L)$ is a t_1-int-soft filter if and only if $(\tilde{f}, L)$ is a t_2-int-soft filter.*

Proof. $(\tilde{f}, L)$ is a t_1-int-soft filter $\Longleftrightarrow L/\tilde{f} \in \mathbb{B}(\mathrm{t}_1) \Longleftrightarrow L/\tilde{f} \in \mathbb{B}(\mathrm{t}_2) \Longleftrightarrow (\tilde{f}, L)$ is a t_2-int-soft filter. □

Remark 3. *The above results reveal the general principles concerning the relations among t-int-soft filters. Their relations are consistent with those among the corresponding quotient algebras. Since we are familiar with the relations among these algebras, we can easily obtain the relations among t-int-soft filters. We have the following results.*

Theorem 17. *Let L be a residuated lattice. If $(\tilde{f}, L)$ is a Boolean-int-soft filter, then $(\tilde{f}, L)$ is a MV-int-soft filter, an involution-int-soft filter, a Heyting-int-soft filter, a Rl-monoid-int-soft filter, a MTL-int-soft filter, and a BL-int-soft filter.*

Theorem 18. *Let L be a residuated lattice. If $(\tilde{f}, L)$ is a MV-int-soft filter, then $(\tilde{f}, L)$ is a MTL-int-soft filter, an involution-int-soft filter, a Rl-monoid-int-soft filter, and a BL-int-soft filter.*

Theorem 19. *Let L be a residuated lattice. If $(\tilde{f}, L)$ is a Heyting-int-soft filter filter, then $(\tilde{f}, L)$ is a Rl-monoid-int-soft filter.*

Theorem 20. *Let L be a residuated lattice. Then, $(\tilde{f}, L)$ is a Boolean-int-soft filter if and only if $(\tilde{f}, L)$ is a MV-int-soft filter and a Heyting-int-soft filter.*

Theorem 21. *Let L be a residuated lattice. Then, $(\tilde{f}, L)$ is a MV-int-soft filter if and only if $(\tilde{f}, L)$ is an involution-int-soft filter and a Rl-monoid-int-soft filter.*

Theorem 22. *Let L be a residuated lattice. Then, $(\tilde{f}, L)$ is a BL-int-soft filter if and only if $(\tilde{f}, L)$ is a MTL-int-soft filter and a Rl-monoid-int-soft filter.*

6. Conclusions and Future Work

In this paper, we proposed the notion of t-int-soft filters on residuated lattices. In our framework, the "generator" of t-int-soft filters was given. As long as there are t-filters, we can obtain the corresponding t-int-soft filters. The characterizations of t-int-soft filters were derived. The general principles of investigating the relations among t-int-soft filters were derived. The research embodies the relationship between t-filters and t-int-soft filters. The thoughts and methods in this paper can be completely applied to special cases of residuated lattices, and the corresponding int-soft filters can be introduced and their characterizations can be obtained on these logical algebras.

Additionally, in [34–36], Chishty and Jun et al., respectively, discussed the properties of uni-soft filters on residuated lattices and MTL-algebras. In our future work, we will characterize the common features of uni-soft filters on residuated lattices.

Author Contributions: The author, H.Z., contributed mainly in the introduction and investigation of the properties of the t-int-soft filters. The solution process was discussed a number of times with the second author, M.L. She also provided the technical help regarding the write-up of manuscript.

Funding: This work is supported by the National Natural Science Foundation of China (No. 11701540, 61773019).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Feller, W. An introduction to probability theory and its applications II. *Phys. Today* **1967**, *20*, 76. [CrossRef]
2. Zadeh, L.A. Fuzzy sets. *Inf. Control.* **1965**, *8*, 338–353. [CrossRef]
3. Pawlak, Z. Rough sets. *Int. J. Comput. Inf. Sci.* **1982**, *11*, 341–356. [CrossRef]
4. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
5. Molodtsov, D. Soft set theory—First results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
6. Ali, M.I.; Feng, F.; Liu, X. On some new operations in soft set theory. *Comput. Math. Appl.* **2009**, *57*, 1547–1553. [CrossRef]
7. Maji, P.K.; Biswas, R.; Roy, A.R. Soft set theory. *Comput. Math. Appl.* **2003**, *45*, 555–562. [CrossRef]
8. Sezgin, A.; Atagün, A.O. On operations of soft sets. *Comput. Math. Appl.* **2011**, *61*, 1457–1467. [CrossRef]
9. Som, T. On soft relation and fuzzy soft relation. *J. Fuzzy Math.* **2008**, *16*, 677–687.
10. Zhan, J.; Jun, Y.B. Soft BL-algebras based on fuzzy sets. *Comput. Math. Appl.* **2010**, *59*, 2037–2046. [CrossRef]
11. Kharal, A. Distance and similarity measures for soft sets. *New Math. Nat. Comput.* **2010**, *6*, 321–334. [CrossRef]
12. Majumdar, P.; Samanta, S.K. Similarity measure of soft sets. *New Math. Nat. Comput.* **2008**, *4*, 1–12. [CrossRef]
13. Qin, K.; Hong, Z. On soft equality. *J. Comput. Appl. Math.* **2010**, *234*, 1347–1355. [CrossRef]
14. Maji, P.K.; Roy, A.R.; Biswas, R. An application of soft sets in a decision making problem. *Comput. Math. Appl.* **2002**, *44*, 1077–1083. [CrossRef]
15. Xiao, Z.; Gong, K.; Zou, Y. A combined forecasting approach based on fuzzy soft sets. *J. Comput. Appl. Math.* **2009**, *228*, 326–333. [CrossRef]
16. Ward, M.; Dilworth, R.P. Residuated lattice. *Proc. Nat. Acad. Sci. USA* **1938**, *24*, 162–164. [CrossRef] [PubMed]
17. Buşneag, D.; Piciu, D. Semi-G-filters, Stonean filters, MTL-filters, divisible filters, BL-filters and regular filters in residuated lattices. *Iran. J. Fuzzy Syst.* **2016**, *13*, 145–160.
18. Buşneag, D.; Piciu, D. A new approach for classification of filters in residuated lattices. *Fuzzy Sets Syst.* **2015**, *260*, 121–130. [CrossRef]
19. Van Gasse, B.; Deschrijver, G.; Cornelis, C. Filters of residuated lattices and triangle algebra. *Inf. Sci.* **2010**, *180*, 3006–3020. [CrossRef]
20. Kondo, M. Characterization of some types of filters in commutative residuated lattices. *Far East J. Math. Sci.* **2010**, *2*, 193–203.
21. Ma, Z.M.; Hu, B.Q. Characterizations and new subclasses of I-filters in residuated lattices. *Fuzzy Sets Syst.* **2014**, *247*, 92–107. [CrossRef]
22. Muresan, C. Dense elements and classes of residuated lattices. *Bull. Math. Soc. Sci. Math.* **2010**, *53*, 11–24.
23. Shen, J.G.; Zhang, X.H. On filters of residuated lattice. *Chin. Q. J. Math.* **2006**, *180*, 443–447.
24. Víta, M. Why are papers about filters on residuated structures (usually) trivial? *Inf. Sci.* **2014**, *276*, 387–391. [CrossRef]
25. Zhang, X.; Zhou, H.; Mao, X. IMTL (MV)-filters and fuzzy IMTL (MV)-filters of residuated lattices. *J. Intell. Fuzzy Syst.* **2014**, *26*, 589–596.
26. Zhu, Y.; Xu, Y. On filter theory of residuated lattices. *Inf. Sci.* **2010**, *180*, 3614–3632. [CrossRef]
27. Jun, Y.B.; Ahn, S.S.; Lee, K.J. Intersection-soft filters in R0-Algebras. *Discret. Dyn. Nat. Soc.* **2013**, 950897. [CrossRef]
28. Jun, Y.B.; Ahn, S.S.; Lee, K.J. Implicative int-soft filters of R0-Algebras. *Discret. Dyn. Nat. Soc.* **2013**, *4*, 1–7. [CrossRef]
29. Lin, C.; Ma, Z. Int-Soft filters and their congruences on residuated lattices. *Op. Res. Fuzziol.* **2017**, *7*, 37–44. [CrossRef]

30. Jun, Y.B.; Ahn, S.S.; Lee, K.J. Classes of int-soft filters in residuated lattices. *Sci. World J.* **2014**, *2014*, 595160. [CrossRef] [PubMed]
31. Hájek, P. *Metamathematics of Fuzzy Logic*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1998.
32. Turunen, E. *Mathematics Behind Fuzzy Logic*; Springer: Heidelberg, Germany, 1999.
33. Zhang, H.; Li, Q. The relations among fuzzy t-filters on residuated lattices. *Sci. World J.* **2014**, *2014*, 894346. [CrossRef] [PubMed]
34. Chishty, G.M.; Al-Roqi, A. Filter theory in MTL-algebras based on Uni-soft property. *Bull. Iran. Math. Soc.* **2017**, in press.
35. Jun, J.B.; Lee, K.J.; Park, C.H.; Roh, E.H. Filters of residuated lattices based on soft set theory. *Commun. Korean Math. Soc.* **2015**, *30*, 155–168. [CrossRef]
36. Jun, J.B.; Song, S.Z. Uni-soft filters and uni-soft G-filters in residuated lattices. *J. Comput. Anal. Appl.* **2016**, *20*, 319–334.