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Article

On the Domain of the Fibonacci Difference Matrix

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Abstract: Matrix $\hat{F}$ derived from the Fibonacci sequence was first introduced by Kara (2013) and the spaces $l_p(F)$ and $l_\infty(F)$; ($1 \leq p < \infty$) were examined. Then, Başarır et al. (2015) defined the spaces $c_0(F)$ and $c(F)$ and Candan (2015) examined the spaces $c(F(r,s))$ and $c_0(F(r,s))$. Later, Yaşar and Kayaduman (2018) defined and studied the spaces $cs(F(s,r))$ and $bs(F(s,r))$. In this study, we built the spaces $cs(F)$ and $bs(F)$. They are the domain of the matrix F on cs and bs, where F is a triangular matrix defined by Fibonacci Numbers. Some topological and algebraic properties, isomorphism, inclusion relations and norms, which are defined over them are examined. It is proven that $cs(F)$ and $bs(F)$ are Banach spaces. It is determined that they have the γ, β, α -duals. In addition, the Schauder base of the space $cs(F)$ are calculated. Finally, a number of matrix transformations of these spaces are found.

Keywords: matrix transformations; Fibonacci numbers; sequence spaces; Fibonacci double band matrix; γ, β, α -duals

1. Introduction

Cooke [1] formulated the theory of infinite matrices in the book "Infinite Matrices and Sequence Spaces". Many researchers have investigated infinite matrices after the publication of this book in 1950. In most of these studies, the domain of infinite matrices on a sequence space was studied. In this study, we address the question: What are the properties of the domain of the Fibonacci band matrix on sequence spaces *bs* and *cs*? The domain of the Fibonacci band matrix creates a new sequence space. We handle algebraic properties of this new space in order to determine its duals and its place among other known spaces, and to characterize the matrix transformations of this space.

One difficulty of this study is to determine whether the new space is the contraction or the expansion, or the overlap of the original space. Another difficulty is to determine the matrix transformations on this space and into this space. For the first problem, we give a few inclusion theorems. For the second problem, we use the matrix transformation between the standard sequence spaces and two theorems.

Generating a new sequence space and researching on its properties have been important in the studies on the sequence space. Some researchers examined the algebraic properties of the sequence space while others investigated its place among other known spaces and its duals, and characterized the matrix transformations on this space.

We can create a new sequence space by using the domain of infinite matrices. Ng-Lee [2] first investigated the domain of an infinite matrix in 1978. In the same period, Wang [3] created a new sequence space by using another infinite matrix. Many researchers such as Malkovsky [4], Altay, and Başar [5] followed these studies. This topic was studied intensively after 2000.

Leonardo Fibonacci invented Fibonacci numbers. He introduced Fibonacci numbers originated from a rabbit problem. These numbers create a number sequence:

$$1,1,2,3,5,8,13,21,34,55,89,....$$

This sequence has important properties and applications in various fields.

Let us indicate the Fibonacci sequence by (f_n). f_n is defined as

$$f_n = f_{n-1} + f_{n-2} \ (n \geq 2)$$

with $f_0 = f_1 = 1$. The golden ratio is

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \phi.$$

Let us indicate the set of all real-valued sequences with w and list some subspaces of w called standard sequence spaces.

$$c = \left\{ r = (r_k) \in w : \lim_{k\to\infty} |r_k - p| = 0 \ \text{ for some } p \in \mathbb{C} \right\},$$

$$c_0 = \left\{ r = (r_k) \in w : \lim_{k\to\infty} |r_k| = 0 \right\},$$

$$bs = \left\{ r = (r_k) \in w : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} r_k \right| < \infty \right\},$$

$$cs_0 = \left\{ r = (r_k) \in w : \lim_{n\to\infty} \left| \sum_{k=0}^{n} r_k \right| = 0 \right\},$$

$$l_\infty = \left\{ r = (r_k) \in w : \sup_{k\in\mathbb{N}} |r_k| < \infty \right\},$$

$$bv = \left\{ r = (r_k) \in w : \sum_{k=0}^{\infty} |r_k - r_{k-1}| < \infty \right\},$$

$$bv_0 = bv \cap c_0$$

$$l_p = \left\{ r = (r_k) \in w : \sum_{k=0}^{\infty} |r_k|^p < \infty, \ 0 < p < \infty \right\},$$

$$l_1 = \left\{ r = (r_k) \in w : \sum_{k=0}^{\infty} |r_k| < \infty \right\},$$

Now let us take real valued infinite matrix $T = (t_{nk})$, where t_{nk} is a real number for every $n,k \in \mathbb{N}$ Let A and B be sequence spaces. Sequence $Tx = \{T_n(x)\}$ is T-transform of a for every $a = (a_k) \in A$. Here, $Ta \in B$ and

$$T_n(a) = \sum_k t_{nk} a_k \tag{1}$$

and $T_n(a) \to t$ (t exists for every $n \in \mathbb{N}$). Then, T is called a matrix transformation from A to B.

Now let us take infinite matrix T and sequence space δ to define domain of infinite matrix T. The domain of the matrix T on δ is characterized by

$$\delta_T = \{x = (x_k) \in w \colon Tx \in \delta\}. \tag{2}$$

Many reserachers have studied the domain of a matrix on a sequence space. For more detailed information on these new sequence spaces, see references [6–26].

The Fibonacci difference matrix F was first introduced by Kara [27] in 2013. F is derived from (f_n). In this study, Kara [27] defined the spaces $l_p(F)$ and $l_\infty(F)$; $(1 \le p < \infty)$. After this study the $c_0(F(r,s))$ and $c(F(r,s))$ was introduced by Candan [28], in 2015, where $F(r,s)$ is a generalized Fibonacci matrix. Candan and Kara [19] introduced and examined $l_p(F(r,s))$; $(1 \le p < \infty)$. In 2018, Yaşar and Kayaduman [29] defined and examined $cs(F(s,r))$ and $bs(F(s,r))$ and Kayaduman and Yaşar [30] studied spaces $bs(N^t)$ and $cs(N^t)$, where N^t is a Nörlund matrix.

Let δ be a sequence space. The γ, β, α -duals of δ are defined, respectively, as follows

$$\delta^\gamma = \{x = (x_k) \in w : xs = (x_k s_k) \in bs \text{ for all } s \in \delta\},$$

$$\delta^\beta = \{x = (x_k) \in w : xs = (x_k s_k) \in cs \text{ for all } s \in \delta\},$$

$$\delta^\alpha = \{x = (x_k) \in w : xs = (x_k s_k) \in l_1 \text{ for all } s \in \delta\}.$$

In this study, spaces $cs(F)$ and $bs(F)$ are introduced and the related notations are given in Section 2. In addition, some topological and algebric properties, isomorphism, inclusion relations and norms which are defined over them are examined. The γ, β, α -duals of these spaces are determined in Section 3. The Schauder base of space $cs(F)$ are calculated. Finally, many matrix transformations of these spaces are found. In the last section, the results and previous studies and the working hypotheses are discussed.

A detailed literature review was performed before this study was started. Scans were made on related articles, magazines, and books. As a result of these scans, the part related to our subject was synthesized and the results were noted. These results were then applied to our problem area. Finally, the results of this study were obtained.

2. Results

2.1. The Domain of Fibonacci Difference Matrix F on Bounded and Convergent Series

In this section, $cs(F)$ and $bs(F)$ are introduced. Related notations are given. In addition, some topological and algebric properties, isomorphism, inclusion relations, and norms defined over them are examined.

For similar studies, see refeerences [19] and [27–34].

Let spaces $cs(F)$ and $bs(F)$ be the domain of the matrix F on cs and bs, where $F = \{f_{nk}\}$ infinite matrix is defined by (f_n)

$$f_{nk} = \begin{cases} -\dfrac{f_{n+1}}{f_n}, & k = n-1 \\ \dfrac{f_n}{f_{n+1}}, & k = n \\ 0, & 0 \le k < n-1 \text{ or } n < k \end{cases}$$

for all $k, n \in \mathbb{N}$. Then we inroduce $cs(F)$ and $bs(F)$ as

$$cs(F)=\left\{x=(x_k)\in w:\left(\sum_{k=0}^{n}\left(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right)\right)_n\in c\right\}$$

$$bs(F)=\left\{x=(x_k)\in w:\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\left(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right)\right|<\infty\right\}.$$

We can see $cs(F) = (cs)_F$ and $bs(F) = (bs)_F$ by using Equation (2).

Let the inverse matrix of F be F^{-1}. For all $k,n\in\mathbb{N}$, $F^{-1} = \{F^{-1}{}_{nk}\}$ is found as

$$f_{nk}^{-1}=\begin{cases}\dfrac{f_{n+1}^2}{f_k f_{k+1}}, & 0\le k\le n\\ 0, & k>n.\end{cases}\tag{3}$$

Let us take sequence $x = (x_n)$. If $y = Fx$, then we calculate as

$$y_n=(Fx)_n=\begin{cases}x_0, & n=0\\ \dfrac{f_n}{f_{n+1}}x_n-\dfrac{f_{n+1}}{f_n}x_{n-1}, & n\ge 1.\end{cases}\tag{4}$$

Herefrom, if we calculate inverse of F, then we find that $x = F^{-1}y$ and

$$x_n=\sum_{k=0}^{n}\frac{f_{n+1}^2}{f_k f_{k+1}}y_k.\tag{5}$$

Now, let us give some theorems related to our study.

Theorem 1. $bs(F)$ is a linear space.

Proof. The proof is left to the reader since it is easy to show. □

Theorem 2. $cs(F)$ is a linear space.

Proof. The proof is left to the reader since it is easy to show. □

Theorem 3. $bs(F)$ is a normed space with:

$$\|x\|=\sup_{n\in\mathbb{N}}\left|\sum_{j=0}^{n}\left(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right)\right|\tag{6}$$

Proof. The proof is left to the reader since it is easy to show. □

Theorem 4. $cs(F)$ is a normed space with Equation (6).

Proof. The proof is left to the reader since it is easy to show. □

Theorem 5. $bs(F)$ is isomorphic to bs.

Proof. Let us take T: $bs(F)\to bs$ mentioned Equation (4) by $x\to y = Tx = Fx$. It is easy to see that T is linear and injective.

We must find T is surjective. Let $y = (y_n)\in bs$. By using Equation (5) and Equation (6), we see

$$\begin{aligned}\|x\|&=\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\left(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right)\right|\\&=\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\left(\frac{f_k}{f_{k+1}}\left(\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_j f_{j+1}}\,y_j\right)-\frac{f_{k+1}}{f_k}\left(\sum_{k=0}^{k-1}\frac{f_{k+1}^2}{f_j f_{j+1}}\,y_j\right)\right)\right|\\&=\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}y_k\right|=\|y\|_{bs}<\infty.\end{aligned}$$

We see that $x \in bs(F)$. Hence, T is surjective. In addition, $bs(F)$ and bs izometric because $\|x\|_{bs(F)} = \|y\|_{bs}$. □

Theorem 6. $cs(F)$ is isomorphic to cs.

Proof. The proof can be made similar to Theorem 5, so it is left to the reader. □

Theorem 7. $bs(F)$ is a Banach space with Equtaion (6).

Proof. It is easy to see the norm conditions are ensured. Let a Cauchy sequence $x^i = (x_k^i)$ in $bs(F)$ for each $i \in \mathbb{N}$. For all $k \in \mathbb{N}$, we have

$$y_k^i = \frac{f_k}{f_{k+1}} x_k^i - \frac{f_{k+1}}{f_k} x_{k-1}^i$$

from Equation (4). For all $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ such that

$$\begin{aligned}\|x^i - x^m\|_{bs(F)} &= \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left(\frac{f_k}{f_{k+1}} (x_k^i - x_k^m) - \frac{f_{k+1}}{f_k} (x_{k-1}^i - x_{k-1}^m) \right) \right| \\ &= \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} (y_k^i - y_k^m) \right| = \|y^i - y^m\|_{bs} < \varepsilon\end{aligned}$$

for all $i, m \geq n_0$. $y_i \to y$ $(i \to \infty)$ such that $y \in bs$ exists, since bs is complete. Since bs and $bs(F)$ are isomorphic, $bs(F)$ is complete. It hereby is a Banach space. □

Theorem 8. $cs(F)$ is a Banach space with Equation (6).

Proof. It is easy to see the norm conditions are ensured. Let a Cauchy sequence $x^i = (x_k^i)$ in $cs(F)$ for each $i \in \mathbb{N}$. For all $k \in \mathbb{N}$, we have

$$y_k^i = \frac{f_k}{f_{k+1}} x_k^i - \frac{f_{k+1}}{f_k} x_{k-1}^i$$

from Equation (4). For all $\varepsilon > 0$, there is $n_0 = n_0(\varepsilon)$ such that

$$\begin{aligned}\|x^i - x^m\|_{cs(F)} &= \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left(\frac{f_k}{f_{k+1}} (x_k^i - x_k^m) - \frac{f_{k+1}}{f_k} (x_{k-1}^i - x_{k-1}^m) \right) \right| \\ &= \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} (y_k^i - y_k^m) \right| = \|y^i - y^m\|_{cs} < \varepsilon\end{aligned}$$

for all $i, m \geq n_0$. $y_i \to y$ $(i \to \infty)$ such that $y \in cs$ exists, since cs is complete. Since cs and $cs(F)$ are isomorphic, $cs(F)$ is complete. It hereby is a Banach space. □

Now, let $R = (r_{nk})$ infinite matrix. Let us list the following:

$$\sup_{n \in \mathbb{N}} \sum_k |r_{nk}| < \infty, \tag{7}$$

$$\lim_k r_{nk} = 0 \text{ for each } n \in \mathbb{N}, \tag{8}$$

$$\sup_m \sum_k \left| \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) \right| < \infty, \tag{9}$$

$$\lim_n \sum_k r_{nk} = p \text{ for each k} \in \mathrm{N},\ p \in \mathbb{C}, \tag{10}$$

$$\sup_n \sum_k \left| r_{nk} - r_{n,k+1} \right| < \infty, \tag{11}$$

$$\lim_n r_{nk} = a_k \text{ for each k} \in \mathbb{N},\ a_k \in \mathbb{C}, \tag{12}$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (r_{nk} - r_{n,k+1}) \right| < \infty, \tag{13}$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (r_{nk} - r_{n,k-1}) \right| < \infty, \tag{14}$$

$$\lim_n (r_{nk} - r_{n,k+1}) = a \text{ for each k} \in \mathbb{N},\ a \in \mathbb{C}, \tag{15}$$

$$\lim_n \sum_k \left| r_{nk} - r_{n,k+1} \right| = \sum_k \left| \lim_n (r_{nk} - r_{n,k+1}) \right|, \tag{16}$$

$$\sup_n \left| \lim_k r_{nk} \right| < \infty, \tag{17}$$

$$\lim_n \sum_k \left| r_{nk} - r_{n,k+1} \right| = 0 \text{ uniformly in } n, \tag{18}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) \right| = 0, \tag{19}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) \right| = \sum_k \left| \sum_n (r_{nk} - r_{n,k+1}) \right|, \tag{20}$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} [(r_{nk} - r_{n,k+1}) - (r_{n-1,k} - r_{n-1,k+1})] \right| < \infty, \tag{21}$$

$$\sup_{m \in \mathbb{N}} \left| \lim_k \sum_{n=0}^{m} r_{nk} \right| < \infty, \tag{22}$$

$$\exists a_k \in \mathbb{C} \ni \sum_n r_{nk} = a_k \text{ for each } k \in \mathbb{N}, \tag{23}$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} [(r_{nk} - r_{n-1,k}) - (r_{n,k-1} - r_{n-1,k-1})] \right| < \infty. \tag{24}$$

The collection of all finite subsets of $\mathbb{N}$ denoted by $\mathcal{F}$.

Lemma 9. Let us suppose infinite matrix $R = (r_{nk})$. Then,

(1) $R=(r_{nk})\in(bs,l_\infty)$ iff Equations (11) and (8) hold [35].
(2) $R=(r_{nk})\in(cs,c)$ iff Equations (11) and (12) hold [36].
(3) $R=(r_{nk})\in(bs,l_1)$ iff Equations (13) and (8) hold [37].
(4) $R=(r_{nk})\in(cs,l_1)$ iff Equation (14) holds [35].
(5) $R=(r_{nk})\in(bs,c)$ iff Equations (8), (15) and (16) hold [37].
(6) $R=(r_{nk})\in(cs,l_\infty)$ iff Equations (17) and (11) hold [35].
(7) $R=(r_{nk})\in(bs,c_0)$ iff Equations (18) and (8) hold [35].
(8) $R=(r_{nk})\in(bs,cs_0)$ iff Equations (19) and (8) hold [38].
(9) $R=(r_{nk})\in(bs,cs)$ iff Equations (20) and (8) hold [38].
(10) $R=(r_{nk})\in(bs,bv)$ iff Equations (21) and (8) hold [38].
(11) $R=(r_{nk})\in(bs,bs)$ iff Equations (8) and (9) hold [38].
(12) $R=(r_{nk})\in(cs,cs)$ iff Equations (9) and (10) hold [39].
(13) $R=(r_{nk})\in(bs,bv_0)$ iff Equations (21), (18) and (21) hold [35].
(14) $R=(r_{nk})\in(cs,c_0)$ iff Equations (11) and (12) hold with $a_k = 0$ for all $k\in\mathbb{N}$ [40].
(15) $R=(r_{nk})\in(cs,bs)$ iff Equations (9) and (22) hold [38].
(16) $R=(r_{nk})\in(cs,cs_0)$ iff Equations (9) and (23) hold with $a_k = 0$ for all $k\in\mathbb{N}$ [38].
(17) $R=(r_{nk})\in(cs,bv)$ iff Equation (24) holds [38].
(18) $R=(r_{nk})\in(cs,bv_0)$ iff Equations (24) and (12) hold with $a_k = 0$ for all $k\in\mathbb{N}$ [35].

Theorem 10. $bs(F) \supset bs$ is valid.

Proof. Suppose $x\in bs$. If we show that F is an element of (bs,bs) then x is element of $bs(F)$. For this, F must provide Equations (8) and (9). Since $\lim_k f_{nk} = 0$ for each $n \in \mathbb{N}$, Equation (8) is provided.

If we examine Equation (9), we find

$$\sup_m \sum_k \left| \sum_{n=0}^{m} (f_{nk} - f_{n,k+1}) \right| = 0. \quad \square$$

Theorem 11. $bs(F) \supset \ell_\infty$ is not valid.

Proof. Suppose $x = (x_k) = (f^2_{k+1})$. Then $y = Fx = (1,0,0....)\in bs$. On the other hand, $f^2_{k+1}\to\infty$ as $k\to\infty$. It is clear $x\in bs(F)$, but $x \notin \ell_\infty$. This result completes the proof. □

Theorem 12. $cs(F) \supset cs$ is valid.

Proof. Suppose $x \in cs$. If we show that F is element of (cs,cs) then x is element of $cs(F)$. For this, F must provide Equations (10) and (9). Equation (9) has been provided from the Theorem 10. If we look at the Equation (10) then, for each $k \in \mathbb{N}$,

$$\lim_n \sum_k f_{nk} = \lim_n \left(\frac{f_n}{f_{n+1}} - \frac{f_{n+1}}{f_n} \right) = \frac{1}{\phi} - \phi = l$$

such that $l \in \mathbb{C}$ exists. □

Theorem 13. $cs(F) \supset c$ is not valid.

Proof. Let $x = (x_k) = (f^2_{k+1})$. Then $y = Fx = (1,0,0....) \in cs$. On the other hand, $f^2_{k+1} \to \infty$ as $k \to \infty$. It is clear $x \in cs(F)$, but $x \notin c$. This result completes the proof. □

Theorem 14. $cs(F) \subset bs(F)$ is valid.

Proof. If $x \in cs(F)$, $y = Fx \in cs$. Hence, $\sum_k Fx \in c$. Since $c \subset \ell_\infty$, $\sum_k Fx \in l_\infty$. Hence, $Fx \in bs$. That is, $x \in bs(F)$. This result completes the proof. □

Let us take normed space A and let $(a_k) \in A$. If there is only one scalar sequence (v_k) such that $y = \sum_{k=0}^{\infty} v_k a_k$ and $\lim_{n\to\infty} \left\| y - \sum_{k=0}^{n} v_k a_k \right\| = 0$ then (a_k) is called a Schauder base for A.

Now, let us give corallary releated to Schauder basis.

Corallary 15. Let a sequence $u^{(k)} = \{u_n^{(k)}\}_{n\in\mathbb{N}}$ in $cs(F)$ be for each $k \in \mathbb{N}$ and

$$u_n^{(k)} = \begin{cases} \dfrac{f_{n+1}^2}{f_k f_{k+1}}, & 0 \le k \le n \\ 0, & n < k. \end{cases}$$

Then $\{u_n^{(k)}\}_{n\in\mathbb{N}}$ is a base for $cs(F)$. Every $x \in cs(F)$ can write as a single $x = \sum_{k=0}^{\infty} y_k u^k$ such that $y_k = (\hat{F}x)_k$.

2.2. The Duals of cs(F) and bs(F) and Matrix Transformations

Let us give the two lemmas to use in the next stage.

Lemma 16. Let infinite matrix $C = (c_{nk})\frac{1}{2}$ and $a = (a_n) \in w$. Let us take $C = aF^{-1}$, that is,

$$c_{nk} = \begin{cases} a_n f_{nk}^{-1}, & 0 \le k \le n \\ 0, & n < k \end{cases}$$

for all $k, n \in \mathbb{N}$, $\delta \in \{cs, bs\}$. Then, $a \in \{\delta(F^{-1})\}^{\alpha}$ iff $C \in \{\delta, l_1\}$.

Proof. Let $x = (x_n)$ and $a = (a_n)$ elements of w. $y = (y_n)$ such that $y = Fx$ which is defined in Equation (4). If we use to Equation (4), then

$$a_n x_n = a_n (F^{-1} y)_n = (Cy)_n . \tag{25}$$

$ax = (a_n x_n) \in \ell_1$ with $x = (x_n) \in \mu(F)$ iff $Cy \in \ell_1$ with $y \in \lambda$. Consequently, $C \in (\mu, \ell_1)$. □

Lemma 17. [41] Let us take $a = (a_k) \in w$ and infinite matrix $C = (c_{nk})$. Let the inverse matrix $H = (h_{nk})$ of the triangular matrix $G = (g_{nk})$ is given by

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j h_{jk}, & 0 \le k \le n \\ 0, & n < k. \end{cases}$$

Then, for any sequence space δ,

$$\delta_G^{\gamma} = \{ a = (a_k) \in w : C \in (\delta, l_\infty) \},$$

$$\delta_G^{\beta} = \{ a = (a_k) \in w : C \in (\delta, c) \}.$$

If we consider Lemma 9, Lemma 16 and Lemma 17 together, the following is obtained;

Corollary 18. Let us take $r = (r_k) \in w$ and infinite matrix $A = (a_{nk})$ and $B = (b_{nk})$ such that

$$a_{nk} = \begin{cases} r_n f_{nk}^{-1}, & 0 \le k \le n \\ 0, & n < k \end{cases} \text{ and } b_{nk} = \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} r_j .$$

If we take $d_1, d_2, d_3, d_4, d_5, d_6, d_7$ and d_8 as follows:

$$d_1 = \left\{ r = (r_k) \in w : \sup_{N, K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (a_{nk} - a_{n,k+1}) \right| < \infty \right\},$$

$$d_2 = \left\{ r = (r_k) \in w : \sup_{N, K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (a_{nk} - a_{n,k-1}) \right| < \infty \right\},$$

$$d_3 = \left\{ r = (r_k) \in w : \lim_k c_{nk} = 0 \right\},$$

$$d_4 = \left\{ r = (r_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_n (b_{nk} - b_{n,k+1}) = \alpha \text{ } for \text{ } all \text{ } k \in \mathbb{N} \right\},$$

$$d_5 = \left\{ r = (r_k) \in w : \lim_n \sum_k \left| b_{nk} - b_{n,k+1} \right| = \sum_k \left| \lim_n (b_{nk} - b_{n,k+1}) \right| \right\},$$
$$d_6 = \left\{ r = (r_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_n b_{nk} = \alpha \ \ for\, all\, k \in \mathbb{N} \right\},$$
$$d_7 = \left\{ r = (r_k) \in w : \sup_{n \in \mathbb{N}} \sum_k \left| b_{nk} - b_{n,k+1} \right| < \infty \right\},$$
$$d_8 = \left\{ r = (r_k) \in w : \sup_{n \in \mathbb{N}} \left| \lim_k b_{nk} \right| < \infty \right\}.$$

then,

(1) $\{bs(F)\}^{\alpha} = d_1$
(2) $\{cs(F)\}^{\alpha} = d_2$
(3) $\{bs(F)\}^{\beta} = d_3 \cap d_4 \cap d_5$
(4) $\{cs(F)\}^{\beta} = d_6 \cap d_7$
(5) $\{bs(F)\}^{\gamma} = d_3 \cap d_7$
(6) $\{cs(F)\}^{\gamma} = d_7 \cap d_8$.

Theorem 19. Let $\mu \in \{cs, bs\}$ and $\lambda \subset w$. Then, $U = (u_{nk}) \in (\mu(F), \lambda)$ iff

$$V^m = (v_{nk}^{(m)}) \in (\mu, c) \text{ for all } n \in \mathbb{N} \tag{26}$$

$$V = (v_{nk}) \in (\mu, \lambda), \tag{27}$$

where

$$v_{nk}^{(m)} = \begin{cases} \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} u_{nj}, & 0 \le k \le m \\ 0, & m < k \end{cases} \tag{28}$$

and

$$v_{nk} = \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} u_{nj} \tag{29}$$

for all k, m, $n \in \mathbb{N}$.

Proof. Necessity part: Let us take that $A = (a_{nk}) \in (\mu(F), \lambda)$ and $x = (x_k) \in \mu(F)$. If we use Equation (5), then we find

$$\begin{aligned} \sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m} a_{nk} \sum_{k=0}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} y_j \\ &= \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nk} y_k = \sum_{k=0}^{m} d_{nk}^{(m)} y_k = D_n^{(m)}(y). \end{aligned} \tag{30}$$

According to the hypothesis, for each $m\in\mathbb{N}$, $A_m(x) \in c$. Then, $V^{(m)} \in c$ for each $m\in\mathbb{N}$ and $V^{(m)} \in (\mu, c)$. $Ax = Vy$ if we consider for $m \to \infty$ from Equation (30). As a result, we find $V = (v_{nk}) \in (\mu, \lambda)$.

Sufficient part: Suppose that Equation (26) and Equation (27) are satisfied and $x = (x_k) \in \mu(F)$ be. By using Corollary 18 and Equations (26) and (30), we have that $y = Fx \in \mu$ and

$$V_n^{(m)}(y) = \sum_{k=0}^{m} v_{nk}^{(m)} y_k = \sum_{k=0}^{m} a_{nk} x_k = A_n^{(m)}(x) \in c.$$

Hence, $A=(a_{nk})_{k\in\mathbb{N}}$ exists. $Ax = Vy$ if we consider for $m\to\infty$ from Equation (30). Consequently, we find $A = (a_{nk}) \in (\mu(F), \lambda)$. □

Theorem 20. Let $\mu \in\{bs,cs\}$ and $\lambda \subset w$ and $U = (u_{nk})$ and $B = (b_{nk})$ be infinite matrices. Let

$$b_{nk} := \frac{f_n}{f_{n+1}} u_{nk} - \frac{f_{n+1}}{f_n} u_{n-1,k}. \tag{31}$$

Then, $U \in (\lambda, \mu(\hat{F}))$ iff $B \in (\lambda, \mu)$.

Proof. Let $z = (z_k) \in \lambda$ and Equation (31) exist. Then, we have

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{k=0}^{m} \left(\frac{f_n}{f_{n+1}} a_{nk} - \frac{f_{n+1}}{f_n} a_{n-1,k} \right) z_k. \tag{32}$$

If we take $m\to\infty$ to Equation (32), we have that $(Bz)_n = (F(Az))_n$. Consequently, $Az\in\mu(F)$ iff $Bz\in\mu$. That is, $B \in (\lambda, \mu)$.□

Let us give almost convergent sequences space, which was first defined by Lorentz [42]. Let $t = (t_k)\in\ell_\infty$. t is almost convergent to limit ℓ iff $\lim_{m\to\infty} \sum_{k=0}^{m} \frac{t_{n+k}}{m+1} = \alpha$ uniformly in n. It is denoted by $\hat{c} - \lim t = \alpha$. In addition, $\hat{c}s$ and $\hat{c}_0$ mean the spaces of almost convergent series and almost null sequences, respectively. $\hat{c}_0$ and $\hat{c}$ are

$$\hat{c}_0 = \left\{ x = (x_k) \in l_\infty : \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n \right\},$$

$$\hat{c} = \left\{ x = (x_k) \in l_\infty : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \alpha \text{ uniformly in } n \right\}.$$

Now, let us take infinite matrix $R = (r_{nk})$ and list the following:

$$\exists a_k \in \mathbb{C} \ni f - \lim r_{nk} = a_k \text{ for each k} \in \mathbb{N}, \tag{33}$$

$$\lim_q \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \Delta \left[\sum_{j=0}^{n+i} (r_{jk} - a_k) \right] \right| = 0 \text{ uniformly in } n, \tag{34}$$

$$\sup_{n \in \mathbb{N}} \sum_k \left| \Delta \left[\sum_{j=0}^{n} r_{jk} \right] \right| < \infty, \tag{35}$$

$$\exists a_k \in \mathbb{C} \ni f - \lim \sum_{j=0}^{n} r_{jk} = a_k \text{ for each k} \in \mathbb{N}, \tag{36}$$

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=0}^{n} r_{jk} \right| < \infty, \tag{37}$$

$$\exists a_k \in \mathbb{C} \ni \sum_n \sum_k r_{nk} = a_k \text{ for each k} \in \mathbb{N}, \tag{38}$$

$$\lim_n \sum_k \left| \Delta \left[\sum_{j=0}^{n} (r_{jk} - a_k \right] \right| = 0, \tag{39}$$

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=0}^{n} r_{jk} \right|^p < \infty, \ q = \frac{p}{p-1}, \tag{40}$$

$$\sup_{m,n \in \mathbb{N}} \left| \sum_{n=0}^{m} r_{nk} \right| < \infty, \tag{41}$$

$$\sup_{m,l \in \mathbb{N}} \left| \sum_{n=0}^{m} \sum_{k=l}^{\infty} r_{nk} \right| < \infty, \tag{42}$$

$$\sup_{m,l \in \mathbb{N}} \left| \sum_{n=0}^{m} \sum_{k=0}^{l} r_{nk} \right| < \infty, \tag{43}$$

$$\lim_m \sum_k \left| \sum_{n=m}^{\infty} r_{nk} \right| = 0, \tag{44}$$

$$\sum_n \sum_k r_{nk} \text{ convergent} \tag{45}$$

$$\lim_{m \to \infty} \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) = a \text{ for each } k \in \mathbb{N} \ a \in \mathbb{C} \tag{46}$$

Lemma 21. Let infinite matrix $R = (r_{nk})$ for all $k,n \in \mathbb{N}$. Then,

(1) $R=(r_{nk})\in(\hat{c},cs)$ iff Equation (23) and Equations (37)–(39) hold [43].

(2) $R=(r_{nk})\in(cs,\hat{c})$ iff Equation (11) and Equation (33) hold [44].

(3) $R=(r_{nk})\in(bs,\hat{c})$ iff Equations (8), (11) and (33)–(34) hold [45].

(4) $R=(r_{nk})\in(bs,\hat{c}s)$ iff Equations (8) and (34)–(36) hold [45].

(5) $R=(r_{nk})\in(cs,\hat{c}s)$ iff Equation (35) and Equation (36) hold [44].

(6) $R=(r_{nk})\in(l_\infty,bs)=(c,bs)=(c_0,bs)$ iff Equation (37) holds [38].

(7) $R=(r_{nk})\in(l_p,bs)$ iff Equation (40) holds [46].

(8) $R=(r_{nk})\in(l,bs)$ iff Equation (41) holds [38].

(9) $R=(r_{nk})\in(bv,bs)$ iff Equation (42) holds [38].

(10) $R=(r_{nk})\in(bv_0,bs)$ iff Equation (43) holds [46].

(11) $R=(r_{nk})\in(l_\infty,cs)$ iff Equation (44) holds [38].

(12) $R=(r_{nk})\in(c,cs)$ if and only Equations (10), (37) and (45) hold [38].

(13) $R=(r_{nk})\in(cs_0,cs)$ iff Equations (9) and (46) hold [38].

(14) $R=(r_{nk})\in(l_p,cs)$ iff Equations (10) and (40) hold [46].

(15) $R=(r_{nk})\in(l,cs)$ iff Equations (10) and (41) hold [46].

(16) $R=(r_{nk})\in(bv,cs)$ if and only Equations (10), (41) and (43) hold [38].

(17) $R=(r_{nk})\in(bv_0,cs)$ iff Equations (10) and (43) hold [46].

Now, suppose v_{nk} and $v_{nk}^{(m)}$ which mentioned Equation (28) and Equation (29) and give the following equations

$$\lim_k v_{nk}^{(m)} = 0 \text{ for each } n \in \mathbb{N}, \tag{47}$$

$$\lim_n \sum_k \left| v_{nk}^{(m)} - v_{n,k+1}^{(m)} \right| = 0 \text{ uniformly in } n, \tag{48}$$

$$\exists v_k \in \mathbb{C} \ni \lim_{n\to\infty} (v_{nk}^{(m)} - v_{n,k+1}^{(m)}) = v_k \text{ for each } k \in \mathbb{N}, \tag{49}$$

$$\lim_{k\to\infty} v_{nk} = 0 \text{ for each } n \in \mathbb{N}, \tag{50}$$

$$\sup_n \sum_k \left| v_{nk} - v_{n,k+1} \right| < \infty, \tag{51}$$

$$\exists v_k \in \mathbb{C} \ni \lim_{n\to\infty} (v_{nk} - v_{n,k+1}) = d_k \text{ for each } k \in \mathbb{N}, \tag{52}$$

$$\exists l \in \mathbb{C} \ni \lim_{n\to\infty} \sum_k \left| v_{nk} - v_{n,k+1} \right| = l \text{ uniformly in } n, \tag{53}$$

$$\sup_{m\in\mathbb{N}} \sum_k \left| \sum_{n=0}^{m} (v_{nk} - v_{n,k+1}) \right| < \infty, \tag{54}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (v_{nk} - v_{n,k+1}) \right| = 0, \tag{55}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (v_{nk} - v_{n,k+1}) \right| = \sum_k \left| \sum_n (v_{nk} - v_{n,k+1}) \right| = 0, \tag{56}$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} [(v_{nk} - v_{n,k+1}) - (v_{n-1,k} - v_{n-1,k+1})] \right| < \infty, \tag{57}$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (v_{nk} - v_{n-1,k}) \right| < \infty. \tag{58}$$

$$\exists v_k \in \mathbb{C} \ni \lim_n \ \mathrm{v}_{nk}^{(m)} = v_k \ \text{ for each } k \in \mathbb{N}, \tag{59}$$

$$\sup_n \sum_k \left| v_{nk}^{(m)} - v_{n,k+1}^{(m)} \right| < \infty, \tag{60}$$

$$\sup_n \left| \lim_k v_{nk} \right| < \infty, \tag{61}$$

$$\exists v_k \in \mathbb{C} \ni \lim_{n \to \infty} \ v_{nk} = v_k \quad \text{for each } k \in \mathbb{N}, \tag{62}$$

$$\sup_{m \in \mathbb{N}} \left| \lim_k \sum_{n=0}^{m} v_{nk} \right| < \infty, \tag{63}$$

$$\sup_{m \in \mathbb{N}} \sum_k \left| \sum_{n=0}^{m} (v_{nk} - v_{n,k-1}) \right| < \infty, \tag{64}$$

$$\exists v_k \in \mathbb{C} \ni \sum_n v_{nk} = v_k \ \text{ for each } k \in \mathbb{N} \tag{65}$$

$$\sup_{N,K \in \mathcal{F}} \sum_{n \in \mathbb{N}} \left| \sum_{k \in \mathbb{N}} (v_{nk} - v_{n,k-1}) \right| < \infty, \tag{66}$$

$$\exists v_k \in \mathbb{C} \ni f - \lim v_{nk} = v_k \ \text{ for each } k \in \mathbb{N}, \tag{67}$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} [(v_{nk} - v_{n-1,k}) - (v_{n,k-1} - v_{n-1,k-1})] \right| < \infty, \tag{68}$$

$$\lim_q \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \Delta \left[\sum_{j=0}^{n+i} (v_{jk} - l_k) \right] \right| = 0 \text{ uniformly in } n, \tag{69}$$

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=0}^{n} v_{jk} \right| < \infty, \tag{70}$$

$$\exists l \in \mathbb{C} \ni \sum_n \sum_k v_{nk} = l \tag{71}$$

$$\lim_n \sum_k \left| \Delta \left[\sum_{j=0}^{n} (v_{jk} - l_k \right] \right| = 0, \tag{72}$$

$$\sup_{n \in \mathbb{N}} \sum_k \left| \Delta \left[\sum_{j=0}^{n} v_{jk} \right] \right| < \infty, \tag{73}$$

$$\exists v_k \in \mathbb{C} \ni f - \lim \sum_{j=0}^{n} v_{jk} = v_k \text{ for each } k \in \mathbb{N}. \tag{74}$$

If we consider Theorem 19, Theorem 20, Lemma 9, and Lemma 21, then we give the below conclusions.

Corollary 22. Let us take $U = (u_{nk})$ mentioned in Theorem 19. Then,

(1) $U = (u_{nk}) \in (bs(F), c_0)$ iff Equations (47) and (49) hold and Equation (53) holds with $l = 0$.

(2) $U = (u_{nk}) \in (bs(F), cs_0)$ iff Equations (47)–(50) and Equation (55) hold.

(3) $U = (u_{nk}) \in (bs(F), c)$ iff Equations (47)–(50) and Equations (52) and (53) hold.

(4) $U = (u_{nk}) \in (bs(F), cs)$ iff Equations (47)–(50) and Equation (56) hold.

(5) $U = (u_{nk}) \in (bs(F), l_\infty)$ iff Equations (47)–(51) hold.

(6) $U = (u_{nk}) \in (bs(F), bs)$ iff Equations (47)–(50) and Equation (54) hold.

(7) $U = (u_{nk}) \in (bs(F), l_1)$ iff Equations (47)–(50) and Equation (58) hold.

(8) $U = (u_{nk}) \in (bs(F), bv)$ iff Equations (47)–(50) and Equation (57) hold.

(9) $U = (u_{nk}) \in (bs(F), bv_0)$ iff Equations (57) and (47)–(49) and Equation (51) hold and Equation (53) also holds with $l = 0$.

Corollary 23. Let us take $U = (u_{nk})$ mentioned in Theorem 19. Then,

(1) $U = (u_{nk}) \in (cs(F), c_0)$ iff Equations (59), (60), Equation (51) hold and Equation (62) also holds with $v_k = 0$ for all $k \in \mathbb{N}$.

(2) $U = (u_{nk}) \in (cs(F), cs_0)$ iff Equations (59), (60), Equation (54) hold and Equation (65) also holds with $v_k = 0$ for all $k \in \mathbb{N}$.

(3) $U = (u_{nk}) \in (cs(F), c)$ iff Equations (59), (60), Equation (51) and Equation (62) hold.

(4) $U=(u_{nk})\in(cs(F),cs)$ iff Equations (59), (60), Equation (64) and Equation (65) hold.

(5) $U=(u_{nk})\in(cs(F),l_\infty)$ iff Equations (51) and (59)–(61) hold.

(6) $U=(u_{nk})\in(cs(F),bs)$ iff Equations (59), (60), (54) and (63) hold.

(7) $U=(u_{nk})\in(cs(F),l_1)$ iff Equations (59), (60) and (66) hold.

(8) $U=(u_{nk})\in(cs(F),bv)$ iff Equations (59), (60) and (68) hold.

(9) $U=(u_{nk})\in(cs(F),bv_0)$ iff Equation (59), (60) and (62) hold and Equation (68) holds with $v_k=0$ for all $k\in\mathbb{N}$.

Corollary 24. Let us take $U=(u_{nk})$ mentioned Theorem 19. Then,

(1) $U=(u_{nk})\in(bs(F),\hat{c})$ iff Equations (47)–(51), (67) and (69) hold.

(2) $U=(u_{nk})\in(bs(F),\hat{c}_0)$ iff Equations (47)–(51) hold and Equations (68) and (69) also hold with $v_k=0$ in Equation (67) and $l_k=0$ in Equation (69).

(3) $U=(u_{nk})\in(cs(F),\hat{c})$ iff Equations (59), (60), (67) and (51) hold.

(4) $U=(u_{nk})\in(cs(F),\hat{c}_0)$ iff Equations (51), (59), (60) hold and Equation (67) also holds with $v_k=0$.

(5) $U=(u_{nk})\in(bs(F),\hat{c}s)$ iff Equations (69), (73), (74) and (47)–(50), hold.

(6) $U=(u_{nk})\in(cs(F),\hat{c}s)$ iff Equations (73), (74), (59) and (60) hold.

Corollary 25. Let us take $U=(u_{nk})$ mentioned Theorem 20. Then,

(1) $U=(u_{nk})\in(l_p,bs(F))$ iff Equation (40) holds with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(2) $U=(u_{nk})\in(l_\infty,bs(F))=(c,bs(F))=(c_0,bs(F))$ iff Equation (37) holds with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(3) $U=(u_{nk})\in(l_1,bs(F))$ iff Equation (41) holds with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(4) $U=(u_{nk})\in(bv,bs(F))$ iff Equation (42) holds with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(5) $U=(u_{nk})\in(bv_0,bs(F))$ iff Equation (43) holds b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(6) $U=(u_{nk})\in(l_\infty,cs(F))$ iff Equation (44) holds with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(7) $U=(u_{nk})\in(c,cs(F))$ iff (10), Equations (37) and (47) hold b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(8) $U=(u_{nk})\in(cs_0,cs(F))$ iff Equations (9) and (46) hold with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(9) $U=(u_{nk})\in(l_p,cs(F))$ iff Equations (10) and (40) hold with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(10) $U=(u_{nk})\in(l,cs(F))$ iff Equations (10) and (41) hold with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(11) $U=(u_{nk})\in(bv,cs(F))$ iff Equations (10), (41) and (43) hold with b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(12) $U=(u_{nk})\in(bv_0,cs(F))$ iff Equations (10) and (43) hold b_{nk} instead of r_{nk}, where b_{nk} is defined by Equation (31).

(13) $U=(u_{nk})\in(\hat{c},cs(F))$ iff Equation (65) and Equations (70)–(72) hold with b_{nk} instead of v_{nk}, where b_{nk} is defined by Equation (31).

3. Discussion

Kızmaz [47] first introduced the difference sequence operator in 1981. Generalized difference sequence spaces were characterized and investigated by Kirişçi and Başar [4] in 2010. Kara [27] first defined the Fibonacci Difference Matrix F, which created the Fibonacci sequence (f_n) in 2013. He also introduced the new sequence spaces $\ell_p(F)$ and $\ell_\infty(F)$; where $1 \le p < \infty$. The spaces $c(F(r,s))$ and $c_0(F(r,s))$ were introduced by Candan [28] in 2015. In 2015, the sequence space $\ell_p(F(r,s))$ was introduced and studied by Candan and Kara [19]; where $1 \le p \le \infty$. In addtion, a class of compact operators on $\ell_p(F)$ and $\ell_\infty(F)$ was characterized by Kara et al. [32], where $1 \le p < \infty$.

In the present study, we introduced the domain of a triangular infinite matrix on a sequences space. We described spaces $cs(F)$ and $bs(F)$, where F, cs, and bs are the Fibonacci Difference Matrix, convergent and bounded series, respectively. It was demonstrated that $bs(F)$ are the linear spaces, and given that $cs(F)$ is linear space in Theorem 6. without proof and, they have the same norm

$$\|x\|=\sup_{n\in\mathbb{N}}\left|\sum_{j=0}^{n}\left(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right)\right|$$

where $x \in cs(F)$ or $x \in bs(F)$. It was found that they are Banach spaces. In addition, inclusions theorems were examined and found. Finally, the γ, β, α -duals of them were calculated. Finally, some matrix transformations as a main result were given.

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