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Stanley Depth of Edge Ideals of Some Wheel-Related Graphs

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Abstract: Stanley depth is a geometric invariant of the module and is related to an algebraic invariant called depth of the module. We compute Stanley depth of the quotient of edge ideals associated with some familiar families of wheel-related graphs. In particular, we establish general closed formulas for Stanley depth of quotient of edge ideals associated with the m th-power of a wheel graph, for $m \geq 3$, gear graphs and anti-web gear graphs.

Keywords: stanley depth; edge ideal; m th-power of wheel graph; gear graph; anti-web gear graph

1. Introduction

Richard P. Stanley is famous for his contribution to combinatorial aspects of algebra and geometry. Simplicial complexes are critical objects in combinatorics; two of the most important such complexes are partitionable and Cohen–Macaulay complexes. Stanley introduced the famous conjecture relating these two complexes and hence, introduced the notion of Stanley depth in 1982 in [1]. Stanley Depth is a geometric invariant of module. The Stanley conjecture is interesting in the sense that it compares a combinatorial invariant with a homological invariant of module. Stanley depth gained attention of algebraists in 2006 when J. Herzog and D. Popescu studied this conjecture. Afterwards, a number of articles have been published in which this conjecture has been discussed for different cases. Let $S = K[x_1, x_2, \dots, x_n]$ be a polynomial ring in n variables over a field K and M be finitely generated $\mathbb{Z}^n$ -graded S -module. For a homogeneous element $q \in M$ and a subset $Z \subseteq \{x_1, x_2, \dots, x_n\}$, $qK[Z]$ denotes the K -subspace of M generated by all homogeneous elements of the form qv , where v is a monomial in $K[Z]$. The space $qK[Z]$ is $\mathbb{Z}^n$ -graded K -subspace and is called *Stanley space* of dimension $|Z|$ if it is a free $K[Z]$ -module, where $|Z|$ denotes the cardinality of Z . A decomposition of M as a finite direct sum of $\mathbb{Z}^n$ -graded K -vector spaces is called Stanley decomposition.

$$\mathcal{D} : M = \bigoplus_{j=1}^s q_j K[Z_j]$$

where each $q_j K[Z_j]$ is a Stanley space of M . The number

$$\text{sdepth}(\mathcal{D}) = \min\{|Z_j| : j = 1, 2, 3, \dots, s\}$$

is called Stanley depth of decomposition $\mathcal{D}$ and the quantity

$$\text{sdepth}(M) = \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called Stanley depth of M .

Richard P. Stanley [1] in his article “Linear Diophantine equation and local cohomology” stated this conjecture, in a generalized way, as follows:

Let R be a finitely generated $\mathbb{N}^n$ -graded K -algebra ($R_0 = K$) and let M be a finitely generated $\mathbb{Z}^n$ -graded R -module. Then there exist finitely many subalgebras $S_1, \dots, S_t$ of R , each generated by algebraically independent $\mathbb{N}^n$ -homogeneous elements of R , and there exist $\mathbb{Z}^n$ -homogeneous elements $\eta_1, \dots, \eta_t$ of M , such that

$$M = \bigoplus_{i=1}^t \eta_i S_i$$

where $\dim(S_i) \geq \text{depth}(M)$ for all i , and where $\eta_i S_i$ is a free S_i -module and where $\bigoplus_{i=1}^t$ is finite direct sum of modules $\eta_i S_i$. The direct sum of modules is a new module defined as $\bigoplus_{i \in I} M_i = \{(m_i \in M_i) \mid i \in I \mid m_i = 0 \text{ for but finitely many } i\}$.

In [2], the authors found an upper bound for the Stanley depth of $S/Q \cap Q'$, where S is polynomial ring and Q, Q' are monomial primary ideals and also showed that this conjecture holds for $Q_1 \cap Q_2$ and $S/Q_1 \cap Q_2 \cap Q_3$, where Q_1, Q_2, Q_3 are monomial irreducible ideals. In [3], authors found upper bound for Stanley depth of edge ideals of k -partite complete graph and s -uniform complete bipartite hypergraph. In [4], authors found upper bound for the Stanley depth of edge ideals of complete graph and complete bipartite graph when some variables are added as generators. In [5], the authors gave the lower and upper bound for the cyclic module associated with the complete k -partite hyper-graph. Duval et al. disproved this conjecture in [6]. A method to compute Stanley depth for the module M , where M is of the form $M = I/J$, where $J \subset I \subset S$ are monomial ideals, was given by Herzog, Vlodoiu and Zheng in [7]. Rinaldo developed this method into an algorithm in [8] which is implemented in CoCoA [9]. In [10], Rauf proved some important basic results regarding depth and Stanley depth of multi-graded module. Stanley depth of the edge ideal associated with some graphs have been subject matter of recent results. In [11], Cimpoeas computed Stanley depth of the edge ideal of line and cycles. Iqbal et al. computed Depth and sdepth of edge ideals of square paths and square cycles in [12]. Stanley depth of edge ideals of general powers of path ideals is computed in [13]. In [14], Cimpoeas proved several inequalities regarding computation of Stanley depth. Let $I(P_n)$ denote the edge ideal associated with path on n vertices. Then bounds and values for Stanley depth and depth for module $S/I(P_n)$ are given in [11–13]. Stanley depth of the quotient of edge ideal associated with wheel graph and square wheel graph is computed in [15]. In paper [16], Biro et al. discussed interval partition associated with partially ordered set which is required to find Stanley depth, and also how this interval partition is used to find out Stanley depth. Cimpoeas, in [17] discussed Stanley depth of modules having small number of generators. For the friendly introduction to commutative algebra, we refer readers to [18].

We set $S_m = K[x_1, x_2, x_3, \dots, x_m]$, $\text{supp}(u) = \{i : x_i \mid u\}$ and $\text{supp}(J) = \{i : x_i \mid v \text{ for some } v \in G(J)\}$ where $G(J)$ denotes the unique minimal set of monomial generators of the monomial ideal J for the rest of this article.

2. Example

We explain Stanley decomposition and Stanley depth with the help of an example. We present some graphical representations which are used to construct a particular Stanley decomposition which are explained briefly in [7].

Let $S = K[x_1, x_2]$ and $I = (x_1^3, x_1^2 x_2, x_2^2)$ be a monomial ideal. Since we know that one module can have many Stanley decompositions, see for example Figure 1.

Figure 1. Decomposition D_1 .

In the Figure 1, decomposition D_1 of I is shown graphically given as,

$$D_1 : I = x_1^3 K[x_1] \oplus x_1^2 x_2 K[x_1] \oplus x_2^2 K[x_1, x_2].$$

Now we give another decomposition to explain further. In the Figure 2, decomposition D_2 of I is shown graphically as

$$D_2 : I = x_2^2 K[x_2] \oplus x_1 x_2^2 K[x_2] \oplus x_1^2 x_2 K[x_2] \oplus x_1^3 K[x_1, x_2].$$

Figure 2. Decomposition D_2 .

Clearly the $sdepth(I) = 1$ because in decomposition D_2 of I , Stanley spaces have dimensions 1, 1, 1 and 2 respectively. So the smallest in all these dimensions is 1 and called Stanley depth of I . For further and detailed reading, we refer to [7].

The Stanley decomposition of S/I is given as

$$D : S/I = K \oplus x_1 K \oplus x_1^2 K \oplus x_2 K \oplus x_1 x_2 K,$$

hence, we see that $sdepth(S/I) = 0$.

Definition 1. Edge ideal: Let $G = (V(G), E(G))$ be a graph with edge set $E(G)$ and vertex set $V(G) = \{1, 2, \dots, n\}$ then the edge ideal associated with G is the square free monomial ideal of S , defined as

$$I(G) = (x_l x_k : \{l, k\} \in E(G)).$$

Definition 2. m th-power of wheel graph: Let $n \geq 5$ and $\mathcal{W}_n^m$ be a m th-power of wheel on n vertices, then the edge ideal of m th-power of Wheel is

$$(I(W_n^m)) = (x_1x_n, x_2x_n, x_3x_n, \dots, x_{n-1}x_n, x_1x_2, x_2x_3, \dots, x_{n-1}x_1, x_1x_3, x_2x_4, \dots, x_{n-1}x_2, \\ x_1x_4, x_2x_5, \dots, x_{n-1}x_3, \dots, x_1x_m, x_2x_{m+1}, \dots, x_{n-1}x_{m-1}, x_1x_{m+1}, x_2x_{m+2}, \dots, x_{n-1}x_m).$$

An example of graph of $I(W_7^2)$ is given in Figure 3.

Definition 3. Gear graph: The gear graph G_n , where $n \in \{5, 7, 9, \dots\}$ consists of cycle of $n - 1$ vertices and the vertex n joins with vertices $1, 3, 5, 7, \dots, n - 1$.

The edge ideal $I(G_n)$ of the gear graph is defined as

$$I(G_n) = (x_1x_2, x_2x_3, x_3x_4, \dots, x_{n-2}x_{n-1}, x_{n-1}x_1, x_1x_n, x_3x_n, x_5x_n, \dots, x_{n-2}x_n).$$

The graph of $I(G_{11})$ is given in Figure 4.

Figure 3. Graph of $I(W_7^2)$.

Figure 4. Graph of $I(G_{11})$.

Definition 4. Anti-web gear graph: The anti-web gear graph denoted by AWG_n , where $n \in \{9, 11, 13, \dots\}$ consists of cycle of $n - 1$ vertices and also the edges joining vertices having distance two between them on this cycle. And the vertex n joins with vertices $1, 3, 5, 7, \dots, n - 2$. The order of this graph is n .

The edge ideal $I(AWG_n)$ of the Gear graph is defined as

$$I(AWG_n) = (x_1x_2, x_2x_3, x_3x_4, \dots, x_{n-2}x_{n-1}, x_{n-1}x_1, x_1x_3, x_2x_4, x_3x_5, \dots, x_{n-2}x_1, x_{n-1}x_2, \\ x_1x_n, x_3x_n, x_5x_n, \dots, x_{n-2}x_n).$$

The graph of $I(AWG_9)$ is given in Figure 5.

Figure 5. Graph of $I(AWG_9)$.

Following important Lemmas will play significant role in our computation. A short exact sequence of Modules is $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ where arrows are maps such that the third map is surjective, second map is injective along with the property that kernel of the third map is same as that of image of second map.

Lemma 1 ([10]). Let $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be a short exact sequence of $\mathbb{Z}^n$ -graded S -modules. Then

$$\text{sdepth}(V) \geq \min\{\text{sdepth}(U), \text{sdepth}(W)\}.$$

Lemma 2 ([12]). Let $J \subset S$ be a square free monomial ideal such that $\text{supp}(J) = [n]$, let $w = x_{j_1} x_{j_2} x_{j_3} \dots x_{j_t} \in S/J$, such that, $x_i w \in J$, for all $i \in [n] \setminus \text{supp}(w)$. Then $\text{sdepth}(S/J) \leq t$.

For example is $S = K[x_1, x_2, x_3, \dots, x_n]$ is a polynomial ring in n variables over a field K and $I(P_n^3)$ is a cube of path whose ideal is given as

$$I(P_n^3) = (x_1 x_2, x_2 x_3, \dots, x_{n-1} x_n, x_1 x_3, x_2 x_4, \dots, x_{n-2} x_n, x_1 x_4, x_2 x_5, \dots, x_{n-3} x_n).$$

Then Stanley decomposition for the following cases by [7] is given as

$$\text{If } n = 3. \text{ Then, } S/I(P_3^3) = K[x_1] \oplus x_2 K[x_2] \oplus x_3 K[x_3].$$

$$\text{If } n = 4. \text{ Then, } S/I(P_4^3) = K[x_1] \oplus x_2 K[x_2] \oplus x_3 K[x_3] \oplus x_4 K[x_4].$$

$$\text{If } n = 5. \text{ Then, } S/I(P_5^3) = K[x_1, x_5] \oplus x_2 K[x_2] \oplus x_3 K[x_3] \oplus x_4 K[x_4].$$

In this paper we are interested in computing Stanley depth of the quotient of m th-power of Wheel graph, gear graph and anti-web gear graph.

3. Main Result

In this section we are going to compute our main results.

Theorem 1. Let $n \geq 6$, and $I(W_n^m)$ be an ideal of wheel graph on n vertices, then $\text{sdepth}(S/I(W_n^m)) = 1$.

Proof. The ideal of m th-power of Wheel graph is given as

$$(I(W_n^m)) = (x_1 x_n, x_2 x_n, x_3 x_n, \dots, x_{n-1} x_n, x_1 x_2, x_2 x_3, \dots, x_{n-1} x_1, x_1 x_3, x_2 x_4, \dots, x_{n-1} x_2, x_1 x_4, x_2 x_5, \dots, x_{n-1} x_3, \dots, x_1 x_m, x_2 x_{m+1}, \dots, x_{n-1} x_{m-1}, x_1 x_{m+1}, x_2 x_{m+2}, \dots, x_{n-1} x_m).$$

Now consider a short exact sequence by taking functions $\psi : (I(\mathcal{W}_n^m) : x_n) \rightarrow (I(\mathcal{W}_n^m))$ such that $\psi(r) = x_n r$ is a multiplication by x_n and $\varphi : (I(\mathcal{W}_n^m)) \rightarrow (I(\mathcal{W}_n^m), x_n)$ such that $\varphi(r) = r$. Throughout this paper we use similar kind of functions to make our sequence of modules short exact.

$$0 \rightarrow S/(I(\mathcal{W}_n^m) : x_n) \rightarrow S/(I(\mathcal{W}_n^m)) \rightarrow S/(I(\mathcal{W}_n^m), x_n) \rightarrow 0.$$

by Lemma 1

$$\text{sdepth}(S/(I(\mathcal{W}_n^m))) \geq \min\{\text{sdepth}(S/(\mathcal{W}_n^m) : x_n), \text{sdepth}(S/(I(\mathcal{W}_n^m), x_n))\}.$$

Here

$$(I(\mathcal{W}_n^m) : x_n) = (x_1, x_2, x_3, x_4, \dots, x_{n-1}). \text{ So } (S/(I(\mathcal{W}_n^m) : x_n)) \cong K[x_n].$$

$$\text{Hence } \text{sdepth}(S/(I(\mathcal{W}_n^m) : x_n)) = 1.$$

And

$$(I(\mathcal{W}_n^m), x_n) = (x_1 x_2, x_2 x_3, \dots, x_{n-1} x_1, x_1 x_3, x_2 x_4, \dots, x_{n-1} x_2, \\ x_1 x_4, x_2 x_5, \dots, x_{n-1} x_3, \dots, x_1 x_m, x_2 x_{m+1}, \dots, x_{n-1} x_{m-1}, x_1 x_{m+1}, x_2 x_{m+2}, \dots, x_{n-1} x_m, x_n).$$

$$\text{where } S/(I(\mathcal{W}_n^m), x_n) \cong S_{n-1}/I(C_{n-1}^m)$$

$$\text{and } \text{sdepth}(S/(I(\mathcal{W}_n^m), x_n)) = \text{sdepth}(S_{n-1}/I(C_{n-1}^m)) \geq \left\lceil \frac{(n-1)-m}{2m+1} \right\rceil \text{ by [19].}$$

Here $\left\lceil \frac{(n-1)-m}{2m+1} \right\rceil \geq 1$, because if $n-1 \leq m$ then we have two cases. If $n-1 = 2k$, where $k \geq 3$ then we can check that $I(C_{2k}^m) = I(C_{2k}^k)$. And if $n-1 = 2k+1$, then can see that $I(C_{2k+1}^m) = I(C_{2k+1}^k)$, so we get the desired result for both these cases. So $\text{sdepth}(S/(I(\mathcal{W}_n^m) : x_n)) = 1 \leq \text{sdepth}(S/(I(\mathcal{W}_n^m), x_n))$. This implies that $\text{sdepth}(S/(I(\mathcal{W}_n^m))) \geq \text{sdepth}(S/(I(\mathcal{W}_n^m) : x_n)) = 1$. We get the other inequality by Proposition 2.7 in [14]. Hence $\text{sdepth}(S/(I(\mathcal{W}_n^m))) = \text{sdepth}(S/(I(\mathcal{W}_n^m) : x_n)) = 1$. $\square$

Theorem 2. Let $n \geq 5$ and is an odd number, then $\text{sdepth}(S/I(G_n)) \geq \left\lceil \frac{n-2}{3} \right\rceil$.

Proof. We have

$$I(G_n) = (x_1 x_2, x_2 x_3, x_3 x_4, \dots, x_{n-2} x_{n-1}, x_{n-1} x_1, x_1 x_n, x_3 x_n, x_5 x_n, \dots, x_{n-2} x_n).$$

Consider the following short exact sequence

$$0 \rightarrow S/(I(G_n) : x_n) \rightarrow S/I(G_n) \rightarrow S/(I(G_n), x_n) \rightarrow 0.$$

By Lemma 1

$$\text{sdepth}(S/I(G_n)) \geq \min\{\text{sdepth}(S/(I(G_n) : x_n)), \text{sdepth}(S/(I(G_n), x_n))\}.$$

Here

$$(I(G_n) : x_n) = (x_1 x_2, x_2 x_3, x_3 x_4, \dots, x_{n-2} x_{n-1}, x_{n-1} x_1, x_n).$$

So $S/(I(G_n) : x_n) \cong S_{n-1}/I(C_{n-1})$. Hence by Theorem 1.9 of [11] $\text{sdepth}(S/(I(G_n) : x_n)) \geq \left\lceil \frac{n-2}{3} \right\rceil$. And

$$(I(G_n), x_n) = (x_1, x_3, x_5, \dots, x_{n-2}).$$

So we have $S/(I(G_n), x_n) \cong K[x_2, x_4, x_6, \dots, x_{n-1}, x_n]$.
Hence $\text{sdepth}(S/(I(G_n), x_n)) \geq \left\lceil \frac{n}{2} \right\rceil$.

So we get $\text{sdepth}(S/(I(G_n))) \geq \text{sdepth}(S/(I(G_n), x_n))$, $\text{sdepth}(S/(I(G_n))) \geq \lceil \frac{n-2}{3} \rceil$. $\square$

Theorem 3. Let $n \geq 5$, then $\text{sdepth}(S/(I(G_n))) = \lceil \frac{n}{3} \rceil$, for $n \equiv 0 \pmod{3}$.

The $\text{sdepth}(S/(I(G_n))) = \lceil \frac{n}{3} \rceil - 1$, for $n \equiv 1 \pmod{3}$.

And when $n \equiv 2 \pmod{3}$, then $\lceil \frac{n-2}{3} \rceil \leq \text{sdepth}(S/(I(G_n))) \leq \lceil \frac{n}{3} \rceil$.

Proof. We prove this result by the following three cases.

Case 1 : When $n = 3q$, where $q \in \{3, 5, 7, \dots\}$. Then by Theorem 2 we can see $\text{sdepth}(S/(I(G_n))) \geq \lceil \frac{n}{3} \rceil$.

Now we have to show the other inequality.

Since $w = x_1 x_4 x_7 \dots x_{(3(q-2)+1)} x_{3(q-1)+1} \in S/(I(G_n))$, but $x_i w \in I(G_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(G_n))) \leq q = \lceil \frac{n}{3} \rceil$.

Case 2 : When $n = 3q + 1$, where $q \in \{2, 4, 6, \dots\}$. Then by Theorem 2 we can see $\text{sdepth}(S/(I(G_n))) \geq \lceil \frac{(3q+1)-2}{3} \rceil = q = \lceil \frac{n}{3} \rceil - 1$.

Now to prove the other inequality.

Since $w = x_1 x_4 x_7 \dots x_{(3(q-1)+1)} x_{3q} \in S/(I(G_n))$, but $x_i w \in I(G_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(G_n))) \leq q = \lceil \frac{n}{3} \rceil - 1$.

Case 3 : When $n = 3q + 2$, where $q \in \{1, 3, 5, 7, \dots\}$. Then by Theorem 2 we can see $\text{sdepth}(S/(I(G_n))) \geq \lceil \frac{n-2}{3} \rceil$.

To show the other inequality.

Since $w = x_1 x_4 x_7 \dots x_{(3(q-1)+1)} x_{3q} \in S/(I(G_n))$, but $x_i w \in I(G_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(G_n))) \leq q + 1 = \lceil \frac{n}{3} \rceil$. $\square$

Theorem 4. Let $n \geq 9$ and is an odd number, then $\text{sdepth}(S/I(AWG_n)) \geq \lceil \frac{n-3}{5} \rceil$.

Proof. We have

$$I(AWG_n) = (x_1 x_2, x_2 x_3, x_3 x_4, \dots, x_{n-2} x_{n-1}, x_{n-1} x_1, x_1 x_3, x_2 x_4, \dots, x_{n-2} x_1, x_{n-1} x_2, x_1 x_n, x_3 x_n, \dots, x_{n-2} x_n).$$

Consider the following short exact sequence

$$0 \rightarrow S/(I(AWG_n) : x_n) \rightarrow S/I(AWG_n) \rightarrow S/(I(AWG_n), x_n) \rightarrow 0.$$

Here

$$(I(AWG_n) : x_n) = (x_1, x_3, x_5, \dots, x_{n-2}).$$

So $S/(I(AWG_n) : x_n) \cong K[x_2, x_4, x_6, \dots, x_{n-1}, x_n]$.

We get $\text{sdepth}(S/(I(AWG_n) : x_n)) \geq \lceil \frac{n}{2} \rceil$, and

$$(I(AWG_n), x_n) = (x_1 x_2, x_2 x_3, x_3 x_4, \dots, x_{n-2} x_{n-1}, x_{n-1} x_1, x_1 x_3, x_2 x_4, \dots, x_{n-2} x_1, x_{n-1} x_2, x_n).$$

So we have $S/(I(AWG_n), x_n) \cong S_{n-1}/I(C_{n-1}^2)$, where $I(C_{n-1}^2)$ is a square cycle on $n-1$ vertices. By using [8,12] we get $\text{sdepth}(S/(I(AWG_n), x_n)) \geq \lceil \frac{n-3}{5} \rceil$. By applying Lemma 1 we have

$$\text{sdepth}(S/(I(AWG_n))) \geq \min\{\text{sdepth}(S/(I(AWG_n) : x_n)), \text{sdepth}(S/(I(AWG_n), x_n))\}.$$

So we get $\text{sdepth}(S/(I(AWG_n))) \geq \text{sdepth}(S/(I(AWG_n), x_n))$.

Hence $\text{sdepth}(S/(I(AWG_n))) \geq \lceil \frac{n-3}{5} \rceil$. $\square$

Theorem 5. Let $n \geq 9$ and is an odd number, then $\text{sdepth}(S/(I(\text{AWG}_n))) = \lceil \frac{n}{5} \rceil$, when $n \equiv 0, 4 \pmod{5}$.
 For $n \equiv 1 \pmod{5}$, the $\text{sdepth}(S/(I(\text{AWG}_n))) = \lceil \frac{n}{5} \rceil - 1$.
 And when $n \equiv 2, 3 \pmod{5}$, then $\lceil \frac{n}{5} \rceil - 1 \leq \text{sdepth}(S/(I(\text{AWG}_n))) \leq \lceil \frac{n}{5} \rceil$.

Proof. We will prove this result by following cases.

Case 1 : When $n = 5q$, where $q \in \{3, 5, 7, \dots\}$. Then by Theorem 4 we can see $\text{sdepth}(S/(I(\text{AWG}_n))) \geq \lceil \frac{n}{5} \rceil$.

Now we have to show the other inequality.

Since $w = x_1x_6x_{11}\dots x_{(5(q-1)+1)} \in S/(I(\text{AWG}_n))$, but $x_iw \in I(\text{AWG}_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(\text{AWG}_n))) \leq q = \lceil \frac{n}{5} \rceil$.

Case 2 : When $n = 5q + 1$, where $q \in \{2, 4, 6, \dots\}$. Then by Theorem 4 we can see $\text{sdepth}(S/(I(\text{AWG}_n))) \geq \lceil \frac{n}{5} \rceil - 1$.

Now to show the other inequality.

Since $w = x_1x_6x_{11}\dots x_{(5(q-1)+1)} \in S/(I(\text{AWG}_n))$, but $x_iw \in I(\text{AWG}_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(\text{AWG}_n))) \leq q = \lceil \frac{n}{5} \rceil - 1$.

Case 3 : When $n = 5q + 2$, where $q \in \{3, 5, 7, \dots\}$. Then by Theorem 4 we can see $\text{sdepth}(S/(I(\text{AWG}_n))) \geq \lceil \frac{n}{5} \rceil - 1$.

Now we have to show the other inequality.

Since $w = x_1x_6x_{11}\dots x_{(5(q-1)+1)}x_{(5(q-1)+2)} \in S/(I(\text{AWG}_n))$, but $x_iw \in I(\text{AWG}_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(\text{AWG}_n))) \leq q + 1 = \lceil \frac{n}{5} \rceil$.

Case 4 : When $n = 5q + 3$, where $q \in \{2, 4, 6, \dots\}$. Then by Theorem 4 we can see $\text{sdepth}(S/(I(\text{AWG}_n))) \geq \lceil \frac{n}{5} \rceil - 1$.

Now we have to show the other inequality. Since $w = x_1x_6x_{11}\dots x_{(5(q-1)+1)}x_{5q} \in S/(I(\text{AWG}_n))$, but $x_iw \in I(\text{AWG}_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(\text{AWG}_n))) \leq q + 1 = \lceil \frac{n}{5} \rceil$.

Case 5 : When $n = 5q + 4$, where $q \in \{1, 3, 5, \dots\}$. Then by Theorem 4 we can see $\text{sdepth}(S/(I(\text{AWG}_n))) \geq \lceil \frac{n}{5} \rceil$.

Now we have to show the other inequality.

Since $w = x_1x_6x_{11}\dots x_{(5(q-1)+1)}x_{(5q)+1} \in S/(I(\text{AWG}_n))$, but $x_iw \in I(\text{AWG}_n)$, for all $i \in [n] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(\text{AWG}_n))) \leq q + 1 = \lceil \frac{n}{5} \rceil$. $\square$

4. Examples

Now we give some examples of our main results.

Example 1. Let $I(\mathcal{W}_7^2)$ be an edge ideal of square wheel graph as shown in Figure 3, then we have to show that $\text{sdepth}(S/I(\mathcal{W}_7^2)) = 1$. The edge ideal of the square wheel graph is given as

$$I(\mathcal{W}_7^2) = (x_1x_7, x_2x_7, x_3x_7, \dots, x_6x_7, x_1x_2, x_2x_3, x_3x_4, \dots, x_6x_1, \\ x_1x_3, x_2x_4, x_3x_5, \dots, x_6x_2).$$

Now consider a short exact sequence

$$0 \rightarrow S/(I(\mathcal{W}_7^2) : x_7) \rightarrow S/(I(\mathcal{W}_7^2)) \rightarrow S/(I(\mathcal{W}_7^2), x_7) \rightarrow 0$$

by Lemma 1

$$\text{sdepth}(S/(I(\mathcal{W}_7^2))) \geq \min\{\text{sdepth}(S/(I(\mathcal{W}_7^2) : x_7)), \text{sdepth}(S/(I(\mathcal{W}_7^2), x_7))\}.$$

Here

$$(I(\mathcal{W}_7^2) : x_7) = (x_1, x_2, x_3, x_4, \dots, x_6).$$

So $(S/(I(\mathcal{W}_7) : x_7)) \cong K[x_7]$.

Hence $\text{sdepth}(S/(I(\mathcal{W}_7^2) : x_7)) = 1$.

And $(I(\mathcal{W}_7^2), x_7) = (x_1x_2, x_2x_3, x_3x_4, \dots, x_6x_1, x_1x_3, x_2x_4, x_3x_5, \dots, x_6x_2, x_7)$

where $S/(I(\mathcal{W}_7^2), x_7) \cong S_6/I(C_6^2)$

and $\text{sdepth}(S/(I(\mathcal{W}_7^2), x_7)) = \text{sdepth}(S_6/I(C_6^2)) \geq \lceil \frac{6-2}{5} \rceil$ by [12].

Here $\lceil \frac{6-2}{5} \rceil \geq 1$.

So we have $\text{sdepth}(S/(I(\mathcal{W}_7^2) : x_7)) = 1 \leq \text{sdepth}(S/(I(\mathcal{W}_7^2), x_7))$.

This implies that $\text{sdepth}(S/(I(\mathcal{W}_7^2))) \geq \text{sdepth}(S/(I(\mathcal{W}_7^2) : x_7)) = 1$.

We get the other inequality from [14] (Proposition 2.7).

Hence,

$$\text{sdepth}(S/(I(\mathcal{W}_7^2))) = \text{sdepth}(S/(I(\mathcal{W}_7^2) : x_7)) = 1.$$

Example 2. Let $I(G_{11})$ is the gear graph on 11 vertices.

Then $3 \leq \text{sdepth}(S/I(G_{11})) \leq 4$.

The graph of $I(G_{11})$ is given in Figure 4

We have

$$I(G_{11}) = (x_1x_2, x_2x_3, x_3x_4, \dots, x_9x_{10}, x_{10}x_1, x_1x_{11}, x_3x_{11}, x_5x_{11}, \dots, x_7x_{11}).$$

Consider the following short exact sequence

$$0 \rightarrow S/(I(G_{11}) : x_{11}) \rightarrow S/I(G_{11}) \rightarrow S/(I(G_{11}), x_{11}) \rightarrow 0.$$

By Lemma 1

$$\text{sdepth}(S/(I(G_{11}))) \geq \min\{\text{sdepth}(S/(I(G_{11}) : x_{11})), \text{sdepth}(S/(I(G_{11}), x_{11}))\}.$$

Here

$$(I(G_{11}), x_{11}) = (x_1x_2, x_2x_3, x_3x_4, \dots, x_9x_{10}, x_{10}x_1, x_{11}).$$

So $S/(I(G_{11}), x_{11}) \cong S_{10}/I(C_{10})$. Hence by Proposition 1.8 of [11] $\text{sdepth}(S/(I(G_{11}), x_{11})) \geq \lceil \frac{9}{3} \rceil = 3$.

And

$$(I(G_{11}) : x_{11}) = (x_1, x_3, x_5, \dots, x_9).$$

So we have $S/(I(G_{11}) : x_{11}) \cong K[x_2, x_4, x_6, \dots, x_{10}, x_{11}]$.

Hence $\text{sdepth}(S/(I(G_{11}) : x_{11})) \geq \lceil \frac{11}{2} \rceil$.

So we get $\text{sdepth}(S/(I(G_{11}))) \geq \text{sdepth}(S/(I(G_{11}), x_{11}))$,

$\text{sdepth}(S/(I(G_{11}))) \geq 3$.

Now we have to show the other inequality.

Since $w = x_1x_4x_7x_9 \in S/(I(G_{11}))$, but $x_iw \in I(G_{11})$, for all $i \in [9] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(G_{11}))) \leq 4$.

Example 3. Let $I(\text{AWG}_9)$ is the anti-web gear graph on 9 vertices, then $\text{sdepth}(S/I(\text{AWG}_9)) = 2$.

The graph of $I(\text{AWG}_9)$ is given in fig 2.3.

We have

$$I(\text{AWG}_9) = (x_1x_2, x_2x_3, x_3x_4, \dots, x_7x_8, x_8x_1, x_1x_3, x_2x_4, \dots, x_7x_1, x_8x_2, \\ x_1x_9, x_3x_9, \dots, x_9x_9).$$

Consider the following short exact sequence

$$0 \rightarrow S/(I(AWG_9) : x_9) \rightarrow S/I(AWG_9) \rightarrow S/(I(AWG_9), x_9) \rightarrow 0.$$

Here

$$(I(AWG_9) : x_9) = (x_1, x_3, x_5, \dots, x_7).$$

So $(S/(I(AWG_9) : x_9)) \cong K[x_2, x_4, x_6, \dots, x_8, x_9]$.

We get $\text{sdepth}(S/(I(AWG_9) : x_9)) \geq \lceil \frac{9}{2} \rceil = 5$.

And

$$(I(AWG_{11}), x_9) = (x_1x_2, x_2x_3, x_3x_4, \dots, x_7x_8, x_8x_1, x_1x_3, \\ x_2x_4, \dots, x_7x_1, x_8x_2, x_9).$$

So we have $(S/(I(AWG_9), x_9)) \cong S_8/I(C_8^2)$, where $I(C_8^2)$ is a square cycle on 8 vertices. By using (Ishaq et al., 2017) we get $\text{sdepth}(S/(I(AWG_9), x_9)) \geq \lceil \frac{6}{5} \rceil = 2$.

By applying Lemma 1 we have

$$\text{sdepth}(S/(I(AWG_9))) \geq \min\{\text{sdepth}(S/(I(AWG_9) : x_9)), \text{sdepth}(S/(I(AWG_9), x_9))\}.$$

So we get $\text{sdepth}(S/(I(AWG_9))) \geq \text{sdepth}(S/(I(AWG_9), x_9))$.

Hence $\text{sdepth}(S/(I(AWG_9))) \geq \lceil \frac{6}{5} \rceil = 2$.

Now to show the other inequality.

Since $w = x_1x_6 \in S/(I(AWG_9))$, but $x_iw \in I(AWG_9)$, for all $i \in [9] \setminus \text{supp}(w)$, therefore by Lemma 2, $\text{sdepth}(S/(I(AWG_9))) \leq 2$.

5. Conclusions and Open Problems

In this article, we have computed the Stanley depth of the quotient of edge ideals associated with some familiar families of wheel-related graphs. In particular we establish general closed formulas for Stanley depth of quotient of edge ideals associated with m th-power of wheel graph, for $m \geq 3$, gear graphs and anti-web gear graphs and arrive at the following closed formulas.

In the following theorem, we give exact value of Stanley depth of m th – power of wheel graph.

Theorem 6. Let $n \geq 6$, and $I(\mathcal{W}_n^m)$ be an ideal of wheel graph on n vertices, then $\text{sdepth}(S/I(\mathcal{W}_n^m)) = 1$.

Next we give the lower bound for the sdepth of anti-web gear graph having odd numbers of vertices.

Theorem 7. Let $n \geq 9$ and is an odd number, then $\text{sdepth}(S/I(AWG_n)) \geq \lceil \frac{n-3}{5} \rceil$.

In the following theorem, we give the lower and upper bounds for sdepth of gear graph for particular cases.

Theorem 8. Let $n \geq 5$, then $\text{sdepth}(S/(I(G_n))) = \lceil \frac{n}{3} \rceil$, for $n \equiv 0 \pmod{3}$.

The $\text{sdepth}(S/(I(G_n))) = \lceil \frac{n}{3} \rceil - 1$, for $n \equiv 1 \pmod{3}$.

And when $n \equiv 2 \pmod{3}$, then $\lceil \frac{n-2}{3} \rceil \leq \text{sdepth}(S/(I(G_n))) \leq \lceil \frac{n}{3} \rceil$.

Now in next theorem we give lower bound for odd number of vertices in gear graph.

Theorem 9. Let $n \geq 5$ and is an odd number, then $\text{sdepth}(S/I(G_n)) \geq \lceil \frac{n-2}{3} \rceil$.

Following result describes sharp tight bounds for the sdepth of Anti-web gear graph having odd number of vertices.

Theorem 10. Let $n \geq 9$ and is an odd number, then $\text{sdepth}(S/(I(\text{AWG}_n))) = \lceil \frac{n}{5} \rceil$, when $n \equiv 0, 4 \pmod{5}$.
 For $n \equiv 1 \pmod{5}$, the $\text{sdepth}(S/(I(\text{AWG}_n))) = \lceil \frac{n}{5} \rceil - 1$.
 And when $n \equiv 2, 3 \pmod{5}$, then $\lceil \frac{n}{5} \rceil - 1 \leq \text{sdepth}(S/(I(\text{AWG}_n))) \leq \lceil \frac{n}{5} \rceil$.

At the same time, we pose natural open problems about the exact values of Stanley depth of the quotient of edge ideal associated with m th- power of gear and anti-web gear graphs.

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