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Consensus-Based Multi-Person Decision Making with Incomplete Fuzzy Preference Relations Using Product Transitivity

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Abstract: In this paper, a consensus-based method for multi-person decision making (MPDM) using product transitivity with incomplete fuzzy preference relations (IFPRs) is proposed. Additionally, an average aggregation operator has been used at the first level to estimate the missing preference values and construct the complete fuzzy preference relation (FPR). Then it is confirmed to be product consistent by using the transitive closure formula. Following this, weights of decision makers (DMs) are evaluated by merging consistency weights and predefined priority weights (if any). The consistency weights for the DMs are estimated through product consistency investigation of the information provided by each DM. The consensus process determines whether the selection procedure should be initiated or not. The hybrid comprises of a quitting process and feedback mechanism, and is used to enhance the consensus level amongst DMs in case of an inadequate state. The quitting process arises when some DMs decided to leave the course, and is common in MPDM while dealing with a large number of alternatives. The feedback mechanism is the main novelty of the proposed technique which helps the DMs to improve their given preferences based on this consistency. At the end, a numerical example is deliberated to measure the efficiency and applicability of the proposed method after the comparison with some existing models under the same assumptions. The results show that proposed method can offer useful comprehension into the MPDM process.

Keywords: fuzzy set; multi-person decision making (MPDM); fuzzy preference relation (FPR); incomplete fuzzy preference relation (IFPR); product transitivity

1. Introduction

Decision making is an investigative and rational procedure which is used to select a course of action amongst various alternative situations. Every decision making procedure yields a final choice (option) and outputs a judgment of choice by means of some scientifically-based techniques. During daily life, everyone has to face decision making situations: commonly known situations are shopping, to decide what to eat and what to vote for in an election etc. In our context, the selection of the best alternative(s) from a fixed set of possible alternatives is the objective.

Decision making is not only the situation for an individual where he/she gives a pairwise comparison of alternatives, but some decision making situations have to be explained by a group of decision makers (DMs) who work together to evaluate the best alternative(s) from a set of feasible alternatives. The procedure to solve decision making problem(s) with multiple DMs is called group decision making (GDM), also known as multi-person decision making (MPDM). Preference relation is the most common representation format used in MPDM because it is a valuable tool in modeling decision processes, when we have to combine DMs' preferences into group preferences [1].

In fuzzy framework, a DM assigns a numerical value fitting in [0, 1] to each pair of alternatives which reflects the preference degree of one alternative over the other. A first and natural question straightway arises while allocating the values: which conditions have to be satisfied in order to achieve consistent results in final ranking?

Moreover, consistency is an important issue to face while informations are provided by the DMs, and is associated with the transitive property. Numerous procedures on consistency measure and enhancement of preference values have been offered in a successive way. In 1999, Xu and Wei [2] suggested an algorithm in order to enhance the consistency of preference opinions under multiplicative transitivity to acceptable level. In 2006, Ma et al. [3] proposed a scheme to find the weak transitivity and inconsistency of a preference relation and put forward a method to increase consistency-based on additive transitivity. In 2007, Herrera–Viedma et al. [4] acquainted with a description of the consistency measure using the additive transitivity property and suggested a technique to construct a consistent preference format. In 2008, Dong et al. [5] defined the consistency index of linguistic preference opinions and proposed a consistency-based method to handle linguistic preference relations. In 2011, Ergu et al. [6] anticipated a technique to measure the consistency level for multiplicative preference relations. In 2012, Siraj et al. [7] presented an algorithm-based method to enhance ordinal consistency after the identification and elimination of intransitivity of multiplicative preference relations. In 2014, Liu et al. [8] defined the consistent triangular fuzzy reciprocal preference relations and deliberated numerous properties of consistency for triangular fuzzy reciprocal preference relations. In 2014, Wu and Chiclana [9] provided a novel consensus model for GDM problems carrying with incomplete intuitionistic reciprocal preference relations and applied to estimate unknown preferences using multiplicative consistency. In 2015, Xia and Chen [10] defined the consistency index by constructing the nearest consistent preference matrix on an abelian linearly ordered group from an inconsistent one, together with two consistency improving methods. In 2016, Marasini et al. [11] set forth an intuitionistic fuzzy sets (IFS) approach to the problem of students' satisfaction of university teaching and potential advantages of the IFS perspective with respect to other non-fuzzy approaches are provided. They also applied IFS to questionnaire analysis, with a focus on the construction of membership, non-membership and uncertainty functions [12]. In 2018, Kerre et al. [13] presented a multiplicative consistency-based GDM method with reciprocal fuzzy preference relations in an incomplete environment.

To handle a MPDM problem appropriately, two main processes play a crucial role:

(i) consensus process,
(ii) selection process.

The first process is an iterative process that comprises of numerous consensus rounds, where the DMs admit to negotiate diverse opinions to have an acceptable level, but an undisputed or full consensus is often not achievable in practice [14]. After the attainment of DMs' opinions is close enough, the selection process initiates to rank and select suitable alternative(s) from a given set of feasible alternatives. In order to help the DMs to reach an acceptable consensus level in MPDM, many consensus models with different forms have been proposed in literature. In 2002, Herrera–Viedma et al. [15] proposed a consensus-based method for GDM problems in different preference formats, utility values, and multiplicative preference relations. In 2013, Xia et al. [16] examined the multiplicative transitivity-based consensus of reciprocal preference matrices and presented an algorithm to improve consensus level for given preferences. In 2013, Palomares et al. [17] proposed a consensus-based model to incorporate group's approach towards consensus by means of an extension of OWA aggregation operators. To provide a general framework for existing methods, in 2015, Xia and Chen [10] defined a consensus index of individual pairwise comparison matrices and developed two consensus improving methods by introducing a general aggregation operator based on an abelian linearly ordered group. In 2016, Zhang et al. [18] developed a consensus building method based on multiplicative consistency for GDM with incomplete reciprocal preference relations (IRPRs).

In a fuzzy preference context to handle MPDM problems, preference relations are always supposed to be complete, but there may arise some situations in which DMs express their preferences in incomplete fashion due to time constraint, lack of knowledge and limited expertise regarding the problem etc. The incomplete fuzzy preference relations (IFPRs) have been widely used in MPDM problems, in literature, various measures have been taken to determine unknown preference values. Such as, in 2008, a least squared technique was presented by Gong [19] to evaluate the priority vector for GDM using incomplete preference relations. In 2013, a logarithmic least squares method was proposed by Xu et al. to estimate the priority weights in GDM dealing with IFPRs and develop the acceptable fuzzy consistency ratio [20]. In 2015, Xu et al. [21] presented a least deviation method to detremine the priority weights for GDM in IRPRs environment. In 2015, the trust-based consensus model and aggregation method for GDM were investigated by Wu et al. [22] in the context of incomplete linguistic information.

In this paper, we put forward a consensus-based hybrid technique for MPDM after getting motivation from the work of Zhang et al. [18]. They used multiplicative transitivity property, i.e., $\frac{r_{ik}}{r_{ki}} = \frac{r_{ij}r_{jk}}{r_{ji}r_{kj}}$, $j \neq i \neq k \in \{1,2,3,...,n\}$, to estimate the missing preferences and make the information consistent accordingly where r_{ik} represents the preference degree of alternative i over alternative j. We believe that the multiplicative transitivity condition is very hard to satisfy while decimal numbers are being used and an alternative is not allowed to be fully preferred over another, i.e., $r_{ik} \in (0,1)$ rather than $r_{ik} \in [0,1]$. We also observed that the aggregated preference relation constructed in [18]

$$P = \begin{bmatrix} 0.50 & 0.43 & 0.58 & 0.74 \\ 0.57 & 0.50 & 0.75 & 0.83 \\ 0.42 & 0.25 & 0.50 & 0.74 \\ 0.26 & 0.17 & 0.26 & 0.50 \end{bmatrix}$$

carries inconsistent data-based on multiplicative transitivity which definitely have impact on the final ranking of alternatives. For instance, $i = 1$ and $k = 3$ implies that $\frac{r_{i3}}{r_{3i}} = \frac{r_{1j}r_{j3}}{r_{j1}r_{3j}}$ must be satisfied for intermediate values of j, i.e., $j = 2,4$. But from P we get: $\frac{r_{i3}}{r_{3i}} = 1.380952381$ and $\frac{r_{12}r_{23}}{r_{21}r_{32}} = 2.263157895$, $\frac{r_{14}r_{43}}{r_{41}r_{34}} = 1$, which results in inconsistent matrix. These are some issues raised in [18]. In our proposed method, we modify the technique given by Zhang et al. using product transitivity to overcome the above said drawbacks. Hence, this article presents a new approach to handle MPDM problems based on consistency and consensus analysis while incomplete information is provided. At the first stage, estimate the unknown preferences of IFPRs based on the product transitivity. Then, we construct the modified consistent fuzzy preference relations (FPRs) against the DMs which satisfy the product consistency and measure the level of consistency accordingly. The degrees of importance are assigned to DMs based on consistency weights aggregated with predefined weights (if they exist), otherwise, the consistency weights are used as weights of DMs. The proposed method provides us with a valuable way for consensus building in GDM based on product consistency with incomplete preference relations.

The rest of the paper is organized as: in Section 2, some preliminaries are given to support this paper. In Section 3, a procedure is demonstrated to estimate the missing values in IFPRs based on the product transitivity and construct the modified consistent matrices accordingly. In Section 4, the proposed GDM process is detailed. In Section 5, an example is given to illustrate the realism and achievability of the proposed technique. At the end, some comparison and conclusions are given.

2. Preliminaries

In 1965, Zadeh introduced fuzzy set theory [23], designated with a number between 0 and 1, to cope with imprecise and uncertain information working in complex situation.

Definition 1 ([23])**.** *A fuzzy set A on the universe of discourse X is a mapping from X to $[0,1]$, and is denoted by $A = \{(x, A(x))\}$. For any $x \in X$, the value $A(x)$ is called the degree of membership of x in A, i.e., $A(x)$ =Degree$(x \in A)$, and the map $A : X \to [0,1]$ is called a membership function.*

Definition 2 ([1])**.** *An FPR R on a set X of alternatives $X = \{x_1, x_2, ..., x_n\}$ is characterized by a membership function $R(x_i, x_k) = r_{ik}$, satisfying $r_{ik} + r_{ki} = 1$ for all $i, k \in \{1, 2, 3, .., n\}$, and is conveniently denoted by matrix*

$$R = (r_{ik})_{n\times n} = \begin{bmatrix} 0.5 & r_{12} & . & . & r_{1n} \\ r_{21} & 0.5 & . & . & r_{2n} \\ . & . & . & & . \\ . & . & & . & . \\ r_{n1} & r_{n2} & . & . & 0.5 \end{bmatrix}, \tag{1}$$

where r_{ik} shows the degree of preference of alternative x_i over alternative x_k, and all $r_{ii} = 0.5$. If $r_{ik} = 0.5$, this indicates that there is no difference between the two alternatives; if $r_{ik} > 0.5$, it implies that alternative x_i is superior to alternative x_k.

Definition 3 ([4])**.** *An IFPR $R = (r_{ik})_{n\times n}$ carries at least one unknown preference value r_{ik} for which the expert does not have a clear idea of the degree of preference of alternative x_i over the alternative x_k.*

Definition 4. *An FPR $R = (r_{ik})_{n\times n}$ on a finite set X of alternatives is said to be product transitive if*

$$r_{ik} \geq r_{ij}.r_{jk} \tag{2}$$

holds for all intermediate alternatives x_j with $j \neq i, k$. The product transitivity assures the product consistency of FPR.

3. Repairing IFPR

In this section, a new procedure to evaluate the missing preference degrees based on product transitivity has been put forward. Likewise, the proposed technique is used to construct a product consistent FPR. In order to evaluate unknown preference degrees in an IFPR $R = (r_{ik})_{n\times n}$, the pairs of alternatives for known and unknown preference values are signified in form of following sets:

$$K_p = \{(i,k) | r_{ik} \text{ is known}\}, \tag{3}$$

$$U_p = \{(i,k) | r_{ik} \text{ is unknown}\}, \tag{4}$$

where K_p represents the set of pairs of alternatives with known preference values while U_p denotes the set of pairs of alternatives with unknown preference values. The preference value of alternative x_i over x_k belongs to $[0,1]$ (i.e., $r_{ik} \in [0,1]$). We can define following set of intermediate alternative x_j which can be used to determine the unknown preference value r_{ik} of alternative x_i over alternative x_k

$$W_{ik} = \{j \neq i, k \mid (i,j) \in K_p, (j,k) \in K_p \text{ and } (i,k) \in U_p\}, \tag{5}$$

for $1 \leq i \leq n$, $1 \leq j \leq n$ and $1 \leq k \leq n$. Based on (5), the aggregated value (global value) of r_{ik} is obtained by using the *average* aggregation operator, and is the degree of preference of alternative x_i over the alternative x_k, given as

$$r_{ik} = \begin{cases} \underset{j \in W_{ik}}{Ave}(r_{ij}.r_{jk}) & \text{if } |W_{ik}| \neq 0 \\ 0.5 & \text{otherwise} \end{cases}, \tag{6}$$

where $|W_{ik}|$ is the cardinality of the set W_{ik}. Aggregation is for purposed so as to use different pieces of information simultaneously in order to reach a conclusion. The value r_{ki} can be evaluated by using

$$r_{ki} = 1 - r_{ik}, \tag{7}$$

after having the value of r_{ik}. New sets of the pairs of alternatives for known and unknown preference values are determined by

$$K'_p = K_p \cup \{(i,k)\}, \tag{8}$$

$$U'_p = U_p - \{(i,k)\}. \tag{9}$$

After having a complete FPR, it needs to be a fully product consistent FPR $\widetilde{R} = (\widetilde{r}_{ik})_{n\times n}$ which can be obtained by calculating $\widetilde{r}_{ik}$ from

$$\widetilde{r}_{ik} = \max_{j\neq i,k}(r_{ik},\ r_{ij}.r_{jk}), \tag{10}$$

such that $\widetilde{R}$ is stable with $\widetilde{r}_{ik} + \widetilde{r}_{ki} = 1$. It remains working until (2) is satisfied.

4. Iterative Procedure for GDM

This section comprises of a step-by-step procedure for GDM based on product consistency. An explanatory example is given to validate the method. Suppose that there are n alternatives $x_1, x_2, ..., x_n$ and m DMs $D_1, D_2, ..., D_m$. Let R^q be the fuzzy preference relation for the expert D_q shown as follows:

$$R^q = \left(r^q_{ik}\right)_{n\times n} = \begin{bmatrix} 0.5 & r^q_{12} & . & . & r^q_{1n} \\ r^q_{21} & 0.5 & . & . & r^q_{2n} \\ . & . & . & & . \\ . & . & & . & . \\ r^q_{n1} & r^q_{n2} & . & . & 0.5 \end{bmatrix},$$

where $r^q_{ik} \in [0,1]$ is the preference value given by DM D_q for alternative x_i over x_k, $r^q_{ik} + r^q_{ki} = 1$, $1 \le i \le n, 1 \le k \le n$ and $1 \le q \le m$ and $r^q_{ii} = 0.5$, for all $i \in \{1,2,...,n\}$ as an alternative cannot be preferred on itself. Due to time constraints or lack of information and complexity, some of the preference values are missing while data is provided by the DMs. The proposed GDM method consists of several states as follows.

4.1. Repairing IFPRs

Initially, to estimate the unknown preference values in IFPR R^q provided by the DM D_q, the sets K^q_p and U^q_p of pairs of alternatives for known and unknown preferences are introduced as in (3) and (4) respectively. After this, the missing preference values are estimated by using (5)–(9) to construct the complete FPR R^q.

4.2. Consistency Analysis

The product consistent FPRs $\widetilde{R}^q$, for $q = 1,2,3,...,m$, can be constructed with the use of (10) after evaluating the missing preference degrees. We can then approximate the level of consistency of an FPR R^q based on its similarity with the corresponding product transitivity-based $\widetilde{R}^q$ by computing their distances [18].

1. Product consistency index (PCI) of a pair of alternatives is estimated by using

$$PCI(r^q_{ik}) = 1 - \left|r^q_{ik} - \widetilde{r}^q_{ik}\right|. \tag{11}$$

Apparently, the higher the value of $PCI(r_{ik}^q)$, the more consistent r_{ik}^q is with respect to the rest of the preference values involving alternatives x_i and x_k.

2. PCI associated to a particular alternative x_i, $1 \leq i \leq n$, of an FPR is estimated as

$$PCI(x_i) = \frac{1}{(n-1)} \sum_{k=1 \neq i}^{n} PCI(r_{ik}^q), \quad (12)$$

with $PCI(x_i) \in [0,1]$. When $PCI(x_i) = 1$ all the preferences involving the alternative x_i are fully consistent, otherwise, the lower $PCI(x_i)$ is, the more inconsistent these preference values are.

3. Finally, we evaluate PCI of an FPR R^q by taking the average of all PCI of alternatives x_i as

$$PCI(R^q) = \frac{1}{n} \sum_{i=1}^{n} PCI(x_i), \quad (13)$$

and $PCI(R^q) \in [0,1]$. When $PCI(R^q) = 1$ the preference relation R^q is fully consistent, otherwise, the lower $PCI(R^q)$ the more inconsistent R^q is.

As soon as the PCI is computed in three levels using expressions (11)–(13), it is rational to assign higher weights to the DMs against the preference relations with larger consistency degrees respectively. Hence, consistency weights can be assigned to the experts by using the relation

$$Cw(D_q) = \frac{PCI(R^q)}{\sum_{q=1}^{m} PCI(R^q)}. \quad (14)$$

4.3. Assigning Weights to Experts

Final priority weights are allocated to DMs by emerging respective predefined priority weights and consistency weights by using

$$w(D_q) = \frac{\lambda_q \times Cw(D_q)}{\sum_{q=1}^{m} \lambda_q \times Cw(D_q)}, \quad (15)$$

where $\lambda_q, 1 \leq q \leq m$, are the predefined priority weights of DMs and $\sum_{q=1}^{m} w(D_q) = 1$. If the priority weight vector is not given, then consistency weights will be taken as the priority weights of DMs.

4.4. Consensus Measures

After getting FPRs in complete form, we measure the consensus degree amongst DMs. In this manner, similarity matrices $S^{qr} = (s_{ik}^{qr})_{n \times n}$ for every pair of DMs (D_q, D_r) $(q = 1, 2, ..., m-1;$ $r = q+1, ..., m)$ are to be determined and defined as

$$s_{ik}^{qr} = 1 - \left| r_{ik}^q - r_{ik}^r \right|. \quad (16)$$

Then the collective similarity matrix $S = (s_{ik})_{n \times n}$ is constructed after aggregating all the similarity matrices by using following relation

$$s_{ik} = \frac{2}{m(m-1)} \sum_{q=1}^{m-1} \sum_{r=q+1}^{m} s_{ik}^{qr}. \quad (17)$$

To compute the degree of consensus amongst the DMs, the following three states are to be faced [18].

1. At first level, the consensus degree amongst DMs on a pair of alternative (x_i, x_k), denoted by cod_{ik}, is measured as
$$cod_{ik} = s_{ik}. \tag{18}$$
2. At second level, the consensus degree amongst DMs on alternative x_i, denoted by CoD_i, is estimated using
$$CoD_i = \frac{1}{(n-1)} \sum_{k=1, k \neq i}^{n} s_{ik}. \tag{19}$$
3. At third level, the consensus degree amongst DMs on the relation, denoted by CoR, is determined as
$$CoR = \frac{1}{n} \sum_{i=1}^{n} CoD_i. \tag{20}$$

Once the global consensus level among all the experts is reached, it requires us to compare it with a threshold consensus degree η (say), generally settled in advance depending upon the nature of problem. If $CoR \geq \eta$, this shows that an acceptable level of consensus has been obtained, and the decision process begins. Otherwise, the consensus degree is not stable, and the feedback mechanism along with the quitting process originates.

4.5. Quitting Process

This process is concerned with the DMs when some of them no longer wish to participate in GDM and decide to leave the group. Usually, this situation arises when there is a large number of alternatives where the DMs can vary over time. Once a DM $D_k, k = 1, 2, ..., m$, decides to quit the process, the priority weights of the remaining DMs will be affected and updated as
$$w_{new}(D_q) = w(D_q) + \frac{w(D_k)}{l}, \tag{21}$$
where l represents the remaining number of DMs participating in decision making.

4.6. Feedback Mechanism

The central aim of the feedback mechanism is to provide comprehensive knowledge to experts, so as to change their opinions acceptably to enhance the consensus degree. When consensus is not sufficiently high, then we have to identify the preference values that are to be changed, and the following formula helps us in this regard:
$$R^q = \{(i,k) \mid cod_{ik} < CoR \text{ and } r^q_{ik} \text{ is known,}\} \tag{22}$$
for $i, k \in \{1, ..., n\}$. The system recommends that the corresponding expert has to increase in value if it is smaller than the mean value of the valuations of the rest of experts, or decrease it if it is greater than the mean [18].

4.7. Accumulation Phase

It may quite frequently happen that the preference degree set forth by each DM is weighted differently. As soon as the weights for the DMs are estimated, their preferences need to be accumulated into a global one. We determine the collective matrix R^c against all DMs using weighted average formula
$$R^c = (r^c_{ik})_{n \times n} = \left(\sum_{q=1}^{m} w(D_q) \times \tilde{r}^q_{ik} \right)_{n \times n}, \tag{23}$$
where $1 \leq i \leq n, 1 \leq k \leq n$.

4.8. Selection Phase

Once a satisfactory consensus level amongst all DMs is reached, the selection process starts in order to rank and select the best alternative. For a consistent FPR $\widetilde{R} = (\widetilde{r}_{ik})_{n\times n}$, the ranking value $Rv(x_i)$ of alternative x_i, $i = 1, .., n$, is defined by:

$$Rv(x_i) = \frac{2}{n(n-1)} \sum_{k=1,k\neq i}^{n} \widetilde{r}_{ik},\ i = 1, .., n \tag{24}$$

with $\sum_{i=1}^{n} Rv(x_i) = 1$.

5. Numerical Example

This section deals with a numerical example taken from [18] in order to demonstrate the process of the proposed method and its effectiveness. Consider that four DMs D_1, D_2, D_3 and D_4 from different fields are requested to select the best alternative out of four alternatives x_1, x_2, x_3, x_4. The four DMs provide their FPRs as follows:

$$R^1 = \begin{bmatrix} 0.5 & 0.6 & r^1_{13} & r^1_{14} \\ 0.4 & 0.5 & 0.7 & r^1_{24} \\ r^1_{31} & 0.3 & 0.5 & 0.9 \\ r^1_{41} & r^1_{42} & 0.1 & 0.5 \end{bmatrix},\ R^2 = \begin{bmatrix} 0.5 & 0.6 & 0.7 & r^2_{14} \\ 0.4 & 0.5 & r^2_{23} & 0.7 \\ 0.3 & r^2_{32} & 0.5 & r^2_{34} \\ r^2_{41} & 0.3 & r^2_{43} & 0.5 \end{bmatrix},$$

$$R^3 = \begin{bmatrix} 0.5 & 0.3 & 0.5 & 0.75 \\ 0.7 & 0.5 & 0.8 & 0.6 \\ 0.5 & 0.2 & 0.5 & 0.8 \\ 0.25 & 0.4 & 0.2 & 0.5 \end{bmatrix},\ R^4 = \begin{bmatrix} 0.5 & 0.4 & 0.55 & 0.65 \\ 0.6 & 0.5 & 0.8 & 0.75 \\ 0.45 & 0.2 & 0.5 & 0.7 \\ 0.35 & 0.25 & 0.3 & 0.5 \end{bmatrix}.$$

The threshold consensus level η settled in advance is 0.80. Now, we perform the following steps to evaluate the result.

Step-i: Repairing IFPRs

Initially, all the missing preference values need to be determined using (6). For instance, taking R^1, the sets of pairs of alternatives for known and unknown preference values are determined as follows:

$$\begin{aligned} K^1_v &= \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}, \\ U^1_v &= \{(1,3), (3,1), (1,4), (4,1), (2,4), (4,2)\}. \end{aligned}$$

Using (5)–(9), we get the complete FPR R^1 against DM D_1 as:

$$R^1 = \begin{bmatrix} 0.5000 & 0.6000 & 0.4200 & 0.3780 \\ 0.4000 & 0.5000 & 0.7000 & 0.3906 \\ 0.5800 & 0.3000 & 0.5000 & 0.9000 \\ 0.6220 & 0.6094 & 0.1000 & 0.5000 \end{bmatrix}.$$

Similarly, the complete form of R^2 can be obtained and given as

$$R^2 = \begin{bmatrix} 0.5000 & 0.6000 & 0.7000 & 0.4200 \\ 0.4000 & 0.5000 & 0.2800 & 0.7000 \\ 0.3000 & 0.7200 & 0.5000 & 0.3150 \\ 0.5800 & 0.3000 & 0.6850 & 0.5000 \end{bmatrix}.$$

Step-ii: Consistency analysis

Consistency analysis is initiated to assign consistency weights to the DMs. In this manner, all complete FPRs are to be transformed into their product consistent forms using (10) and are given as

$$\widetilde{R}^1 = \begin{bmatrix} 0.5000 & 0.5940 & 0.4200 & 0.3780 \\ 0.4060 & 0.5000 & 0.6555 & 0.6300 \\ 0.5800 & 0.3445 & 0.5000 & 0.7388 \\ 0.6220 & 0.3700 & 0.2612 & 0.5000 \end{bmatrix},\ \widetilde{R}^2 = \begin{bmatrix} 0.5000 & 0.5940 & 0.7000 & 0.4200 \\ 0.4060 & 0.5000 & 0.4795 & 0.6555 \\ 0.3000 & 0.5205 & 0.5000 & 0.3643 \\ 0.5800 & 0.3445 & 0.6357 & 0.5000 \end{bmatrix},$$

$$\widetilde{R}^3 = \begin{bmatrix} 0.5000 & 0.3000 & 0.5000 & 0.7480 \\ 0.7000 & 0.5000 & 0.7120 & 0.6400 \\ 0.5000 & 0.2880 & 0.5000 & 0.7437 \\ 0.2520 & 0.3600 & 0.2563 & 0.5000 \end{bmatrix}, \widetilde{R}^4 = \begin{bmatrix} 0.5000 & 0.4000 & 0.5500 & 0.6500 \\ 0.6000 & 0.5000 & 0.8000 & 0.7500 \\ 0.4500 & 0.2000 & 0.5000 & 0.7000 \\ 0.3500 & 0.2500 & 0.3000 & 0.5000 \end{bmatrix}.$$

The significant PCI values for the given FPRs are determined using (11)–(13) as

$$PCI(R1) = 0.9248;\ PCI(R2) = 0.9501;$$
$$PCI(R3) = 0.9689;\ PCI(R4) = 1.$$

Finally, the consistency weights to the experts are computed by using (14) as

$$Cw(D_1) = 0.2406,\ Cw(D_2) = 0.2472,$$
$$Cw(D_3) = 0.2521,\ Cw(D_4) = 0.2601.$$

Step-iii: Weights to experts

Primarily, the priority weights to DMs are not assigned, hence, the consistency weights will be taken as the priority weights of the DMs:

$$w(D_1) = 0.2406,\ w(D_2) = 0.2472,$$
$$w(D_3) = 0.2521,\ w(D_4) = 0.2601.$$

Step-iv: Consensus measures

After evaluating missing preference values in IFPRs, a collective similarity matrix is constructed by aggregating the different similarity matrices amongst the DMs using (16) and (17). Then, the consensus levels are measured at the three states using (18)–(20).

$$S^{12} = \begin{bmatrix} 1.0000 & 1.0000 & 0.7200 & 0.9580 \\ 1.0000 & 1.0000 & 0.5800 & 0.6906 \\ 0.7200 & 0.5800 & 1.0000 & 0.4150 \\ 0.9580 & 0.6906 & 0.4150 & 1.0000 \end{bmatrix}, S^{13} = \begin{bmatrix} 1.0000 & 0.7000 & 0.9200 & 0.6280 \\ 0.7000 & 1.0000 & 0.9000 & 0.7906 \\ 0.9200 & 0.9000 & 1.0000 & 0.9000 \\ 0.6280 & 0.7906 & 0.9000 & 1.0000 \end{bmatrix},$$

$$S^{14} = \begin{bmatrix} 1.0000 & 0.8000 & 0.8700 & 0.7280 \\ 0.8000 & 1.0000 & 0.9000 & 0.6406 \\ 0.8700 & 0.9000 & 1.0000 & 0.8000 \\ 0.7280 & 0.6406 & 0.8000 & 1.0000 \end{bmatrix}, S^{23} = \begin{bmatrix} 1.0000 & 0.7000 & 0.8000 & 0.6700 \\ 0.7000 & 1.0000 & 0.4800 & 0.9000 \\ 0.8000 & 0.4800 & 1.0000 & 0.5150 \\ 0.6700 & 0.9000 & 0.5150 & 1.0000 \end{bmatrix},$$

$$S^{24} = \begin{bmatrix} 1.0000 & 0.8000 & 0.8500 & 0.7700 \\ 0.8000 & 1.0000 & 0.4800 & 0.9500 \\ 0.8500 & 0.4800 & 1.0000 & 0.6150 \\ 0.7700 & 0.9500 & 0.6150 & 1.0000 \end{bmatrix}, S^{34} = \begin{bmatrix} 1.0000 & 0.9000 & 0.9500 & 0.9000 \\ 0.9000 & 1.0000 & 1.0000 & 0.8500 \\ 0.9500 & 1.0000 & 1.0000 & 0.9000 \\ 0.9000 & 0.8500 & 0.9000 & 1.0000 \end{bmatrix}.$$

1. *On pair of alternatives*:

$$CoD = \begin{bmatrix} 1.0000 & 0.8167 & 0.8517 & 0.7757 \\ 0.8167 & 1.0000 & 0.7233 & 0.8036 \\ 0.8517 & 0.7233 & 1.0000 & 0.6908 \\ 0.7757 & 0.8036 & 0.6908 & 1.0000 \end{bmatrix}.$$

2. *On alternatives*:

$$CoD_1 = 0.8147,\ CoD_2 = 0.7812,$$
$$CoD_3 = 0.7553,\ CoD_4 = 0.7567.$$

3. *On relation*:

$$CoR = 0.7770.$$

Now, the threshold consensus degree η settled in advance is compared with global consensus degree CoR of the relation; $CoR < \eta$. This indicates that the given consensus level is not acceptable amongst the DMs.

Step-v: Quitting process

Suppose the DM D_2 decides to leave the process, then the updated priority weights of DMs are taken under the use of (21) as

$$w_{new}(D_1) = 0.3230,\ w_{new}(D_3) = 0.3345,\ w_{new}(D_4) = 0.3425.$$

Step-vi: Feedback mechanism

The DMs are asked to enhance their preferences using (22), based on the average value of the preferences provided by the DMs D_1, D_3 and D_4, given as

$$R_{av}^{134} = \begin{bmatrix} 0.5000 & 0.4333 & 0.4900 & 0.5927 \\ 0.5667 & 0.5000 & 0.7667 & 0.5802 \\ 0.5100 & 0.2333 & 0.5000 & 0.8000 \\ 0.4073 & 0.4198 & 0.2000 & 0.5000 \end{bmatrix}.$$

Suppose these DMs accepted the suggestions and enhance their preference relations accordingly, as

$$R_{new}^1 = \begin{bmatrix} 0.5 & 0.45 & r_{13}^1 & r_{14}^1 \\ 0.55 & 0.5 & 0.75 & r_{24}^1 \\ r_{31}^1 & 0.25 & 0.5 & 0.8 \\ r_{41}^1 & r_{42}^1 & 0.2 & 0.5 \end{bmatrix},\ R_{new}^3 = \begin{bmatrix} 0.5 & 0.4 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.6 \\ 0.5 & 0.2 & 0.5 & 0.8 \\ 0.4 & 0.4 & 0.2 & 0.5 \end{bmatrix},\ R_{new}^4 = \begin{bmatrix} 0.5 & 0.4 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.6 \\ 0.5 & 0.2 & 0.5 & 0.75 \\ 0.4 & 0.4 & 0.25 & 0.5 \end{bmatrix}.$$

Initially, all the unknown preference values are estimated, as explained in Section 3, and construct the complete FPR R_{new}^1 as

$$R_{new}^1 = \begin{bmatrix} 0.5000 & 0.4500 & 0.3375 & 0.2700 \\ 0.5500 & 0.5000 & 0.7500 & 0.3743 \\ 0.6625 & 0.2500 & 0.5000 & 0.8000 \\ 0.7300 & 0.6257 & 0.2000 & 0.5000 \end{bmatrix}.$$

Then, consensus states are re-evaluated to estimate the census degree amongst DMs in this round. In this way, the DMs reach a final global consensus level, with $CoR = 0.9036 > \eta$. Therefore, the entire process enters into the accumulation phase to get a collective product consistent FPR R^c.

Step-vii: Accumulation phase

$$R^c = \begin{bmatrix} 0.5000 & 0.4162 & 0.4475 & 0.4934 \\ 0.5838 & 0.5000 & 0.6976 & 0.6134 \\ 0.5525 & 0.3024 & 0.5000 & 0.7303 \\ 0.5066 & 0.3866 & 0.2697 & 0.5000 \end{bmatrix}. \tag{25}$$

Step-viii: Selection phase

The relation R^c obtained in (25) is clearly product consistent, therefore, (24) results in ranking value $R_v(x_i)$ of alternative x_i, $1 \le i \le 4$ as follows:

$$R_v(x_1) = 0.2262,\ R_v(x_2) = 0.3158,$$
$$R_v(x_3) = 0.2642,\ R_v(x_4) = 0.1938,$$

where $\sum_{i=1}^{4} Rv(x_i) = 1$ and $R_v(x_2) > R_v(x_3) > R_v(x_1) > R_v(x_4)$, therefore, the ranking order of alternatives x_1, x_2, x_3 and x_4 is: $x_2 > x_3 > x_1 > x_4$, and the best option is x_2.

6. Comparison

To clearly validate the productivity of our proposed scheme, we compare results with two other existing models under the same norms. These models are the consensus building model [18], and a goal-programming model [24]. The two ranking lists are provided as

$$\text{(a)} \quad x_2 > x_1 > x_3 > x_4,$$
$$\text{(b)} \quad x_2 > x_1 > x_4 > x_3.$$

The result depicted in the first model was different from ours, as the intermediate alternatives x_1 and x_3 had changed while x_2 was the best alternative based on the DMs' D_3 and D_4 opinions with high importance level. The second model resulted in different ordering regarding alternatives x_1, x_3 and x_4, but the alternative x_2 was at the common best place. There may be three reasons that created differences in the ranking order. Firstly, data may be inconsistent based on the under-considered transitivity condition and, definitely, a different result could have been created. Secondly, different techniques were used to evaluate missing values. Thirdly, the corresponding parameters regarding different models could have been evaluated in various ways and results may be affected. However, the quitting process was introduced while the consensus-attainment process was being conducted. We think that the proposed technique provides better, consistent information based on product transitivity rather than multiplicative transitivity. Thus, it can deal with the MPDM situations more suitably, and can accelerate the consensus process.

7. Conclusions

In this paper, an hybrid consensus procedure for MPDM with IFPRs based on product transitivity is proposed. After evaluating the missing preference values, the transitive closure formula is used to get the matrices product to be consistent. The weights of the DMs are obtained from the consistency analysis and, rationally, the DMs with a high level of consistency should have large weights assigned to them, in order to carry more importance in the aggregation process. Additionally, a feedback mechanism is proposed to accelerate the execution of a higher consensus level. After getting a satisfactory consensus state amongst DMs, the entire process entered into the selection phase to rank all the alternatives to choose the best one. An example is provided to highlight the efficiency and feasibility of the proposed method, and results in comparison with some existing models are given. The results established the practicability of the method, which can help us to gain a greater insight into the MPDM process.

Some of the key advantages of the set forth technique are: (1) product transitivity being used to evaluate the unknown preference values of FPRs in this paper. As compared with some other methods based on consistency measures, product transitivity provides better information and consistency accordingly. (2) The final priority weights of the DMs are based on consistency weights and predefined weights (if they exist), which plays an important role in evaluating the consistency indices of DMs' opinions. (3) The quitting process allowed DMs to leave the MPDM process, rather than staying on in the procedure until the problem is solved. (4) The modified feedback mechanism enables the DMs to think in different directions to reach the consensus amongst them. We think that there are only a few techniques of such kinds presented in the literature to deal with MPDM in an incomplete FPRs' environment.

In future, we aim to apply the proposed model in multi-criteria decision making problems.

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