mathematics

Article

Analytic Solution of a Class of Singular Second-Order Boundary Value Problems with Applications

Hoda S. Ali, Elham Alali, Abdelhalim Ebaid * and Fahad M. Alharbi

Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia; h_ali@ut.edu.sa (H.S.A.); dr.elham.alali@gmail.com (E.A.); falharbi@ut.edu.sa (F.M.A.)

* Correspondence: aebaid@ut.edu.sa or halimgamil@yahoo.com; Tel.: +966-55-288-4106

Received: 15 December 2018; Accepted: 3 February 2019; Published: 14 February 2019

Abstract: Recently, it was observed that the concentration/heat transfer of pure/nano fluids are finally governed by singular second-order boundary value problems with exponential coefficients. These coefficients were transformed into polynomials and therefore the governing equations become singular in a new independent variable. Unfortunately, the published approximate solutions in the literature suffer from some weaknesses as showed by one of the present coauthors. Hence, the exact solution for such types of problems becomes a challenge. In this paper, a straightforward approach is presented to obtaining the exact solution for such class of singular second-order boundary value problems. The results are also applied to some selected problems within the literature. Accordingly, the published solutions are recovered as special cases of the present ones.

Keywords: ordinary differential equation; hypergeometric series; singular boundary value problem; Laplace transform; nanofluid

1. Introduction

The field of boundary value problems frequently arises in many real-life applications. Recently, it was shown that the flow and heat transfer of nanofluids are governed by a system of partial differential equations which are often transformed to a set of ordinary differential equations by using a similarity variable η [1–8]. This set of ordinary differential equations is originally subjected to boundary conditions at infinity. Over the past few decades, many authors [9–12] resorted to numerical and semi-analytical methods to treat various problems in the finite and infinite domain. The main difficulty that they have faced was the boundary condition at infinity. The solution to these types of BVPs (boundary value problems) is therefore a challenge.

However, the series methods suffer some weakness because the conditions at infinity cannot be directly imposed into the series solution. Therefore, many authors have applied the Pade technique as a method of solution, however, it needs massive computational work to obtain accurate solutions. Additionally, it has recently been shown by one of the present coauthors [13–15] that some of the approximate methods in the literature suffer from some weaknesses and inaccuracy. It should be noted that, the boundary conditions at infinity can be transformed to new finite ones by applying a substitution such as $t = e^{-\beta\eta}, \beta > 0$ [16]. With this substitution, the coefficients in the original ordinary differential equations (of exponential orders $e^{-\beta\eta}, \beta > 0$) are transformed to polynomial ones. Therefore, the coefficients of the final governing ordinary differential equations become polynomials. For example, Hamad [7] derived the ODE (ordinary differential equation):

$$\tau y''(\eta) + \frac{1}{\beta}\left(1 - e^{-\beta\eta}\right)y'(\eta) = 0, \tag{1}$$

subject to the boundary conditions:

$$y = 1 \text{ at } \eta = 0, \text{and } y = 0 \text{ at } \eta \to \infty \tag{2}$$

to describe the heat transfer of carbon nanotubes over a stretching sheet. Interpretations of the parameters β and τ, were given in [1]. On using the new variable $t = e^{-\beta\eta}$, Equation (1) is reduced to an ODE of polynomial coefficients:

$$y''(t) + \left(\frac{m-1}{mt} + \frac{1}{m}\right)y'(t) = 0, \qquad m = \beta^2\tau. \tag{3}$$

subject to:

$$y(0) = 0,\ y(1) = 1. \tag{4}$$

An additional example was discussed by Kameswaran et al. (Equations (32) and (33) in Reference [2]) in which the following ODE of polynomial coefficients was obtained for the temperature of nanofluids:

$$y''(t) + \left(\frac{1-\lambda_1 Pr^*}{t} - \lambda_1\right)y'(t) + \left(\frac{2\lambda_1}{t}\right)y(t) = -\frac{\lambda_2 Ecs^2}{Pr^*}, \tag{5}$$

subject to:

$$y(0) = 0,\ y(-Pr^*) = 1, \tag{6}$$

where the parameters $\lambda_1, \lambda_2, Pr^*, Ec$ and s have been addressed in Reference [2]. Hence, the nanofluids temperature and the nanoparticles concentration are usually special cases of the present generalized class of second-order singular BVPs, given as:

$$y''(t) + \left(\frac{P}{t} + Q\right)y'(t) + \left(\frac{R}{t}\right)y(t) = \alpha t^{n-1}, n > -1, \alpha \in \mathbb{R}, \tag{7}$$

subject to the generalized boundary conditions:

$$y(0) = 0,\ y(\delta) = 1 + \xi y'(\delta),\ \delta, \xi \in \mathbb{R}, \tag{8}$$

at prescribed values for the involved parameters. P, Q, and R are physical parameters which are related to the densities, the thermal conductivities, and the heat capacitances of base fluids and nanofluids [1–8]. The parameter ξ is often used to describe convective heat condition and it takes some particular values according to the physical problem, while δ depends on the final boundary condition of the temperature/concentration of the studied model. Moreover, Equations (7) and (8) reduces to the one studied in Reference [16] at $\xi = 0$ (as a special case). Hence, the exact solution that will be obtained is more general than those previously published by the authors [16], which is the main objective of this paper. Furthermore, the analytical solution will be obtained in general form which means that it will be valid without any restrictions on the coefficients P, Q, and R, and α. Moreover, the analytical solutions of ODEs arise in real-life applications related to fluid temperature/concentration that can be directly obtained via our solution when the coefficients P, Q, and R, and α are assigned. Accordingly, the results may be useful for researchers in this field.

The objective of this paper is to derive a general exact solution for Equations (7) and (8). The paper is organized as follows. In Section 2, the solution of Equations (7) and (8) will be introduced in terms of the hypergeometric function. Section 3 is devoted to obtaining a useful theorem to express the solution in terms of the generalized incomplete gamma function. In Section 4, some applications are discussed in which the current general solution reduces those in the literature as special cases. An example for numerical validation are introduced in Section 5. The conclusion of this paper is presented in Section 6.

2. Analytic Solution

An analytical procedure is presented in this section to solve class (7) and (8). Two separate cases will be considered: the first case is the homogenous BVP (7) and (8), i.e., at $\alpha = 0$, while the second is the in homogenous BVP (7) and (8), i.e., at $\alpha \neq 0$.

2.1. At $\alpha = 0$

This case requires to solve the homogenous equation:

$$y''(t) + \left(\frac{P}{t} + Q\right)y'(t) + \left(\frac{R}{t}\right)y(t) = 0, \tag{9}$$

which can be written as:

$$ty''(t) + (P + Qt)y'(t) + Ry(t) = 0, \tag{10}$$

subject to the BCs (boundary conditions) (8). Following [16], we have:

$$y(t) = \frac{ct^{\mu_1+\mu_2-1}}{\Gamma(\mu_1+\mu_2)}{}_1F_1[\mu_1, \mu_1+\mu_2, -Qt],\ \mu_1 > 0,\ \mu_1+\mu_2 > 1, \tag{11}$$

where c is a constant, ${}_1F_1$ is Kummer's function, and μ_1 and μ_2 are given in terms of P, Q, R by:

$$\mu_1 = 1 - P + \frac{R}{Q},\ \mu_2 = 1 - \frac{R}{Q}. \tag{12}$$

Applying the second BC in Equation (8) on Equation (11), yields:

$$c = \frac{\delta^{1-\mu_1-\mu_2}\Gamma(\mu_1+\mu_2)}{(1-\xi/\delta(\mu_1+\mu_2-1)/\delta)_1F_1[\mu_1, \mu_1+\mu_2, -Q\delta] + (\xi Q\mu_1)_1F_1[1+\mu_1, 1+\mu_1+\mu_2, -Q\delta]}, \tag{13}$$

where:

$$\frac{d}{dt}({}_1F_1[\mu_1, \mu_1+\mu_2, -Qt]) = -(Q\mu_1)_1F_1[1+\mu_1, 1+\mu_1+\mu_2, -Qt], \tag{14}$$

was used to determine c in Equation (13). Inserting Equation (13) into Equation (11), we obtain:

$$y(t) = \frac{(t/\delta)^{\mu_1+\mu_2-1}{}_1F_1[\mu_1, \mu_1+\mu_2, -Qt]}{(1-\xi(\mu_1+\mu_2-1)/\delta)_1F_1[\mu_1, \mu_1+\mu_2, -Q\delta] + (\xi Q\mu_1)_1F_1[1+\mu_1, 1+\mu_1+\mu_2, -Q\delta]}. \tag{15}$$

Later, it will show that solution (15) is equivalent to the existing results in the literature at prescribed values of the parameter δ and the coefficients P, Q, and R.

2.2. At $\alpha \neq 0$

In this general case, the BVP (7) and (8) becomes in homogenous and therefore requires a particular solution in addition to the complementary solution (11). Assuming that $y_c(t)$ and $y_p(t)$ are respectively a complementary solution and the particular solution for Equation (7), then:

$$y(t) = y_c(t) + y_p(t), \tag{16}$$

where the particular solution $y_p(t)$ of Equation (7) was obtained as (see [16])

$$y_p(t) = \frac{\alpha t^{n+1}}{(n+1)(n+P)}, \tag{17}$$

provided that:

$$R = -(n+1)Q,\ (n+P) \neq 0. \tag{18}$$

Accordingly, μ_1 and μ_2 in Equation (12) becomes:

$$\mu_1 = -(n+P),\ \mu_2 = n+2. \tag{19}$$

From Equation (11) and Equations (16)–(19), $y(t)$ reads:

$$y(t) = \frac{\alpha t^{n+1}}{(n+1)(n+P)} + \frac{c t^{1-P}}{\Gamma(2-P)} {}_1F_1[-(P+n), 2-P, -Qt], \tag{20}$$

where $-1 < n < -P$ admits $y(0) = 0$. At this point, we have the solution (14) for the following new form of Equation (7):

$$y''(t) + \left(\frac{P}{t} + Q\right) y'(t) - \left(\frac{(n+1)Q}{t}\right) y(t) = \alpha t^{n-1}, \tag{21}$$

which arises in many real physical problems at certain values of n (> -1), P, and Q, as will be discussed in a later section. The constant c should be calculated again by applying the second BC in Equation (8) on Equation (20) and this gives c as:

$$c = \frac{\delta^{-1+P}\left(1 + \frac{\alpha\delta^n}{n+P}\left(\xi - \frac{\delta}{n+1}\right)\right)\Gamma(2-P)}{(1 - \xi(1-P)/\delta)\,{}_1F_1[-(P+n), 2-P, -Q\delta] - \xi Q(n+P)\,{}_1F_1[1-(P+n), 3-P, -Q\delta]}. \tag{22}$$

It was verified by using a direct substitution of the analytic solution (20) along with (22) to satisfy Equation (21) and the BCs (Equation (8)). Furthermore, Section 4 proves that such analytic solution agree with the existing results in the literature as special cases.

3. Analysis

Here, it should be noted that the current solutions can be expressed in terms of a standard special function, the generalized incomplete gamma function, defined by:

$$\Gamma(a, \tau_0, \tau_1) = \int_{\tau_0}^{\tau_1} \sigma^{a-1} e^{-\sigma} d\sigma. \tag{23}$$

To achieve this task, we introduced the following theorem that relates Kummer's function with the generalized incomplete gamma function.

Theorem 1

For $b = a+3$, we have:

$${}_1F_1[a, b, \tau] = \frac{(-1)^a \Gamma(a+3)}{2\tau^{a+2}\Gamma(a)} \Big(\tau^2 \Gamma(a, 0, -\tau) + 2\tau\Gamma(a+1, 0, -\tau) + \Gamma(a+2, 0, -\tau)\Big), a > 0. \tag{24}$$

Proof. Using the definition of Kummer's function, we have:

$$\begin{aligned} {}_1F_1[a, b, \tau] &= \tfrac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \mu^{a-1}(1-\mu)^{b-a-1} e^{\tau\mu} d\mu, \\ &= \tfrac{\Gamma(a+3)}{2\Gamma(a)} \int_0^1 \mu^{a-1}(1-\mu)^2 e^{\tau\mu} d\mu,\ b = a+3, \\ &= \tfrac{(-1)^a\Gamma(a+3)}{2\tau^{a+2}\Gamma(a)} \int_0^{-\tau} \sigma^{a-1}(\tau+\sigma)^2 e^{-\sigma} d\sigma,\ \sigma = -\tau\mu, \\ &= \tfrac{(-1)^a\Gamma(a+3)}{2\tau^{a+2}\Gamma(a)} \left(\tau^2\Gamma(a, 0, -\tau) + 2\tau\Gamma(a+1, 0, -\tau) + \Gamma(a+2, 0, -\tau)\right), \end{aligned} \tag{25}$$

Through this theorem, some existing results in the literature will be recovered as special cases of the present ones. □

4. Applications

4.1. No Convective Heat Condition: $\xi = 0$ at $\alpha \neq 0$

In Reference [16], the class (7) and (8) were analyzed when the convective heat condition was absent, i.e., at $\xi = 0$. In this case, the constant c in Equation (22) reduces to:

$$c = \frac{\delta^{-1+P}\left(1 - \frac{\alpha\delta^{n+1}}{(n+P)(n+1)}\right)\Gamma(2-P)}{{}_1F_1[-(P+n), 2-P, -Q\delta]}. \tag{26}$$

Implementing Equation (26) into Equation (20), then the solution takes the final form:

$$y(t) = \frac{\alpha t^{n+1}}{(n+1)(n+P)} + \frac{(t/\delta)^{1-P}\left(1 - \frac{\alpha\delta^{n+1}}{(n+P)(n+1)}\right){}_1F_1[-(P+n), 2-P, -Qt]}{{}_1F_1[-(P+n), 2-P, -Q\delta]}, \tag{27}$$

which is the same solution obtained very recently by Ebaid et al. [16].

4.2. Convective Heat Condition: $\xi \neq 0$ at $\alpha = 0$ and $\delta = 1$

Here, the given system (7) and (8) reduces to the following homogenous boundary value problem which was analyzed by Ebaid et al. [17]:

$$y''(t) + \left(\frac{P}{t} + Q\right)y'(t) + \left(\frac{R}{t}\right)y(t) = 0,\; y(0) = 0,\; y(1) = 1 + \xi y'(1). \tag{28}$$

In this case, the BVP (Equation (28)) is homogenous and its solution is obtained from Equation (15) by setting $\delta = 1$, which leads to:

$$y(t) = \frac{t^{\mu_1+\mu_2-1}{}_1F_1[\mu_1, \mu_1+\mu_2, -Qt]}{(1-\xi(\mu_1+\mu_2-1)){}_1F_1[\mu_1, \mu_1+\mu_2, -Q] + (\xi Q\mu_1){}_1F_1[1+\mu_1, 1+\mu_1+\mu_2, -Q]}. \tag{29}$$

By substituting μ_1 and μ_2 given in Equation (12) into Equation (29), we obtain:

$$y(t) = \frac{t^{1-P}{}_1F_1\left[1-P+\frac{R}{Q}, 2-P, -Qt\right]}{(1-\xi(1-P)){}_1F_1\left[1-P+\frac{R}{Q}, 2-P, -Q\right] + \xi(Q(1-P)+R){}_1F_1\left[2-P+\frac{R}{Q}, 3-P, -Q\right]}, \tag{30}$$

which agrees with the obtained results by Ebaid et al. [17].

4.3. Marangoni Boundary Layer Temperature: $\xi = 0$ at $\alpha = 0$ and $\delta = -1$

Aly and Ebaid [8] obtained the following temperature equation for the Marangoni boundary layer of a nanofluid in a porous medium:

$$ty''(t) + (l - mt)y'(t) + 2my(t) = 0, \tag{31}$$

subject to:

$$y(0) = 0,\; y(-1) = 1. \tag{32}$$

On comparing the BVP, Equations (31) and (32), with the present class (Equations (1) and (2)), the values of P, Q, R, α, and δ are assigned as:

$$P = l, Q = -m, R = 2m, \alpha = 0, \xi = 0, \delta = -1. \tag{33}$$

Hence,

$$\mu_1 = -1 - l, \mu_2 = 3, \tag{34}$$

and $y(t)$ in Equation (15) becomes:

$$y(t) = \frac{(-t)^{1-l} {}_1F_1[-1-l, 2-l, mt]}{{}_1F_1[-1-l, 2-l, -m]}, l < -1. \tag{35}$$

Now, using Theorem 1 at $a = -1-l$, $b = 2-l$, and $\tau = mt$, where $b = a+3$ is already satisfied, we obtain:

$$\begin{gathered} {}_1F_1[-1-l, 2-l, mt] = \\ \tfrac{(-1)^{-1-l}\Gamma(2-l)}{2(mt)^{1-l}\Gamma(-1-l)} \left(m^2t^2\Gamma(-1-l, 0, -mt) + 2mt\Gamma(-l, 0, -mt) + \Gamma(-l+1, 0, -mt)\right) \end{gathered} \tag{36}$$

and:

$$\begin{gathered} {}_1F_1[-1-l, 2-l, -m] = \\ \tfrac{(-1)^{-1-l}\Gamma(2-l)}{2(-m)^{1-l}\Gamma(-1-l)} \left(m^2\Gamma(-1-l, 0, m) + 2mt\Gamma(-l, 0, m) + \Gamma(-l+1, 0, m)\right) \end{gathered} \tag{37}$$

Therefore, the final solution is given by:

$$y(t) = \frac{m^2t^2\Gamma(-1-l, 0, -mt) + 2mt\Gamma(-l, 0, -mt) + \Gamma(-l+1, 0, -mt)}{m^2\Gamma(-1-l, 0, m) + 2mt\Gamma(-l, 0, m) + \Gamma(-l+1, 0, m)}, \tag{38}$$

which is also the same obtained solution in [8].

4.4. Magnetohydrodynamic Marangoni Convection: $\xi = 0$ at $\alpha \neq 0$ and $\delta = 1$

Khaled [8] obtained the following temperature equation for the Marangoni boundary layer in the presence of radiation and joule heating:

$$ty''(t) + (l - mt)y'(t) + 2my(t) = -\lambda t, \tag{39}$$

subject to:

$$y(0) = 0,\ y(1) = 1. \tag{40}$$

Comparing Equation (39) with Equation (21) and Equation (40) with Equation (8), the values of P, Q, R, α, n, and δ are as follows:

$$P = l, Q = -m, R = 2m, \alpha = -\lambda, n = 1, \xi = 0, \delta = 1. \tag{41}$$

Hence, the values of μ_1 and μ_2 are the same as in Equation (28). Substituting Equation (41) into Equation (20) and Equation (22), respectively, we get:

$$y(t) = -\frac{\lambda t^2}{2(l+1)} + \frac{ct^{1-l}}{\Gamma(2-l)} {}_1F_1[-(1+l), 2-l, mt], \tag{42}$$

and:

$$c = \frac{\left(1 + \frac{\lambda}{2(1+l)}\right)\Gamma(2-l)}{{}_1F_1[-(1+l), 2-l, m]}. \tag{43}$$

Therefore,

$$y(t) = -\frac{\lambda t^2}{2(l+1)} + \left(1 + \frac{\lambda}{2(l+1)}\right) \frac{t^{1-l} {}_1F_1[-1-l, 2-l, mt]}{{}_1F_1[-1-l, 2-l, m]}, \tag{44}$$

which is the same result that was obtained by Khaled [8].

4.5. Mass Transfer in a Jeffrey Fluid: $\xi = 0$ at $\alpha = 0$ and $\delta = Sc/\beta^2$

The mass transfer equation in a Jeffrey fluid in the presence of heat source/sink was given in Qasim [18] as:

$$ty''(t) + \left(1 - \frac{\mathrm{Sc}}{\beta^2} + t\right)y'(t) - my(t) = 0, \tag{45}$$

subject to:

$$y(0) = 0, y\left(\frac{\mathrm{Sc}}{\beta^2}\right) = 1, \tag{46}$$

where $\beta > 0$ and Sc is the Schmidt parameter. In this case, P, Q, R, α, ξ, and δ are as follows:

$$P = 1 - \frac{\mathrm{Sc}}{\beta^2}, Q = 1, R = -m, \alpha = 0, \xi = 0, \delta = \frac{\mathrm{Sc}}{\beta^2}. \tag{47}$$

Hence, μ_1 and μ_2 are:

$$\mu_1 = \frac{\mathrm{Sc}}{\beta^2} - m, \mu_2 = 1 + m. \tag{48}$$

Accordingly, we have:

$$y(t) = \left(\frac{\beta^2}{\mathrm{Sc}}t\right)^{\frac{\mathrm{Sc}}{\beta^2}} \frac{{}_1F_1\left[\frac{\mathrm{Sc}}{\beta^2} - m, \frac{\mathrm{Sc}}{\beta^2} + 1, -t\right]}{{}_1F_1\left[\frac{\mathrm{Sc}}{\beta^2} - m, \frac{\mathrm{Sc}}{\beta^2} + 1, -\frac{\mathrm{Sc}}{\beta^2}\right]}. \tag{49}$$

This is also the exact solution of Equations (45) and (46) which were verified by substitution. It may be useful to end this section by observing that the exact solution reported by Qasim [18] is given as:

$$y(t) = \left(\frac{\beta^2}{\mathrm{Sc}}t\right)^{\frac{\mathrm{Sc}}{\beta^2}} \frac{{}_1F_1\left[\frac{\mathrm{Sc}}{\beta^2} - m, 2\frac{\mathrm{Sc}}{\beta^2} + 1, -t\right]}{{}_1F_1\left[\frac{\mathrm{Sc}}{\beta^2} - m, 2\frac{2\mathrm{Sc}}{\beta^2} + 1, -\frac{\mathrm{Sc}}{\beta^2}\right]}, \tag{50}$$

which does not satisfy Equations (45) and (46).

5. Example for Numerical Validation

In Reference [19], the authors investigated the laminar boundary layer flow of a viscous incompressible electrically-conducting and radiating fluid past a semi-infinite flat surface in two dimensions. They obtained the governing equations as:

$$z'''(\eta) + z(\eta)z''(\eta) - \left(z'(\eta)\right)^2 - M^2 z'(\eta(t) = 0, \tag{51}$$

$$\left(\frac{1+Nr}{Pr}\right)y''(\eta) - 2z'(\eta)\theta(\eta) + z(\eta)y'(\eta) + Ec\left[M^2\left(z'(\eta)\right)^2 + \left(z''(\eta)\right)^2\right] = 0, \tag{52}$$

$$w''(\eta) + Sc\left[z(\eta)w'(\eta) - 2z'(\eta)w(\eta)\right] = 0, \tag{53}$$

with the BCs:

$$z''(0) = -2(r+1), z(0) = f_w, z'(\infty(t) = 0, \tag{54}$$

$$y(0) = 1,\ y(\infty(t) = 0, \tag{55}$$

$$w(0) = 1,\ w(\infty) = 0, \tag{56}$$

where the parameters Nr, M, Ec, and f_w were well defined by [20]. They solved Equations (51)–(56) numerically using the Runge-Kutta-Fehlberg method along with shooting technique. However, the exact solutions of the above system can be obtained via our approach.

Firstly, we show that Equation (46) and Equation (47) take the form of the present class. The exact solution of Equations (51) and (54) is given as:

$$z(\eta) = a + be^{-\beta\eta}, a = f_w + \frac{2(r+1)}{\beta^2}, b = -\frac{2(r+1)}{\beta^2}, \tag{57}$$

where β is a positive root for the following equation:

$$\beta^3 - f_w\beta^2 - M^2\beta - 2(r+1) = 0. \tag{58}$$

By means of the transformation $t = e^{-\beta\eta}$, Equations (52) and (53) become:

$$y''(t) + \left(\frac{n_1}{t} - m_1\right)y'(t) + \frac{2m_1}{t}y(t) = -\lambda, \tag{59}$$

and:

$$w''(t) + \left(\frac{n_2}{t} - m_2\right)w'(t) + \frac{2m_2}{t}w(t) = 0, \tag{60}$$

where:

$$\Omega = \frac{1+Nr}{Pr}, n_1 = 1 - \frac{a}{\Omega\beta}, m_1 = \frac{b}{\Omega\beta}, \lambda = \frac{Ec\beta^2(\beta^2 + M^2)}{\Omega}, \tag{61}$$

$$n_2 = 1 - \frac{Sca}{\beta}, m_2 = \frac{Scb}{\beta}. \tag{62}$$

The BCs (55) and (56) become:

$$y(0) = 0, \, y(1) = 1. \tag{63}$$

and:

$$w(0) = 0, \, w(1) = 1. \tag{64}$$

Comparing Equation (59) with Equation (21) and Equation (63) with Equation (8) we have $P = n_1, Q = -m_1, n = 1, \alpha = -\lambda$ and $\zeta = 0$. Therefore, $y(t)$ is given from Equation (21) and Equation (22) as:

$$\theta(t) = -\frac{\lambda t^2}{2(n_1+1)} + \left(1 + \frac{\lambda}{2(n_1+1)}\right)\frac{t^{1-n_1}F_{1,1}[-1-n_1, 2-n_1, m_1 t]}{F_{1,1}[-1-n_1, 2-n_1, m_1]}. \tag{65}$$

By repeating the same analysis for the w-equation (Equation (60)) along with the BCs (Equation (64)), we have:

$$w(t) = \left(1 + \frac{\lambda}{2(n_2+1)}\right)\frac{t^{1-n_2}F_{1,1}[-1-n_2, 2-n_2, m_2 t]}{F_{1,1}[-1-n_2, 2-n_2, m_2]}, n_2 < 1. \tag{66}$$

Here, we aim to give some light on the difference between the obtained numerical results by [20] and present exact ones. Table 1 shows the comparisons between the exact results and the approximate ones [20] for various values of the magnetic parameter M. It is observed from this table that the approximate results [20] for $-y'(0)$ and $-w'(0)$ are different than the present exact results. The absolute error increases at higher values of M. This conclusion may confirm the views of the authors.

Table 1. Comparisons between the present results with those of Reference [20] for $-y'(0)$ and $-w'(0)$ at various values of M (magnetic parameter) when $r = 0, Nr = 0, Sc = 0.6, Ec = 0.2, \Pr = 0.78, f_w = 0$.

M	$-y'(0)$			$-w'(0)$		
	Approximate [20]	Exact (present)	Absolute Error	Approximate [20]	Exact (present)	Absolute Error
1	1.01345	0.97809	0.03536	1.05783	1.0055	0.05233
2	0.65609	0.53004	0.12605	0.75964	0.61018	0.14946
3	0.47085	0.25593	0.21492	0.57836	0.34641	0.23195
4	0.38809	0.12334	0.26475	0.48607	0.21084	0.27523

6. Conclusions

In this paper, a generalized analytical solution of a class of singular second-order ordinary differential equations that arises in various applications was obtained. The solution was basically expressed in terms of the hypergeometric series. By means of a theorem, we were able to express the solution in terms of the generalized incomplete gamma function. The obtained results were applied and compared with selected problems in the literature. The current class was reduced to similar published problems at particular choices for the coefficients. Hence, the corresponding solutions in the literature were recovered as special cases of the present generalized solution.

Author Contributions: Formal analysis, project administration, H.S.A.; data curation, E.A.; investigation, data curation, A.E. and F.A.

Funding: The authors would like to acknowledge financial support for this work from the Deanship of Scientific Research (DSR), University of Tabuk, Tabuk, Saudi Arabia, under Grant no. 0018-1439-S.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hamad, M.A.A. Analytical solution of natural convection flow of a nanofluid over a linearly stretching sheet in the presence of magnetic field. *Int. Commun. Heat Mass Transf.* **2011**, *38*, 487–492. [CrossRef]
2. Kameswaran, P.K.; Narayana, M.; Sibanda, P.; Murthy, P.V.S.N. Hydromagnetic nanofluid flow due to a stretching or shrinking sheet with viscous dissipation and chemical reaction effects. *Int. J. Heat Mass Trans.* **2012**, *55*, 7587–7595. [CrossRef]
3. Aly, E.H.; Ebaid, A. Exact analytical solution for suction and injection flow with thermal enhancement of five nanofluids over an isothermal stretching sheet with effect of the slip model: A comparative study. *Abstr. Appl. Anal.* **2013**, *2013*, 721578. [CrossRef]
4. Aly, E.H.; Ebaid, A. New exact solutions for boundary-layer flow of a nanofluid past a stretching sheet. *J. Comput. Theor. Nanosci.* **2013**, *10*, 2591–2594. [CrossRef]
5. Khan, W.A.; Khan, Z.H.; Rahi, M. Fluid flow and heat transfer of carbon nanotubes along a flat plate with Navier slip boundary. *Appl. Nanosci.* **2014**, *4*, 633–641. [CrossRef]
6. Ebaid, A.; al Mutairi, F.; Khaled, S.M. Effect of velocity slip boundary condition on the flow and heat transfer of Cu-water and TiO_2-water nanofluids in the presence of a magnetic field. *Adv. Math. Phys.* **2014**, *2014*, 538950. [CrossRef]
7. Ebaid, A.; al Sharif, M. Application of Laplace transform for the exact effect of a magnetic field on heat transfer of carbon-nanotubes suspended nanofluids. *Z. Nat. A* **2015**, *70*, 471–475. [CrossRef]
8. Aly, E.H.; Ebaid, A. Exact analysis for the effect of heat transfer on MHD and radiation Marangoni boundary layer nanofluid flow past a surface embedded in a porous medium. *J. Mol. Liq.* **2016**, *215*, 625–639. [CrossRef]
9. Boyd, J.P. Pad´e-approximant algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain. *Comput. Phys.* **1997**, *11*, 299–303. [CrossRef]
10. Wazwaz, A.M. The modified decomposition method and Pad´e approximants for solving the Thomas-Fermi equation. *Appl. Math. Comput.* **1999**, *105*, 11–19.
11. Wazwaz, A.M. The modified decomposition method and Pad´e approximants for a boundary layer equation in unbounded domain. *Appl. Math. Comput.* **2006**, *177*, 737–744.

12. Wazwaz, A.M. Pad´e approximants and Adomian decomposition method for solving the Flierl-Petviashivili equation and its variants. *Appl. Math. Comput.* **2006**, *182*, 1812–1818.
13. Ebaid, A.; Aly, E.H. Exact analytical solution of the peristaltic nanofluids flow in an asymmetric channel with flexible walls: Application to cancer treatment. *Comput. Math. Methods Med.* **2013**, *2013*, 825376. [CrossRef] [PubMed]
14. Ebaid, A. Remarks on the homotopy perturbation method for the peristaltic flow of Jeffrey fluid with nano-particles in an asymmetric channel. *Comput. Math. Methods Med.* **2014**, *68*, 77–85. [CrossRef]
15. Ebaid, A.; Khaled, S.M. An Exact Solution for a Boundary Value Problem with Application in Fluid Mechanics and Comparison with the Regular Perturbation Solution. *Abstr. Appl. Anal.* **2014**, *2014*, 172590. [CrossRef]
16. Ebaid, A.; Wazwaz, A.M.; Alali, E.; Masaedeh, B. Hypergeometric Series Solution to a Class of Second-Order Boundary Value Problems via Laplace Transform with Applications to Nanouids. *Commun. Theor. Phys.* **2017**, *67*, 231. [CrossRef]
17. Ebaid, A.; Alali, E.; Saleh, H. The exact solution of a class of boundary value problems with polynomial coefficients and its applications on nanofluids. *J. Assoc. Arab Univ. Basi Appl. Sci.* **2017**, *24*, 156–159. [CrossRef]
18. Khaled, S.M. The exact effects of radiation and joule heating on magnetohydrodynamic Marangoni convection over a flat surface. *Therm. Sci.* **2018**, *22*, 63–72. [CrossRef]
19. Qasim, M. Heat and mass transfer in a Jeffrey fluid over a stretching sheet with heat source/sink. *Alex. Eng. J.* **2013**, *52*, 571–575. [CrossRef]
20. Sreenivasulu, P.; Reddy, N.B.; Reddy, M.G. Effects of radiation on MHD thermosolutal Marangoni convection boundary layer flow with Joule heating and viscous dissipation. *Int. J. Appl. Math. Mech.* **2013**, *9*, 47–65.