The First Eigenvalue Estimates of Warped Product Pseudo-Slant Submanifolds

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Abstract: The aim of this paper is to construct a sharp general inequality for warped product pseudo-slant submanifold of the type $M = M_\perp \times_f M_\theta$, in a nearly cosymplectic manifold, in terms of the warping function and the symmetric bilinear form h which is known as the second fundamental form. The equality cases are also discussed. As its application, we establish a bound for the first non-zero eigenvalue of the warping function whose base manifold is compact.

1. Introduction

The clue of warped product manifolds is related to the generalization of Riemannian products. The role of warped product submanifolds in studying Riemannian geometry was studied actively from the pioneering work of Chen [1]. The work of Chen is about the characterization of CR-warped products in Kaehler manifolds, and derives the inequality for the second fundamental form. In fact, distinct classes of warped product submanifolds of the different kinds of structures were studied by several geometers (see [1–14]). Recently, Ali et al. [15], established general inequalities for warped product pseudo-slants isometrically immersed in nearly Kenmotsu manifolds for mixed, totally geodesic submanifolds. Moreover, some results on the existence of the warped product pseudo-slant submanifolds in a nearly cosymplectic manifold in terms of endomorphisms were proved by Uddin in [14]. We have noticed that the warped product pseudo-slant submanifolds of the form $M_\perp \times_f M_\theta$, and $M_\theta \times_f M_\perp$, is a CR-warped product submanifold with slant angle $\theta = 0$. For contradict that warped product pseudo-slant submanifolds always not generalize CR-warped product submanifold which was show in [13]. However, some interesting inequalities have been obtained by many geometers (see [4,10,12,16–20]) for distinct warped product submanifolds in the different types of ambient manifolds. In [5], Al-Solamy derived the inequality for mixed, totally geodesic warped product pseudo-slant submanifolds of type $M = M_\theta \times_f M_\perp$, in a nearly cosymplectic manifold. On other hand, the warped product pseudo-slant submanifold in a nearly cosymplectic manifold of the type $M = M_\perp \times_f M_\theta$ was studied by Uddin et al. [14]. We consider the non-trivial warped product pseudo-slant submanifold $M = M_\perp \times_f M_\theta$, such that M_θ and $M_\perp$ are proper-slant and anti-invariant submanifolds, respectively. In this case, considering that M is not mixed and totally geodesic, we announce our first result as follows.

Theorem 1. *Assuming $\widetilde{M}$ is a nearly cosymplectic manifold, let $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of $\widetilde{M}$. Then,*

(i)　*The second fundamental form of M is derived as:*

$$||h||^2 \geq \left(\frac{d_1 \cos^2 \theta}{9} \right) ||\nabla \ln f||^2, \tag{1}$$

where $\dim \widetilde{M} = 2n + 1$, $\dim M = m + 1$, $\dim M_\theta = d_1$, & $\dim M_\perp = d_2$. *Further,* $\nabla \ln f$ *is the gradient of* $\ln f$ *over* $M_\perp$;

(ii)　*If the equality sign holds in (1), then* $M_\perp$ *is a totally geodesic submanifold with satisfying conditions:*

$$h(\mathcal{D}^\perp, \mathcal{D}^\theta) \subseteq \Gamma(\varphi T M_\perp), \tag{2}$$

$$\&$$

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subseteq \Gamma(F(TM_\theta)) \ \& \ h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subseteq \Gamma(\varphi T M_\perp). \tag{3}$$

As an application of Theorem 1, the next result comes from the idea of the eigenvalue comparison theorem of Cheng [21], which has proved that M is complete and isometric to the standard unit sphere, provided that $Ric(M) \geq 1$ and $d(M) = \pi$ by using the first non-zero eigenvalue of the Laplacian operator. After that, Mihai [12] obtained the first eigenvalue for CR-warped products into the Sasakian space form. Therefore, we used the method of the maximum principle for the first non-zero eigenvalue λ_1 which is defined in [22], and made use of Theorem 1 to deduce the following.

Theorem 2. *Assuming* $\widetilde{M}$ *is a nearly cosymplectic manifold, let* $M = M_\perp \times_f M_\theta$ *be a warped product pseudo-slant submanifold of* $\widetilde{M}$, *such that* $M_\perp$ *is a compact submanifold of* $\widetilde{M}$. *Then, we have:*

$$\lambda_1 \leq \left(\frac{9 \sec^2 \theta \int_{M_\perp} ||h||^2 dV}{d_1 \int_{M_\perp} (\ln f)^2 dV} \right), \tag{4}$$

where λ_1 *is a first non-zero eigenvalue of the warping function* $\ln f$. *The dimensions are defined in Theorem 1 (ii).*

2. Preliminaries

An odd $(2n + 1)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ is called a *nearly cosymplectic* manifold, if it consists of an endomorphism φ of its tangent bundle $T\widetilde{M}$, a structure vector field ξ, and a 1-form η, which satisfies the following:

$$\varphi^2 = -I + \eta \oplus \xi, \ \ \eta(\xi) = 1, \ \ \eta o \varphi = 0, \ \&, \varphi(\xi) = 0, \tag{5}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \tag{6}$$

$$(\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U = 0, \tag{7}$$

for any vector field U, V on $\widetilde{M}$ such that $\widetilde{\nabla}$ denotes the Riemannian connection with respect to the Riemannian metric g (see [14]). Furthermore, the fundamental 2-form denoted by Φ, i.e., $\Phi(U, V) = g(\varphi U, V)$.

Let the Lie algebras of vector fields tangent and normal to a submanifold M be denoted as $\Gamma(TM)$ and $\Gamma(T^\perp M)$. Moreover, the induced connection on $T^\perp M$ is denoted $\nabla^\perp$, and ∇ is the Levi-Civita connection of M. Thus, the Gauss and Weingarten formulas are defined as:

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{8}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla_U^\perp N, \tag{9}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator, respectively, for the submanifold of M into $\widetilde{M}$, which are related as:

$$g(h(U, V), N) = g(A_N U, V). \tag{10}$$

For any $U \in \Gamma(TM)$, we have:

$$\varphi U = PU + FU, \tag{11}$$

where PU and FU are the tangential and normal components of φU, respectively. Similarly, for any $N \in \Gamma(T^\perp M)$, we have:

$$\varphi N = tN + fN, \tag{12}$$

where tN (resp. fN) are tangential (resp. normal) components of φN. If $h(U, V) = 0$, then M is totally and if $h(U, V) = g(U, V)H$, then M is totally umbilical.

The submanifold M is defined as a slant submanifold if, for each non-zero vector X tangent to M, the angle $\theta(X)$ between φX and $T_p M - <\xi>$ is constant. We shall use Theorem 2.1 [5] in our subsequent proof as a characterization of the slant submanifold.

In this sequel, by using the slant distribution given in [23,24], we shall give the definition of pseudo-slant submanifolds of almost-contact manifolds. For deeper classifications of pseudo-slant submanifolds, we refer to [7]. First, we give the definition of a pseudo-slant submanifold:

Definition 1. *If the tangent bundle TM of submanifold M in an almost-contact manifold is decomposed as $TM = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus <\xi>$, where $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ are slant and anti-invariant distributions such that $P\mathcal{D}^\theta \subseteq \mathcal{D}^\theta$ and $\varphi\mathcal{D}^\perp \subseteq T^\perp M$, respectively—in this case, where $<\xi>$ is a one-dimensional distribution spanned by the structure field ξ, then M is called pseudo-slant submanifold.*

If μ is an invariant subspace of $T^\perp M$, then for the pseudo-slant case, the normal bundle $T^\perp M$ can be decomposed as: $T^\perp M = \varphi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu$.

3. Warped Product Submanifolds of the Form $M_\perp \times_f M_\theta$

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. Then, the warped product of M_1 and M_2 is the Riemannian product $M_1 \times_f M_2 = (M_1 \times_f M_2, g)$ with the metric $g = g_1 + f^2 g_2$ and f being a positive differential function defined on M_1. Then, from Lemma 7.3 [25], we have:

$$\nabla_U W = \nabla_W U = U(\ln f)W, \tag{13}$$

for any vector fields U and W tangent to M_1 and M_2, respectively, where ∇ denote the Levi-Civitas connection on M (see [25]). If the warping function f is constant, then a warped product manifold $M = M_1 \times_f M_2$ is called a simply Riemannian product or a *trivial* warped product manifold. Now, we consider the warped product pseudo-slant submanifold of the form $M = M_\perp \times_f M_\theta$, such that $\xi \in \Gamma(TM_\perp)$ and other cases is leaved because they were studied by Al-Solamy [5], and then we obtain some lemmas for use in our main result.

Lemma 1. *Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold, $\widetilde{M}$. Then:*

$$2g(h(U, W), FPU) = (W \ln f) \cos^2 \theta ||U||^2 + g(h(U, PU), \varphi W) + g(h(W, PU), FU),$$

for any $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$.

Proof. Suppose that $M = M_\perp \times_f M_\theta$ is a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$; then, from (8), we find that:

$$g(h(U, W), FPU) = g(\tilde{\nabla}_W U, FPU).$$

Thus, from (11) and from Theorem 2.1 in [5], we obtain:

$$g(h(U, W), FPU) = -g(\varphi \tilde{\nabla}_W U, PU) + \cos^2 \theta g(\nabla_W U, U).$$

Then, using (13), we can derive:

$$g(h(U, W), FPU) = g((\tilde{\nabla}_W \varphi)U, PW) - g(\tilde{\nabla}_W \phi U, PU) + \cos^2 \theta (W \ln f)||U||^2.$$

Using (7) and the (11), the above equation then becomes:

$$g(h(U, W), FPU) = -g((\tilde{\nabla}_U \varphi)W, PU) - g(\nabla_W PU, PU) - g(\tilde{\nabla}_W FU, PU) + \cos^2 \theta (U \ln f)||W||^2.$$

By using (13), (9), and from (Equation (2.12) in [5]), we derive the following:

$$\begin{aligned} g(h(U, W), FPU) = &- g(\tilde{\nabla}_U \varphi W, PU) - g(\tilde{\nabla}_U W, \varphi PU) \\ &- \cos^2 \theta (W \ln f)||U||^2 + g(h(W, PU), FU) + \cos^2 \theta (W \ln f)||U||^2. \end{aligned}$$

From (9), (11), and Theorem 2.1 [5] for the slant submanifold, it is not hard to come by the following:

$$g(h(U, W), FPU) = g(A_{\varphi W} U, PU) + \cos^2 \theta g(\nabla_U W, U) - g(\tilde{\nabla}_U W, FPU) + g(h(W, PU), FU).$$

Finally, (8) and (13) implies that:

$$2g(h(U, W), FPU) = (W \ln f) \cos^2 \theta ||U||^2 + g(h(U, PU), \varphi W) + g(h(W, PU), FU),$$

which gives our assertion. □

Lemma 2. *Assume that $M = M_\perp \times_f M_\theta$ is a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then:*

$$g(h(U, W), FPU) = 2g(h(U, PU), \varphi W) - g(h(W, PU), FU),$$

for any $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$.

Proof. From (8) and (11), we have:

$$g(h(W, PU), FU) = -g(\varphi \tilde{\nabla}_W PU, U) - g(\tilde{\nabla}_W PU, PU),$$

for $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$. Then, the covariant derivative of φ and (13), and from (Equation (2.12) in [5]), we modified as:

$$g(h(W, PU), FU) = g((\tilde{\nabla}_W \varphi)PU, U) - g(\tilde{\nabla}_W \varphi PU, U) - \cos^2 \theta (W \ln f)||U||^2.$$

Taking into account (7) for the nearly cosymplectic manifold, the virtues of (8), (11), and Theorem 2.1 in [5], it follows that:

$$g(h(W, PU), FU) = g(h(U, W), FPU) - \cos^2 \theta (W \ln f)||U||^2 - g((\tilde{\nabla}_{PU} \varphi)W, X) + \cos^2 \theta g(\nabla_W U, U).$$

Using (13), one can obtain:

$$g(h(W, PU), FU) = g(h(U, W), FPU) - g(\tilde{\nabla}_{PU} \varphi W, U) - g(\tilde{\nabla}_{PU} W, \varphi U).$$

Then, from (9) and (11), it is clear that:

$$2g(h(W, PU), FU) = g(h(U, W), FPU) + g(h(U, PU), \varphi W) - g(\widetilde{\nabla}_{PU} W, PU).$$

Again, from (13) and (Equation (2.12) in [5]), the last equation reduces the new form:

$$2g(h(W, PU), FU) = g(h(U, W), FPU) - (W \ln f) \cos^2 \theta ||U||^2 + g(h(U, PU), \varphi W). \tag{14}$$

Now, rearrange X by PX in (14) and Theorem 2.1 from [5], and it will give us:

$$-2 \cos^2 \theta g(h(U, Z), FPU) = - (W \ln f) \cos^4 ||U||^2 - \cos^2 \theta g(h(W, PU), FU)$$
$$- \cos^2 \theta g(h(PU, U), \varphi W).$$

which implies:

$$2g(h(U, W), FPU) = (W \ln f) \cos^2 ||U||^2 + g(h(W, PU), FU) + g(h(PU, U), \varphi W). \tag{15}$$

From (14) and (15), it concludes that:

$$g(h(W, PU), FU) = 2g(h(U, PU), \varphi W) - g(h(U, W), FPU),$$

which completes the proof of lemma. $\square$

Lemma 3. *On a warped product pseudo-slant submanifold, $M = M_\perp \times_f M_\theta$ of a nearly cosymplectic manifold $\widetilde{M}$. Then:*

$$g(h(PU, U), \varphi W) = g(h(U, W), FPU) - \frac{1}{3} \cos^2 \theta (W \ln f) ||U||^2,$$

for any $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$.

Proof. From Lemmas 1 and 2, we can derive the proof of Lemma 3. $\square$

Lemma 4. *On a warped product pseudo-slant submanifold, $M = M_\perp \times_f M_\theta$ of a nearly cosymplectic manifold $\widetilde{M}$. Then:*

(i) $g(h(W, W), FU) = g(h(W, U), \varphi W),$
(ii) $g(h(W, W), FPU) = g(h(W, PU), \varphi W),$

for any $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$.

Proof. From (8) and (11), we obtain:

$$g(h(W, W), FU) = g(\widetilde{\nabla}_W W, \varphi U) - g(\widetilde{\nabla}_W W, PU),$$

for $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$. It can be extended further by the fact that W and PU are orthogonal, i.e.,

$$g(h(W, W), FU) = -g(\varphi \widetilde{\nabla}_W W, U) + g(\widetilde{\nabla}_W PU, W).$$

We also derived from the covariant derivative of φ and (8), that is:

$$g(h(W, W), FU) = g((\widetilde{\nabla}_W \varphi) W, U) - g(\widetilde{\nabla}_W \varphi W, U) + g(\nabla_W PU, W).$$

Then, from (7) and (13), we obtained:

$$g(h(W,W), FU) = g(A_{\varphi W}W, U) + (W \ln f)g(PU, W),$$

which implies:

$$g(h(W,W), FU) = g(A_{\varphi W}W, U). \tag{16}$$

That is, the first part of the proof of lemma ends up and part (ii) can be constructed by rearranging X by PX in (16). $\square$

Lemma 5. *Let $\widetilde{M}$ be a nearly cosymplectic manifold and $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold in $\widetilde{M}$. Thus,*

(i) $g(h(U,U), \varphi W) = g(h(W,U), FU)$;
(ii) $g(h(PU, PU), \varphi W) = g(h(W, PU), FPU)$;

for any $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$

Proof. Assuming $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$, we have:

$$g(h(U,U), \varphi W) = g(\widetilde{\nabla}_U U, \varphi W) = -g(\varphi \widetilde{\nabla}_U U, W)$$

From the covariant derivative of the tensor field φ, it follows that:

$$g(h(U,U), \varphi W) = g((\widetilde{\nabla}_U \varphi)U, W) - g(\widetilde{\nabla}_U \varphi U, W).$$

Taking into account (7) and (8), we have:

$$g(h(U,U), \varphi W) = g(\widetilde{\nabla}_U W, PU) - g(\widetilde{\nabla}_U FU, W).$$

Thus, from (13) and (9), it can easily be found that:

$$g(h(U,U), \varphi W) = (W \ln f)g(U, PU) + g(A_{FU}U, W).$$

Since U and PU are orthogonal and (10), we arrive at:

$$g(h(U,U), \varphi W) = g(h(U,W), FU), \tag{17}$$

which is the first part of lemma. Then, by switching X by PX in (17), we were able to find the proof of the second part of the lemma. $\square$

4. Main Proof of Inequality for Warped Product of the form $M_\perp \times_f M_\theta$

In this section, we establish a geometric inequality for warped product pseudo-slant submanifolds in terms of the symmetric bilinear form and warping functions with included immersion.

4.1. Proof of Theorem 1

Let $M = M_\perp \times_f M_\theta$ be an $(m+1)$-dimensional warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$, such that M_θ of dimension $d_1 = 2\beta$ and $M_\perp$ of dimension $d_2 = (\alpha + 1)$. We chose the orthonormal frames $\{e_1, e_2, \cdots, e_\alpha, e_{\alpha+1} = \xi, \}$ and $\{e_{\alpha+2} = e_1^*, \cdots, e_{\alpha+\beta+1} = e_\beta^*, e_{\alpha+\beta+2} = e_{\beta+1}^* = sec\theta Pe_1^*, \cdots, e_{\alpha+1+2\beta} = e_{2\beta}^* = sec\theta Pe_\beta^*\}$ of $TM_\perp$ and TM_θ, respectively. Thus, the orthonormal frames of the normal subbundles $\varphi(TM_\perp)$, $F(TM_\theta)$, and μ are denoted by $\{e_{m+1} = \bar{e}_1 = \varphi e_1, \cdots e_{m+\alpha} = \bar{e}_\alpha = \varphi e_\alpha\}$, $\{e_{m+\alpha+1} = \bar{e}_{\alpha+1} = \tilde{e}_1 = csc\theta Fe_1^*, \cdots, e_{m+\alpha+\beta} = \bar{e}_{\alpha+\beta} = \tilde{e}_\beta = $

$csc\theta Fe_{\beta}^*, e_{m+\alpha+\beta+1} = \bar{e}_{\alpha+\beta+1} = \tilde{e}_{\beta+1} = csc\theta sec\theta FPe_1^*, \cdots, e_{m+\alpha+2\beta} = \bar{e}_{\alpha+2\beta} = \tilde{e}_{2\beta} = csc\theta sec\theta FPe_{\beta}^*\}$, and $\{e_{2m-1} = \bar{e}_m, \cdots, e_{2n+1} = \bar{e}_{2(n-m+1)}\}$, respectively. From the definition of the second fundamental form, we have:

$$||h||^2 = ||h(\mathcal{D}^\theta, \mathcal{D}^\theta)||^2 + ||h(\mathcal{D}^\perp, \mathcal{D}^\perp)||^2 + 2||h(\mathcal{D}^\theta, \mathcal{D}^\perp)||^2. \tag{18}$$

After missing the second term of the above equation, we get:

$$||h||^2 \geq ||h(\mathcal{D}^\theta, \mathcal{D}^\theta)||^2 + 2||h(\mathcal{D}^\theta, \mathcal{D}^\perp)||^2.$$

Thus, changes in the above equation into a new form by using the definition of h lead to:

$$||h||^2 \geq \sum_{l=m+1}^{2n+1} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), e_l)^2 + 2 \sum_{l=m+1}^{2n+1} \sum_{r=1}^{2\beta} \sum_{s=1}^{\alpha+1} g(h(e_r^*, e_s), e_l)^2.$$

If we put $h(U, \xi) = 0$ in the second term in the components of $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$, and μ, the proceeding equation can be expressed as:

$$\begin{aligned}
||h||^2 \geq & 2 \sum_{l,s=1}^{\alpha} \sum_{r=1}^{2\beta} g(h(e_r^*, e_s), e_l)^2 + 2 \sum_{l,r=1}^{2\beta} \sum_{s=1}^{\alpha} g(h(e_r^*, e_s), e_l)^2 \\
& + 2 \sum_{l=m}^{2(n-m+1)} \sum_{r=1}^{2\beta} \sum_{s=1}^{\alpha+1} g(h(e_r^*, e_s), e_l)^2 + \sum_{l=1}^{\alpha} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2 \\
& + \sum_{l=\alpha+1}^{2\beta+\alpha} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2 + \sum_{l=m}^{2(n-m+1)} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2.
\end{aligned} \tag{19}$$

Considering only the first, second, and fourth terms, they then imply that:

$$||h||^2 \geq 2 \sum_{l,s=1}^{\alpha} \sum_{r=1}^{2\beta} g(h(e_r^*, e_s), e_l)^2 + 2 \sum_{l,r=1}^{2\beta} \sum_{s=1}^{\alpha} g(h(e_r^*, e_s), e_l)^2 + \sum_{l=1}^{\alpha} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2.$$

Then, using the components of the adapted frame for $F\mathcal{D}^\theta$, we derive:

$$\begin{aligned}
||h||^2 \geq & 2\csc^2\theta \sum_{s=1}^{\alpha} \sum_{l,r=1}^{\beta} g(h(e_r^*, e_s), Fe_l^*)^2 + 2\csc^2\theta \sec^4\theta \sum_{s=1}^{\alpha} \sum_{l,r=1}^{\beta} g(h(Pe_r^*, e_s), FPe_l^*)^2 \\
& + 2 \sum_{l,s=1}^{\alpha} \sum_{r=1}^{\beta} g(h(e_r^*, e_s), e_l)^2 + 2\sec^2\theta \sum_{l,s=1}^{\alpha} \sum_{r=1}^{\beta} g(h(Pe_r^*, e_s), e_l)^2 \\
& + \sum_{l=1}^{\alpha} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2.
\end{aligned}$$

After taking into account Lemmas 4 and 5, the above equation then takes the form:

$$\begin{aligned}
||h||^2 \geq & 2\csc^2\theta \sum_{s=1}^{\alpha} \sum_{l,r=1}^{\beta} g(h(e_r^*, e_l^*), \varphi e_s)^2 + 2\csc^2\theta \sec^4\theta \sum_{s=1}^{\alpha} \sum_{l,r=1}^{\beta} g(h(Pe_r^*, Pe_l^*), \varphi e_s)^2 \\
& + 2 \sum_{l,s=1}^{\alpha} \sum_{r=1}^{\beta} g(h(e_l, e_s), Fe_r^*)^2 + 2\sec^2\theta \sum_{l,s=1}^{\alpha} \sum_{r=1}^{\beta} g(h(e_l, e_s), FPe_r^*)^2 \\
& + \sum_{l=1}^{\alpha} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2.
\end{aligned} \tag{20}$$

Now, if we just consider the last part and the rest of the parts are left, then we have:

$$||h||^2 \geq \sum_{l=1}^{\alpha} \sum_{r,s=1}^{2\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2.$$

Again, using the components of $\mathcal{D}^\theta$, we finish at:

$$||h||^2 \geq \sum_{l=1}^{\alpha} \sum_{r,s=1}^{\beta} g(h(e_r^*, e_s^*), \bar{e}_l)^2 + \sec^2 \theta \sum_{l=1}^{\alpha} \sum_{r,s=1}^{\beta} g(h(Pe_r^*, e_s^*), \bar{e}_l)^2$$

$$+ \sec^2 \theta \sum_{l=1}^{\alpha} \sum_{r,s=1}^{\beta} g(h(e_r^*, Pe_s^*), \bar{e}_l)^2 + \sec^4 \theta \sum_{l=1}^{p} \sum_{r,s=1}^{\beta} g(h(Pe_r^*, Pe_s^*), \bar{e}_l)^2. \tag{21}$$

If we leave the first and last terms of the right-hand side in (21) and just add the remaining terms, then we obtain:

$$||h||^2 \geq 2 \sec^2 \sum_{l=1}^{\alpha} \sum_{r,s=1}^{\beta} g(h(Pe_r^*, e_s^*), \bar{e}_l)^2.$$

With the help of Lemma 3 in the above inequality, we get:

$$||h||^2 \geq \frac{2}{9} \cos^2 \theta \sum_{l=1}^{\alpha} \sum_{s=1}^{\beta} (\bar{e}_l \ln f)^2 g(e_s^*, e_s^*)^2 + 2 \sum_{l=1}^{\alpha} \sum_{s=1}^{\beta} g(h(\bar{e}_l, e_s^*), FPe_s^*)^2$$

$$- \frac{4}{3} \sum_{l=1}^{\alpha} \sum_{s=1}^{\beta} (\bar{e}_l \ln f) g(h(\bar{e}_l, e_s^*), FPe_s^*). \tag{22}$$

Now, we can add and subtract the term $\xi \ln f$ in the first part of (22), and we have

$$||h||^2 \geq \frac{2}{9} \cos^2 \theta \sum_{s=1}^{\alpha+1} \sum_{l=1}^{\beta} (\bar{e}_s \ln f)^2 g(e_l^*, e_l^*)^2 - \frac{2}{9} \cos^2 \theta \sum_{l=1}^{\beta} (\xi \ln f)^2 g(e_l^*, e_l^*)^2.$$

It is known that $\xi \ln f = 0$ for a nearly cosymplectic manifold (see [14]). Thus, the above inequality reduces to:

$$||h||^2 \geq \frac{d_1}{9} \cos^2 \theta ||\nabla \ln f||^2. \tag{23}$$

If (23) holds, then from the missing terms in (22), we obtain some conditions from the second and third terms, i.e., $(\bar{e}_j \ln f) g(h(\mathcal{D}^\perp, \mathcal{D}^\theta), FD^\theta) = 0$. This implies that either $\bar{e}_j \ln f = 0$ or $g(h(\mathcal{D}^\perp, \mathcal{D}^\theta), FD^\theta) = 0$. If we take $\bar{e}_j \ln f = 0$, it means that f is a constant function on M, which is a contradiction for the non-trivial warped product pseudo-slant submanifold. Hence, we get $g(h(\mathcal{D}^\perp, \mathcal{D}^\theta), FD^\theta) = 0$, and also from the third term of (19), we derive:

$$h(\mathcal{D}^\perp, \mathcal{D}^\theta) \subseteq \Gamma(\varphi TM_\perp), \tag{24}$$

which is (2). Because of the terms which were left in (18), $M_\perp$ is a totally geodesic submanifold in $\widetilde{M}$. On the other hand, leaving the terms in (19), (20), and (21) implies that:

$$g(h(\mathcal{D}^\theta, \mathcal{D}^\theta), \varphi \mathcal{D}^\perp) = 0, \quad g(h(\mathcal{D}^\theta, \mathcal{D}^\theta), \mu) = 0,$$

$$\&$$

$$g(h(\mathcal{D}^\perp, \mathcal{D}^\perp), FD^\theta) = 0, \quad g(h(\mathcal{D}^\perp, \mathcal{D}^\perp), \mu) = 0,$$

which means that:

$$h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) \subseteq \Gamma(F(TM_{\theta})) \ \ \& \ \ h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta}) \subseteq \Gamma(\varphi TM_{\perp}). \tag{25}$$

From this, we get the required condition (3). Thus, the equality cases are verified.

4.2. Proof of Theorem 2

Let us assume that f is a non-constant warping function. From the minimum principle on the first eigenvalue λ_1, one obtains: (p. 186 in [22]).

$$\lambda_1 \int_{M^n} (\ln f)^2 dV \leq \int_{M^n} \|\nabla \ln f\|^2 dV. \tag{26}$$

with equality holding if, and only if $\Delta \ln f = \lambda_1 \ln f$. On the other hand, taking the integration of Equation (1) along the volume element dV, we have:

$$\int_{M_{\perp}} ||h||^2 dV \geq \frac{2\beta}{9} \cos^2 \theta \int_{M_{\perp}} ||\nabla \ln f||^2 dV. \tag{27}$$

Thus, (26) and (27) implies that:

$$\int_{M_{\perp}} ||h||^2 dV \geq \frac{2\beta}{9} \cos^2 \theta \lambda_1 \int_{M_{\perp}} (\ln f)^2 dV \tag{28}$$

The above implies Equation (4). This completes the proof of the Theorem.

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