mathematics

Article

Convergence Theorems for Modified Inertial Viscosity Splitting Methods in Banach Spaces

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Received: 6 January 2019; Accepted: 30 January 2019; Published: 8 February 2019

Abstract: In this article, we study a modified viscosity splitting method combined with inertial extrapolation for accretive operators in Banach spaces and then establish a strong convergence theorem for such iterations under some suitable assumptions on the sequences of parameters. As an application, we extend our main results to solve the convex minimization problem. Moreover, the numerical experiments are presented to support the feasibility and efficiency of the proposed method.

Keywords: Banach spaces; viscosity splitting method; inertial method; accretive operators

MSC: 47H09; 47H10; 47H04

1. Introduction

Throughout this paper, we let E be a real Banach space with norm $\|.\|$ and E^* be its dual space. The normalized duality mapping J from E into 2^{E^*} is defined by the following equation:

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|f\|\|x\| = \|x\|^2\} \ \ \forall x \in E.$$

we denote the generalized duality pairing between E and E^* by $\langle .,. \rangle$ and the single-valued duality mapping by j.

The inclusion problem is to find $x \in E$ such that

$$0 \in (A + B)x$$

where $A : E \to E$ is an operator and $B : E \to 2^E$ is a set-valued operator. Please note that on the one hand, this problem takes into account some special cases, such as variational inequalities, convex programming, minimization problem, and split feasibility problem [1–3]. On the other hand, as an important branch of nonlinear functional analysis and optimization theory, it has been studied numerous times in the literature to solve the real-world problem, such as machine learning, image reconstruction, and signal processing; see [4–7] and the references therein.

In 2012, Takashashi et al. [8] studied a Halpern-type iterative method for an α-inverse strongly monotone mapping A and a maximal monotone operator B in a Hilbert space as follows:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n) J^B_{r_n}(x_n - r_n A x_n)),$$

under certain conditions, the algorithm was showed to converge strongly to a solution of $A + B$. Furthermore, Lopez et al. [9] introduced the following method for accretive operators:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J^B_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n),$$

they studied strong convergence theorems for Halpern-type splitting methods in Banach spaces. In 2016, Pholasa et al. [10] extended the above results [8,9] and studied the modified forward-backward splitting methods in Banach spaces:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n) J^B_{r_n}(x_n - r_n Ax_n)),$$

it was proved that x_n converges strongly to a point $z = Q(u)$ under some mild conditions, where Q is the sunny nonexpansive retraction.

Inertial extrapolation is an important technique to speed up the convergence rate [11–14]. Recently, the fast-iterative algorithms by using inertial extrapolation studied by some authors [15–17]. For instance, in 2003, Moudafi et al. [18] studied the following inertial proximal point algorithm of a maximal monotone operator:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n T)^{-1}(y_n). \end{cases}$$

If λ_n is non-decreasing and $\theta_n \in [0, 1)$ is chosen such that

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty,$$

then x_n converges to a zero point of T. In 2015, Lorenz et al. [19] applied inertial extrapolation technique to forward-backward algorithm for monotone operators in Hilbert spaces. They proved that the iterative process defined by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n B)^{-1}(y_n - r_n Ay_n). \end{cases}$$

converges weakly to a solution of the inclusion $0 \in (A + B)(x)$. In 2018, Cholamjiak et al. [20] proposed a Halpern-type inertial iterative method for monotone operators in Hilbert spaces and they proved the strong convergence of the algorithm.

Inspired and motivated by the above-mentioned works, we apply inertial extrapolation algorithms and viscosity approximation to give an extension, and then we study a modified splitting method for accretive operators in Banach spaces. The strong convergence theorems for such iterations are established and some applications including the numerical experiments are presented to support our main theorem.

2. Preliminaries

Recall that a Banach space E is said to be uniformly convex if for any $\epsilon \in (0, 2]$, there exists a $\delta = \delta_E(\epsilon) > 0$ such that $x, y \in E$ with $\|x\| = \|y\| = 1$, and $\|x - y\| \geq \epsilon$, then $\|x + y\|/2 \leq 1 - \delta$. We denote the modulus of smoothness $\rho_E : R^+ \to R^+$ of E as follows:

$$\rho_E(t) = \sup\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \|x\| = 1, \|y\| = 1\},$$

for $1 < q \leq 2$, a Banach space E is said to be q-uniformly smooth if there exists a constant $c_q > 0$ such that $\rho_E(t) \leq c_q t^q, t > 0$. E is said to be uniformly smooth if $\lim_{n\to\infty} \rho_E(t)/t = 0$. It is obvious that q-uniformly smooth Banach space must be uniformly smooth and E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable.

Let I be the identity operator. We denote by $D(A) = \{z \in E : Az \neq \emptyset\}$, $R(A) = \bigcup\{Az : z \in D(A)\}$ the domain and range of an operator $A \subset E \times E$, respectively. A is called accretive if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0, \forall u \in Ax, v \in Ay.$$

An accretive operator A is called α-inverse strongly accretive, if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq \alpha\|u - v\|^2, \forall u \in Ax, v \in Ay.$$

It is well-known that an accretive operator A is m-accretive if $R(I + rA) = E$ for all r >0. If A is an accretive operator which satisfies the range condition, then, for each $r > 0$, the mapping $J_r^A : R(I + rA) \to D(A)$ is defined by $J_r^A = (I + rA)^{-1}$, which is called the resolvent operator of A.

Let C be a nonempty, closed and convex subset of E, and let D be a nonempty subset of C. A mapping $T : C \to D$ is called a retraction of C onto D, if for all $x \in D$, there is $Tx = x$. We called T is sunny if T has the following property: $T(tx + (1-t)Tx) = Tx$ for each $x \in C$ and $t \geq 0$ whenever $tx + (1-t)Tx \in C$. It is known that a sunny nonexpansive retraction is a sunny retraction which is also nonexpansive.

The following lemmas are needed to prove our results.

Lemma 1 ([21])**.** *Let E be a smooth Banach space. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \forall x, y \in E.$$

Lemma 2 ([22])**.** *For any $r > 0$, give $0 < s \leq r$ and $x \in E$, if*

$$T_r := J_r^B(I - rA) = (I + rB)^{-1}(I - rA)$$

then $Fix(T_r) = (A+B)^{-1}(0)$. In addition, there holds the relation

$$\|x - T_s x\| \leq 2\|x - T_r x\|.$$

Lemma 3 ([23])**.** *If a Banach space E is uniformly smooth, then the duality mapping J is single valued and norm-to-norm uniformly continuous on each bounded subset of E.*

Lemma 4 ([21])**.** *Let E be a uniformly smooth Banach space and $T : C \to C$ be a nonexpansive mapping with a fixed point. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$ and $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tf(x) + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define a mapping $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D.*

Lemma 5 ([24])**.** *Assume $\{a_n\} \subset R^+$, $\{\delta_n\} \subset (0,1)$ and $\{b_n\} \subset R$ be the sequences such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n, n \geq 0,$$

(i) If $\Sigma_{n=0}^{\infty}\delta_n = \infty$; (ii) $limsup_{n\to\infty}\frac{b_n}{\delta_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$; then $lim_{n\to\infty}a_n = 0$.

Lemma 6 ([25])**.** *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\tau_n, n \geq 1,$$

$$s_{n+1} \leq s_n - \eta_n + d_n, n \geq 1,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$, $\{\eta_n\}$ *is a sequence of nonnegative real numbers and* $\{\tau_n\}$, $\{d_n\}$ *are real sequences such that*

(i) $\Sigma_{n=0}^{\infty}\gamma_n = \infty$,
(ii) $\lim_{n\to\infty} d_n = 0$,
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 7 ([26]). *Let* A *be a single-valued* α*-isa in a real uniformly convex Banach space with Fréchet differentiable norm. Then, for all* $x, y \in E$ *and given* $s > 0$, *there exists a continuous, strictly increasing and convex function* $\Phi : R^+ \to R^+$ *with* $\Phi(0) = 0$ *such that*

$$\|T_r x - T_r y\|^2 \leq \|x-y\|^2 - r(2\alpha - rk)\|Ax - Ay\|^2 - \Phi(\|(I - J_r^B)(I - rA)x - (I - J_r^B)(I - rA)y\|)$$

where k *is the uniform smoothness coefficient of* E.

Lemma 8 ([27]). *Let* E *be a uniformly convex Banach space. Then, for all* $x, y \in E$ *and* $t \in [0,1]$, *there exists a convex continuous and strictly increasing function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$ *such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|).$$

3. Main Results

Theorem 1. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* $A : E \to E$ *be an* α*-inverse-strongly accretive mapping and* $B : E \to 2^E$ *be an m-accretive operator. Assume that* $\Omega = (A+B)^{-1}(0) \neq \emptyset$. *Let* $f : E \to E$ *be a contraction with coefficient* $\rho \in [0,1)$ *and* $\{\beta_n\} \subset (0,1)$, $\{\alpha_n\}$,$\{\delta_n\}$ *be real number sequences in* $[0,1)$ *and* $r_n \subset (0,+\infty)$. *Define a sequence* $\{x_n\}$ *in* E *as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \delta_n w_n + (1-\delta_n)J_{r_n}^B(w_n - r_n A w_n), \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n. \end{cases} \tag{1}$$

for all $n \in N$, *where* $x_0, x_1 \in E$ *and* $J_{r_n}^B = (I + r_n B)^{-1}$. *Assume that the following conditions hold:*

$$(i)\ \sum_{n=1}^{\infty} \alpha_n\|x_n - x_{n-1}\| < \infty;$$

$$(ii)\ \lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty;$$

$$(iii) 0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < \frac{2\alpha}{k};$$

$$(iv)\ \limsup_{n\to\infty} \delta_n < 1.$$

Then the sequence $\{x_n\}$ *converges strongly to* $z = Q(f(z))$, *where* Q *is the sunny nonexpansive retraction of* E *onto* Ω.

Proof. Let $T_n = J_{r_n}^B(I - r_n A)$, $z = Q(f)$. Then, we have

$$\begin{aligned} \|y_n - z\| &= \|\delta_n w_n + (1-\delta_n)T_n w_n - z\| \\ &\leq \delta_n\|w_n - z\| + (1-\delta_n)\|T_n w_n - z\| \\ &\leq \delta_n\|w_n - z\| + (1-\delta_n)\|w_n - z\| \\ &\leq \|w_n - z\| \end{aligned}$$

$$\leq \|x_n + \alpha_n(x_n - x_{n-1}) - z\|$$
$$\leq \|x_n - z\| + \alpha_n\|(x_n - x_{n-1})\|.$$

In view of Lemma 2, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n f(x_n) + (1-\beta_n)y_n - z\| \\
&\leq \beta_n\|f(x_n) - z\| + (1-\beta_n)\|y_n - z\| \\
&\leq \beta_n\|f(x_n) - f(z)\| + \beta_n\|f(z) - z\| + (1-\beta_n)\|y_n - z\| \\
&\leq \beta_n\rho\|x_n - z\| + \beta_n\|f(z) - z\| + (1-\beta_n)(\|x_n - z\| + \alpha_n\|(x_n - x_{n-1})\|) \\
&= [1 - \beta_n(1-\rho)]\|x_n - z\| + \beta_n\|f(z) - z\| + (1-\beta_n)\alpha_n\|(x_n - x_{n-1})\|.
\end{aligned}$$

From the restriction and Lemma 5, we find that $\{x_n\}$ is bounded. Hence $\{w_n\}$, $\{y_n\}$ are also bounded.

Using the inequality in Lemma 1 and Lemma 8 , we find that

$$\begin{aligned}
\|w_n - z\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - z\|^2 \\
&\leq \|x_n - z\|^2 + 2\alpha_n\langle x_n - x_{n-1}, j(w_n - z)\rangle,
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n f(x_n) + (1-\beta_n)y_n - z\|^2 \\
&= \|\beta_n(f(x_n) - f(z)) + (1-\beta_n)(y_n - z) + \beta_n(f(z) - z)\|^2 \\
&\leq \|\beta_n(f(x_n) - f(z)) + (1-\beta_n)(y_n - z)\|^2 + 2\beta_n\langle f(z) - z, j(x_{n+1} - z)\rangle \\
&\leq \beta_n\|f(x_n) - f(z)\|^2 + (1-\beta_n)\|y_n - z\|^2 \\
&- \beta_n(1-\beta_n)g(\|(f(x_n) - f(z)) - (y_n - z)\|) + 2\beta_n\langle f(z) - z, j(x_{n+1} - z)\rangle \\
&\leq \beta_n\rho^2\|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 + 2\beta_n\langle f(z) - z, j(x_{n+1} - z)\rangle.
\end{aligned} \tag{3}$$

In view of Lemmas 7 and 8, we get

$$\begin{aligned}
\|y_n - z\|^2 &= \|\delta_n w_n + (1-\delta_n)T_n w_n - z\|^2 \\
&\leq \delta_n\|w_n - z\|^2 + (1-\delta_n)\|T_n w_n - z\|^2 \\
&\leq \delta_n\|w_n - z\|^2 + (1-\delta_n)[\|w_n - z\|^2 \\
&- r_n(2\alpha - r_n k)\|Aw_n - Az\|^2 - \Phi(\|(I - J_{r_n}^B)(I - r_n A)w_n - (I - J_{r_n}^B)(I - r_n A)z\|)] \\
&= \|w_n - z\|^2 - (1-\delta_n)r_n(2\alpha - r_n k)\|Aw_n - Az\|^2 \\
&- (1-\delta_n)\Phi(\|w_n - r_n Aw_n - T_n w_n + r_n Az\|).
\end{aligned} \tag{4}$$

Substitute (2), (4) into (3), we get

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \beta_n\rho^2\|x_n - z\|^2 + 2\beta_n\langle f(z) - z, j(x_{n+1} - z)\rangle \\
&+ (1-\beta_n)\|w_n - z\|^2 - (1-\beta_n)(1-\delta_n)r_n(2\alpha - r_n k)\|Aw_n - Az\|^2 \\
&- (1-\beta_n)(1-\delta_n)\Phi(\|w_n - r_n Aw_n - T_n w_n + r_n Az\|) \\
&\leq \beta_n\rho^2\|x_n - z\|^2 + 2\beta_n\langle f(z) - z, j(x_{n+1} - z)\rangle + (1-\beta_n)\|x_n - z\|^2 \\
&+ 2(1-\beta_n)\alpha_n\langle x_n - x_{n-1}, j(w_n - z)\rangle - (1-\beta_n)(1-\delta_n)r_n(2\alpha - r_n k)\|Aw_n - Az\|^2 \\
&- (1-\beta_n)(1-\delta_n)\Phi(\|w_n - r_n Aw_n - T_n w_n + r_n Az\|) \\
&= (1 - \beta_n(1-\rho^2))\|x_n - z\|^2 + 2\beta_n\langle f(z) - z, j(x_{n+1} - z)\rangle \\
&+ 2(1-\beta_n)\alpha_n\langle x_n - x_{n-1}, j(w_n - z)\rangle - (1-\beta_n)(1-\delta_n)r_n(2\alpha - r_n k)\|Aw_n - Az\|^2 \\
&- (1-\beta_n)(1-\delta_n)\Phi(\|w_n - r_n Aw_n - T_n w_n + r_n Az\|).
\end{aligned} \tag{5}$$

We can check that $\beta_n(1-\rho^2)$is in $(0,1)$, by the condition (iii), we can show that $(1-\beta_n)(1-\delta_n)r_n(2\alpha - r_n k)$ is positive. Then, we have

$$\begin{aligned} \|x_{n+1}-z\|^2 \leq\ & (1-\beta_n(1-\rho^2))\|x_n-z\|^2 + 2\beta_n\langle f(z)-z, j(x_{n+1}-z)\rangle \\ & + 2(1-\beta_n)\alpha_n\langle x_n - x_{n-1}, j(w_n - z)\rangle, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \|x_{n+1}-z\|^2 \leq\ & \|x_n-z\|^2 - (1-\beta_n)(1-\delta_n)r_n(2\alpha - r_n k)\|Aw_n - Az\|^2 \\ & - (1-\beta_n)(1-\delta_n)\Phi(\|w_n - r_n Aw_n - T_n w_n + r_n Az\|) \\ & + 2\beta_n\langle f(z)-z, j(x_{n+1}-z)\rangle + 2(1-\beta_n)\alpha_n\langle x_n - x_{n-1}, j(w_n - z)\rangle. \end{aligned} \tag{7}$$

For each $n \geq 1$, let

$$\begin{aligned} & s_n = \|x_n - z\|^2, \gamma_n = \beta_n(1-\rho^2), \\ & \tau_n = \frac{2}{1-\rho^2}\langle f(z)-z, j(x_{n+1}-z)\rangle + \frac{2\alpha_n(1-\beta_n)}{\beta_n(1-\rho^2)}\langle x_n - x_{n-1}, j(w_n - z)\rangle, \\ & \eta_n = (1-\beta_n)(1-\delta_n)r_n(2\alpha - r_n k)\|Aw_n - Az\|^2 \\ & + (1-\beta_n)(1-\delta_n)\Phi(\|w_n - r_n Aw_n - T_n w_n + r_n Az\|), \\ & d_n = 2\beta_n\langle f(z)-z, j(x_{n+1}-z)\rangle + 2(1-\beta_n)\alpha_n\langle x_n - x_{n-1}, j(w_n - z)\rangle. \end{aligned} \tag{8}$$

we find from (6), (7) that

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\tau_n,$$

and also

$$s_{n+1} \leq s_n - \eta_n + d_n.$$

Notice that $\sum_{n=1}^{\infty}\beta_n = \infty$, we see that $\sum_{n=1}^{\infty}\gamma_n = \infty$. By the boundedness of $\{w_n\}$, $\{x_n\}$ and the restriction $\lim_{n\to\infty}\beta_n = 0$, implies that $\lim_{n\to\infty} d_n = 0$.

On the other hand, using Lemma 6, it remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \leq 0$, for any subsequence $\{n_k\} \subset \{n\}$. Let η_{n_k} be a subsequence of η_n such that $\lim_{k\to\infty}\eta_{n_k} = 0$. It follows from the restrictions and the property of ϕ, we derive from (8) the following

$$\lim_{k\to\infty}\|Aw_{n_k} - Az\| = 0 = \lim_{k\to\infty}\|w_{n_k} - r_{n_k}Aw_{n_k} - T_{n_k}w_{n_k} + r_{n_k}Az\| = 0.$$

By the triangle inequality, it turns out that

$$\lim_{k\to\infty}\|T_{n_k}w_{n_k} - w_{n_k}\| = 0.$$

and moreover, since $0 < \liminf_{n\to\infty} r_n$, there exists $\epsilon > 0$, such that $r_n \geq \epsilon$ for all $n > 0$, in view of the inequality in Lemma 2, we have

$$\|T_\epsilon w_{n_k} - w_{n_k}\| \leq 2\|T_{n_k}w_{n_k} - w_{n_k}\|.$$

It turns out that

$$\limsup_{k\to\infty}\|T_\epsilon w_{n_k} - w_{n_k}\| \leq 0.$$

Therefore, we can get

$$\|T_\epsilon w_{n_k} - w_{n_k}\| = 0. \tag{9}$$

Please note that

$$\begin{aligned}\|T_\epsilon w_{n_k} - x_{n_k}\| &\le \|T_\epsilon w_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \\ &\le \|T_\epsilon w_{n_k} - w_{n_k}\| + \alpha_n\|x_{n_k} - x_{n_k-1}\|\end{aligned}$$

We get from condition (i) and (9) that

$$\lim_{k\to\infty} \|T_\epsilon w_{n_k} - x_{n_k}\| = 0. \tag{10}$$

Put $z_t = tf(z_t) + (1-t)T_\epsilon z_t$ for any $t \in (0,1)$. Apply Lemma 4, we get $z_t \to Q(f) = z, t \to 0$. Then we have

$$\begin{aligned}\|z_t - x_{n_k}\|^2 &= \|tf(z_t) + (1-t)T_\epsilon z_t - x_{n_k}\|^2 \\ &= \|t(f(z_t) - x_{n_k}) + (1-t)(T_\epsilon z_t - x_{n_k})\|^2 \\ &\le (1-t)^2\|T_\epsilon z_t - x_{n_k}\|^2 + 2t\langle f(z_t) - x_{n_k}, j(z_t - x_{n_k})\rangle \\ &= (1-t)^2\|T_\epsilon z_t - x_{n_k}\|^2 + 2t\langle f(z_t) - z_t, j(z_t - x_{n_k})\rangle + 2t\langle z_t - x_{n_k}, j(z_t - x_{n_k})\rangle \\ &\le (1-t)^2(\|T_\epsilon z_t - T_\epsilon w_{n_k}\| + \|T_\epsilon w_{n_k} - x_{n_k}\|)^2 \\ &+ 2t\langle f(z_t) - z_t, j(z_t - x_{n_k})\rangle + 2t\langle z_t - x_{n_k}, j(z_t - x_{n_k})\rangle \\ &\le (1-t)^2(\|z_t - w_{n_k}\| + \|T_\epsilon w_{n_k} - x_{n_k}\|)^2 \\ &+ 2t\langle f(z_t) - z_t, j(z_t - x_{n_k})\rangle + 2t\|z_t - x_{n_k}\|^2 \\ &\le (1-t)^2(\|z_t - x_{n_k}\| + \alpha_n\|x_{n_k} - x_{n_k-1}\| + \|T_\epsilon w_{n_k} - x_{n_k}\|)^2 \\ &+ 2t\langle f(z_t) - z_t, j(z_t - x_{n_k})\rangle + 2t\|z_t - x_{n_k}\|^2\end{aligned}$$

This implies that

$$\begin{aligned}\langle z_t - f(z_t), j(z_t - x_{n_k})\rangle \le \frac{(1-t)^2}{2t}&(\|z_t - x_{n_k}\| + \alpha_n\|x_{n_k} - x_{n_k-1}\| \\ &+ \|T_\epsilon w_{n_k} - x_{n_k}\|)^2 + \frac{2t-1}{2t}\|z_t - x_{n_k}\|^2.\end{aligned} \tag{11}$$

From (10), (11) we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle z_t - f(z_t), j(z_t - x_{n_k})\rangle &\le \frac{(1-t)^2}{2t}M^2 + + \frac{2t-1}{2t}M^2 \\ &= \frac{t}{2}M^2 \to 0, as\ t \to 0,\end{aligned} \tag{12}$$

for some $M > 0$ large enough. Since the duality mapping J is norm-to-norm uniformly continuous on bounded sets of E, we see that $\|j(z_t - x_{n_k}) - j(z - x_{n_k})\| \to 0, t \to 0$. Then, we have that

$$\begin{aligned}&\|\langle z_t - f(z_t), j(z_t - x_{n_k})\rangle - \langle z - f(z_t), j(z - x_{n_k})\rangle\| \\ &= \|\langle z_t - z + z - f(z_t), j(z_t - x_{n_k})\rangle - \langle z - f(z_t), j(z - x_{n_k})\rangle\| \\ &\le \|\langle z_t - z, j(z_t - x_{n_k})\rangle\| + \|\langle z - f(z_t), j(z_t - x_{n_k})\rangle - \langle z - f(z_t), j(z - x_{n_k})\rangle\| \\ &\le \|z_t - z\|\|z_t - x_{n_k}\| + \|z - f(z_t)\|\|j(z_t - x_{n_k}) - j(z - x_{n_k})\|\end{aligned} \tag{13}$$

From (12), (13) and let $t \to 0$, we get that

$$\limsup_{k\to\infty} \langle z - f(z), j(z - x_{n_k}) \rangle \le 0. \tag{14}$$

On the other hand, we have

$$\begin{aligned}
&\|y_{n_k} - x_{n_k}\| \\
&= \|\delta_{n_k} w_{n_k} + (1-\delta_{n_k}) T_{n_k} w_{n_k} - x_{n_k}\| \\
&\le \delta_{n_k}\|w_{n_k} - x_{n_k}\| + (1-\delta_{n_k})\|T_{n_k} w_{n_k} - x_{n_k}\| \\
&\le \delta_{n_k}\|w_{n_k} - x_{n_k}\| + (1-\delta_{n_k})\|T_{n_k} w_{n_k} - w_{n_k}\| + (1-\delta_{n_k})\|w_{n_k} - x_{n_k}\| \\
&= \|w_{n_k} - x_{n_k}\| + (1-\delta_{n_k})\|T_{n_k} w_{n_k} - w_{n_k}\| \\
&\le \alpha_{n_k}\|x_{n_k} - x_{n_{k-1}}\| + (1-\delta_{n_k})\|T_{n_k} w_{n_k} - w_{n_k}\|
\end{aligned}$$

$$\begin{aligned}
&\|x_{n_k+1} - x_{n_k}\| \\
&= \|\beta_{n_k} f(x_{n_k}) + (1-\beta_{n_k}) y_{n_k} - x_{n_k}\| \\
&\le \beta_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\beta_{n_k})\|y_{n_k} - x_{n_k}\| \\
&\le \beta_{n_k}\|f(x_{n_k}) - f(z)\| + \beta_{n_k}\|f(z) - x_{n_k}\| + (1-\beta_{n_k})\|y_{n_k} - x_{n_k}\| \\
&\le \beta_{n_k}\|f(x_{n_k}) - f(z)\| + \beta_{n_k}\|f(z) - x_{n_k}\| \\
&+ (1-\beta_{n_k})\alpha_{n_k}\|x_{n_k} - x_{n_{k-1}}\| + (1-\beta_{n_k})(1-\delta_{n_k})\|T_{n_k} w_{n_k} - w_{n_k}\|
\end{aligned}$$

From condition (i),(ii) and (9), we have

$$\lim_{n\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{15}$$

From (14) and (15), we obtain

$$\limsup_{k\to\infty} \langle z - f(z), j(z - x_{n_k+1}) \rangle \le 0.$$

This implies that $\limsup_{k\to\infty} \tau_{n_k} \le 0$ that means by Lemma 6, we get $\lim_{n\to\infty} s_n = 0$. Hence, we see that $x_n \to z, n \to \infty$. This finishes the proof. $\square$

Corollary 1. *Let E be a uniformly convex and uniformly smooth Banach space. Let $A : E \to E$ be an α-inverse-strongly accretive mapping and $B : E \to 2^E$ be an m-accretive operator. Assume that $\Omega = (A+B)^{-1}(0) \neq \emptyset$. Let $\{\beta_n\} \subset (0,1)$, $\{\alpha_n\}$,$\{\delta_n\}$ be real number sequences in $[0,1)$ and $r_n \subset (0,+\infty)$. Define a sequence $\{x_n\}$ in E as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \delta_n w_n + (1-\delta_n) J^B_{r_n}(w_n - r_n A w_n), \\ x_{n+1} = \beta_n u + (1-\beta_n) y_n. \end{cases}$$

for all $n \in N$, where $u, x_0, x_1 \in E$ and $J^B_{r_n} = (I + r_n B)^{-1}$. Assume that the following conditions hold:

$$\begin{aligned}
&(i) \sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| < \infty; (ii) \lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty; \\
&(iii)\ 0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < \frac{2\alpha}{k}; (iv) \limsup_{n\to\infty} \delta_n < 1.
\end{aligned}$$

Then the sequence $\{x_n\}$ converges strongly to $z = Q(u)$, where Q is the sunny nonexpansive retraction of E onto Ω.

Proof. In this case, the map $f : E \to E$ defined by $f(x) = u, \forall x \in E$ is a strict contraction with constant $\rho = 0$. The proof follows from Theorem 1 above. □

Corollary 2. *Let H be a uniformly convex and uniformly smooth Hilbert space. Let $A : H \to H$ be an α-inverse-strongly monotone operator and $B : H \to 2^H$ be a maximal monotone operator. Assume that $\Omega = (A + B)^{-1}(0) \neq \emptyset$. Let $f : H \to H$ be a contraction with coefficient $\rho \in [0, 1)$ and $\{\beta_n\} \subset (0, 1)$, $\{\alpha_n\}$,$\{\delta_n\}$ be real number sequences in $[0, 1)$ and $r_n \subset (0, 2\alpha)$. Define a sequence $\{x_n\}$ in E as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \delta_n w_n + (1 - \delta_n)J^B_{r_n}(w_n - r_n A w_n), \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n. \end{cases}$$

for all $n \in N$, where $x_0, x_1 \in E$ and $J^B_{r_n} = (I + r_n B)^{-1}$. Assume that the following conditions hold:

$$(i) \sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| < \infty; (ii) \lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty;$$
$$(iii)\ 0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < \frac{2\alpha}{k}; (iv) \limsup_{n\to\infty} \delta_n < 1.$$

Then the sequence $\{x_n\}$ converges strongly to $z = P(f(z))$, where P is the metric projection of H onto Ω.

Proof. We only need to replace Banach space E with Hilbert space H in the proof of Theorem 1. □

Corollary 3. *(Convex minimization problem) Let H be a real Hilbert space. Let $f : H \to R$ be a convex and differentiable function with K-Lipschitz continuous gradient ∇f and $g : H \to R$ a convex and lower semi-continuous function which $f + g$ attains a minimizer.. Let$\{\beta_n\} \subset (0, 1)$, $\{\alpha_n\}$,$\{\delta_n\}$ be real number sequences in $[0, 1)$ and $r_n \subset (0, 2\alpha)$. Define a sequence $\{x_n\}$ in H as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \delta_n w_n + (1 - \delta_n)J^{\partial g}_{r_n}(w_n - r_n \nabla f(w_n)), \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n. \end{cases}$$

for all $n \in N$, where $x_0, x_1 \in E$ and $J^B_{r_n} = (I + r_n B)^{-1}$. Assume that the following conditions hold:

$$(i) \sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| < \infty; (ii) \lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty;$$
$$(iii)\ 0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\alpha; (iv) \limsup_{n\to\infty} \delta_n < 1.$$

Then the sequence $\{x_n\}$ converges strongly to a minimizer of $f + g$.

Proof. We get that gradient ∇f is K-Lipschitz continuous, then it is $\frac{1}{K}$ inverse strongly monotone. and g is a convex and lower semi-continuous function, so ∂g is maximal monotone. Thus, let $A = \nabla f$ and $B = \partial g$ in Theorem 1, the conclusion of Theorem 1 still holds. □

4. Applications and Numerical Experiments

In this section, we give a concrete example of the numerical results to support the main theorem. Furthermore, we give it to compare the efficiency of our proposed algorithm with the algorithm of Pholasa et al. [10]. And we also show the algorithm presented in this paper converges more quickly. The whole codes are written by Matlab R2013b. All the results are carried out by personal computer with Intel(R) Core(TM) i7-4710MQ CPU @ 2.50GHz and RAM 8.00GB.

Example 1. *Let l_3 be a uniformly convex and uniformly smooth Banach space, we set $Ax = 5x + (1,1,1,0,0,0...)$ and $Bx = 6x$ where $x = (x_1, x_2, x_3, ...) \in l_3$. We can check that $A : l_3 \to l_3$ is a $\frac{1}{5}$-isa, $B : l_3 \to l_3$ is an m-accretive operator and $R(I + rB) = l_3$ for all $r > 0$. we take $r_n = 0.02$, $\alpha_n = 0.4$ for all $n \in N$. Let $\beta_n = \frac{1}{1000n+1}$, $\delta_n = \frac{1}{200n}$ and $f(x) = \frac{1}{3}x$ be a contraction with coefficient $\rho = \frac{1}{3}$. Starting $x_0 = (1.8, 3.2, 9.6, ...)$, $x_1 = (1.4290014, 2.5542525, 7.6982578, ...)$ and using algorithm (1) in Theorem 1, we obtain the following numerical results.*

From Table 1 we see that $x_{600} = (-0.0909, -0.0909, -0.0909, 0.0000, 0.0000, 0.0000, ...)$ is an approximation of a solution with an error 1.8770214×10^{-9}. And we make the same choices for x_1 as reported in Table 1. In terms of the number of iterations and the errors, we provide the numerical examples to demonstrate the performance and to compare our proposed algorithm with the iterative algorithm with $\alpha_n = 0$.

In these 600 experiments, Table 2 shows that the final approximation solution is the same as Table 1. Figure 1 shows that the number of iterations and errors of our algorithm and the algorithm with $\alpha_n = 0$ for the above initial points. We can see that the convergence of our algorithm is faster than the algorithm of Pholasa et al. [10].

Table 1. Numerical results of Example 1 for iteration process.

n	x_n	$\|x_{n+1} - x_n\|_{l_3}$
1	$(1.4290014, 2.5542525, 7.6982578, 0.0000000, 0.0000000, 0.0000000, ...)$	2.1732179
10	$(-0.0677632, -0.0506892, 0.0273635, 0.0000000, 0.0000000, 0.0000000, ...)$	8.911330×10^{-2}
20	$(-0.0908530, -0.0908328, -0.0907403, 0.0000000, 0.0000000, 0.0000000, ...)$	1.9408448×10^{-4}
30	$(-0.0908917, -0.0908919, -0.0908926, 0.0000000, 0.0000000, 0.0000000, ...)$	6.7808717×10^{-7}
40	$(-0.0908964, -0.0908964, -0.0908964, 0.0000000, 0.0000000, 0.0000000, ...)$	5.1789690×10^{-7}
50	$(-0.0908991, -0.0908991, -0.0908991, 0.0000000, 0.0000000, 0.0000000, ...)$	3.1714100×10^{-7}
60	$(-0.0909009, -0.0909009, -0.0909009, 0.0000000, 0.0000000, 0.0000000, ...)$	2.1345961×10^{-7}
70	$(-0.0909021, -0.0909021, -0.0909021, 0.0000000, 0.0000000, 0.0000000, ...)$	1.5346777×10^{-7}
80	$(-0.0909030, -0.0909030, -0.0909030, 0.0000000, 0.0000000, 0.0000000, ...)$	1.1564503×10^{-7}
90	$(-0.0909037, -0.0909037, -0.0909037, 0.0000000, 0.0000000, 0.0000000, ...)$	9.0267844×10^{-8}
100	$(-0.0909043, -0.0909043, -0.0909043, 0.0000000, 0.0000000, 0.0000000, ...)$	7.2416422×10^{-8}
200	$(-0.0909067, -0.0909067, -0.0909067, 0.0000000, 0.0000000, 0.0000000, ...)$	1.7357133×10^{-8}
300	$(-0.0909075, -0.0909075, -0.0909075, 0.0000000, 0.0000000, 0.0000000, ...)$	7.6097663×10^{-9}
400	$(-0.0909079, -0.0909079, -0.0909079, 0.0000000, 0.0000000, 0.0000000, ...)$	4.2517012×10^{-9}
500	$(-0.0909082, -0.0909082, -0.0909082, 0.0000000, 0.0000000, 0.0000000, ...)$	2.7101525×10^{-9}
600	$(-0.0909083, -0.0909083, -0.0909083, 0.0000000, 0.0000000, 0.0000000, ...)$	1.8770214×10^{-9}

Table 2. Numerical results for iteration process Algorithm (1) with $\alpha_n = 0$ in Example 1.

n	x_n	$\|x_{n+1} - x_n\|_{l_3}$
1	$(1.4290014, 2.5542525, 7.6982578, 0.0000000, 0.0000000, 0.0000000, ...)$	1.9308196
10	$(0.1216527, 0.2789583, 0.9980697, 0.0000000, 0.0000000, 0.0000000, ...)$	2.702205×10^{-1}
20	$(-0.0670158, -0.0493531, 0.031391, 0.0000000, 0.0000000, 0.0000000, ...)$	3.034530×10^{-2}
30	$(-0.0882106, -0.0862275, -0.0771619, 0.0000000, 0.0000000, 0.0000000, ...)$	3.407800×10^{-3}
40	$(-0.0905947, -0.0903721, -0.0893543, 0.0000000, 0.0000000, 0.0000000, ...)$	3.8297598×10^{-4}
50	$(-0.0908649, -0.0908399, -0.0907256, 0.0000000, 0.0000000, 0.0000000, ...)$	4.3220033×10^{-5}
60	$(-0.0908968, -0.0908940, -0.0908812, 0.0000000, 0.0000000, 0.0000000, ...)$	5.0028407×10^{-6}
70	$(-0.0909015, -0.0909012, -0.0908997, 0.0000000, 0.0000000, 0.0000000, ...)$	6.7427458×10^{-7}
80	$(-0.0909028, -0.0909028, -0.0909026, 0.0000000, 0.0000000, 0.0000000, ...)$	1.7183921×10^{-7}
90	$(-0.0909036, -0.0909036, -0.0909036, 0.0000000, 0.0000000, 0.0000000, ...)$	9.9664118×10^{-8}
100	$(-0.0909042, -0.0909042, -0.0909042, 0.0000000, 0.0000000, 0.0000000, ...)$	7.5993686×10^{-8}
200	$(-0.0909067, -0.0909067, -0.0909067, 0.0000000, 0.0000000, 0.0000000, ...)$	1.7674516×10^{-8}
300	$(-0.0909075, -0.0909075, -0.0909075, 0.0000000, 0.0000000, 0.0000000, ...)$	7.6990015×10^{-9}
400	$(-0.0909079, -0.0909079, -0.0909079, 0.0000000, 0.0000000, 0.0000000, ...)$	4.2884018×10^{-9}
500	$(-0.0909082, -0.0909082, -0.0909082, 0.0000000, 0.0000000, 0.0000000, ...)$	2.7286629×10^{-9}
600	$(-0.0909083, -0.0909083, -0.0909083, 0.0000000, 0.0000000, 0.0000000, ...)$	1.8876276×10^{-9}

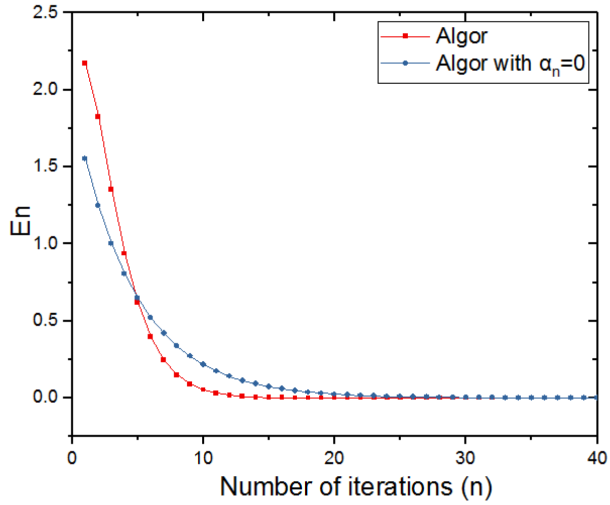

Figure 1. Error plotting of $\|x_{n+1} - x_n\|_{l_3}$.

5. Conclusions

In this paper, we give a modified inertial viscosity splitting algorithm for accretive operators in Banach spaces. The strong convergence theorems are established, and the numerical experiments are presented to support that the inertial extrapolation greatly improves the efficiency of the algorithm. In Theorem 1 and Corollary 1, if $f(x_n) = u$ and A is an inverse strongly monotone operator in Hilbert space, it is the main results of Cholamjiak et al. [20]. In Theorem 1, if $\alpha_n = 0$, $f(x_n) = u$ and E is a uniformly convex and q-uniformly smooth Banach space, it is the main results of Pholasa et al. [10]. Furthermore, some other results are also improved (see [8,9,18,19,26]).

The introduction of the inertial viscosity splitting algorithms sheds new light on inclusion problem. Combined with recent research findings ([4,13,19,20]), Theorem 1 can be further applied to the fixed-point problem, the split feasibility problem and the variational inequality problem. Indeed, it is an important but unsolved problem to choose the optimal inertia parameters α_n in the acceleration algorithm. In the future, more work will be devoted to the wide application of the proposed algorithm and the improvement of its convergence rate.

Author Contributions: conceptualization, C.P. and Y.W.; methodology, C.P. and Y.W.; software, C.P. and Y.W.; validation, C.P. and Y.W.; formal analysis, C.P. and Y.W.; investigation, C.P. and Y.W.;resources, C.P. and Y.W.; data curation, C.P.; writing–original draft preparation,C.P. and Y.W.; writing–review and editing, C.P. and Y.W.; visualization, C.P. and Y.W.; supervision, Y.W.; project administration, Y.W..

Funding: This research received no external funding.

Acknowledgments: This work was supported by the National Natural Science Foundation of China (Grant no. 11671365) and the Natural Science Foundation of Zhejiang Province (Grant no. LY14A010011).

Conflicts of Interest: The authors declare that they have no competing interests.

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