




## Article

# On the $(p, q)$ -Chebyshev Polynomials and Related Polynomials

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**Abstract:** In this paper, we introduce  $(p, q)$ -Chebyshev polynomials of the first and second kind that reduces the  $(p, q)$ -Fibonacci and the  $(p, q)$ -Lucas polynomials. These polynomials have explicit forms and generating functions are given. Then, derivative properties between these first and second kind polynomials, determinant representations, multilateral and multilinear generating functions are derived.

**Keywords:**  $(p, q)$ -Chebyshev polynomials;  $(p, q)$ -Fibonacci polynomials; multilateral generating functions; multilinear generating functions.

**MSC:** 11B39; 11B83; 33C45

## 1. Introduction

For any integer  $n \geq 0$ , the Chebyshev polynomials of the first and second kind  $T_n(x)$  and  $U_n(x)$  are respectively defined as follows:

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x),$$

with the initial values  $T_0(x) = 1$  and  $T_1(x) = x$ , and

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x),$$

with the initial values  $U_0(x) = 1$  and  $U_1(x) = 2x$ . For more information, please see the papers [1–3] and closely related references therein.

These polynomials play a very important role in the study of the theory and applications of mathematics and they are closely related to Fibonacci numbers  $\{F_n\}$  and Lucas numbers  $\{L_n\}$ , which are defined by the second order linear recurrence sequences, for any integer  $n \geq 0$ ,

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_{n+2} = L_{n+1} + L_n,$$

where  $F_0 = 0$ ,  $F_1 = 1$ ,  $L_0 = 2$  and  $L_1 = 1$ , respectively. Many authors have investigated these polynomials and their generalizations [4–12]. In [8], Kim et al. consider sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials and derive Fourier series expansions of functions associated with them. In [9], Kim et al. studied the convolved Fibonacci numbers by using the generating functions of them and gave some new identities for the convolved Fibonacci numbers. In [5], Cigler define  $q$ -analogues of Chebyshev polynomials and derive some

properties of these polynomials. The  $q$ -Chebyshev polynomials of the first kind and second kind are defined by the following recurrences relations, respectively, for any integer  $n \geq 2$ ,

$$T_n(x, s, q) = (1 + q^n)xT_{n-1}(x, s, q) + q^{n-1}sT_{n-2}(x, s, q),$$

with the initial values  $T_0(x, s, q) = 1$ ,  $T_1(x, s, q) = x$  and

$$U_n(x, s, q) = (1 + q^n)xU_{n-1}(x, s, q) + q^{n-1}sU_{n-2}(x, s, q),$$

with the initial values  $U_0(x, s, q) = 1$ ,  $U_1(x, s, q) = (1 + q)x$  and  $x, s$  are real variables in [5].

It is clear that  $U_n(x, -1, 1) = U_n(x)$  and  $T_n(x, -1, 1) = T_n(x)$ . Moreover, in [5], Cigler point out that the  $q$ -Chebyshev polynomials of the first kind are determined as the combinatorial sum

$$T_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j-1}}{(-q; q)_j} s^j x^{n-2j}, \quad n \in \mathbb{N} \cup \{0\}, \quad (1)$$

and the  $q$ -Chebyshev polynomials of the second kind is determined as

$$U_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j}, \quad n \in \mathbb{N} \cup \{0\},$$

where  $(x, q)_n$  is the  $q$ -shifted factorial, that is,  $(x, q)_0 = 1$ ,

$$(x, q)_n = \prod_{i=0}^{n-1} (1 - q^i x)$$

and for integer  $k$ ,  $q$ -binomial coefficient is as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q, q)_n}{(q, q)_{n-k}(q, q)_k}, \quad 0 \leq k \leq n,$$

with  $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$  for  $n < k$ .

Now, we give some definitions related to  $(p, q)$ -integers, for any fixed real number  $0 < q < p \leq 1$  and each non-negative integer  $n$ .  $(p, q)$ -integers are denoted as  $[n]_{p,q}$ , where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Also in Ref. [13],  $(p, q)$ -factorial and  $(p, q)$ -binomial coefficients are defined as follows:

$$[n]_{p,q}! = \begin{cases} [n]_{p,q} [n-1]_{p,q} \cdots [1]_{p,q}, & \text{if } k = 1, 2, \dots, \\ 1, & \text{if } k = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{cases} \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n \geq 0. \end{cases}$$

In Ref. [13], the  $(p, q)$ -binomial coefficients satisfy the following recurrence relations:

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} &= p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}, \\ \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} &= q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}, \end{aligned}$$

with  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{p,q} = 1$ ,  $\begin{bmatrix} n \\ n \end{bmatrix}_{p,q} = 1$  and  $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = 0$  for  $k > n$ .

In Ref. [14], the  $(p, q)$ -shifted factorial is given as

$$((a, b); (p, q))_n = \begin{cases} 1, & n = 0, \\ (a - b)(ap - bq)(ap^2 - bq^2) \dots (ap^{n-1} - bq^{n-1}), & n = 1, 2, \dots \end{cases}$$

Note that, for details of  $(p, q)$ -analysis, one can see [15–18] and  $p \rightarrow 1$  in these properties, we have the property of  $q$ -calculus in [19]. On the other hand, for more details related to  $(p, q)$ -orthogonal polynomials, readers look at the papers in [20,21].

With the help of these generalizations, we will introduce  $(p, q)$ -Chebyshev polynomials of the first and second kind.

## 2. $(p, q)$ -Chebyshev Polynomials

In this section, we will define  $(p, q)$ -Chebyshev polynomials of the first and second kind. Then, we will derive explicit formulas, generating functions and some interesting properties of these polynomials.

**Definition 1.** For any integer  $n \geq 2$  and  $0 < q < p \leq 1$ , the  $(p, q)$ -Chebyshev polynomials of the first kind are defined by the following recurrence relation:

$$T_n(x, s, p, q) = (p^{n-1} + q^{n-1})xT_{n-1}(x, s, p, q) + (pq)^{n-1}sT_{n-2}(x, s, p, q), \quad (2)$$

with the initial values  $T_0(x, s, p, q) = 1$  and  $T_1(x, s, p, q) = x$  and  $x, s$  are real variables.

In the light of this recurrence relation, we will give Table 1:

**Table 1.** Some special cases of the  $(p, q)$ -Chebyshev polynomials of the first kind.

$x$	$s$	$p$	$q$	$T_n(x, s, p, q)$	$(p, q)$ -Chebyshev Polynomials of First Kind
$\frac{x}{2}$	$s$	$p$	$q$	$L_n(x, s, p, q)$	$(p, q)$ -Lucas Polynomials
$x$	$-1$	$1$	$1$	$T_n(x)$	First kind of Chebyshev Polynomials
$\frac{x}{2}$	$1$	$1$	$1$	$\frac{1}{2}L_n(x)$	Lucas Polynomials
$\frac{x}{2}$	$1$	$1$	$1$	$\frac{1}{2}L_n$	Lucas Numbers
$x$	$1$	$1$	$1$	$\frac{1}{2}Q_n(x)$	Pell Lucas Polynomials
$1$	$1$	$1$	$1$	$\frac{1}{2}Q_n$	Pell Lucas Numbers
$\frac{1}{2}$	$2y$	$1$	$1$	$\frac{1}{2}j_n(y)$	Jacobsthal Lucas Polynomials
$\frac{1}{2}$	$2$	$1$	$1$	$\frac{1}{2}j_n$	Jacobsthal Lucas Numbers

**Lemma 1.** The  $(p, q)$ -binomial coefficients satisfy the following identities:

$$(p^{n-j} + q^{n-j}) \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} = (p^n + q^n) \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_{p,q} + (pq)^{-2j+n} (p^j + q^j) \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix}_{p,q}, \quad (3)$$

$$\begin{aligned} (p^{n-j-1} + q^{n-j-1}) \frac{\begin{bmatrix} n \\ p,q \end{bmatrix}}{\begin{bmatrix} n-j \\ p,q \end{bmatrix}} \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} &= (p^{n-1} + q^{n-1}) \frac{\begin{bmatrix} n-1 \\ p,q \end{bmatrix}}{\begin{bmatrix} n-j-1 \\ p,q \end{bmatrix}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_{p,q} \\ &+ (pq)^{-2j+n} (p^j + q^j) \frac{\begin{bmatrix} n-2 \\ p,q \end{bmatrix}}{\begin{bmatrix} n-j-1 \\ p,q \end{bmatrix}} \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix}_{p,q}. \end{aligned} \quad (4)$$

**Proof.** Using these equations,

$$(1 + q^{n-j}) \begin{bmatrix} n-j \\ j \end{bmatrix}_q = (1 + q^n) \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_q + q^{n-2j}(1 + q^j) \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix}_q,$$

$$(1 + q^{n-1}) \frac{[n-1]_q}{[n-j-1]_q} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_q + q^{-2j+n}(1+q^j) \frac{[n-2]_q}{[n-j-1]_q} \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix}_q = (1 + q^{n-j-1}) \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q$$

in [5] and writing  $q \rightarrow \frac{q}{p}$  and  $\begin{bmatrix} n \\ k \end{bmatrix}_{q/p} \rightarrow p^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ , we derive Equations (3) and (4) as  $(p, q)$  analogue of the above equations.  $\square$

**Theorem 1.** The explicit formula of  $(p, q)$ -Chebyshev polynomials of the first kind is as follows:

$$T_n(x, s, p, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (pq)^{j^2} \frac{[n]_{p,q}}{[n-j]_{p,q}} \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j x^{n-2j}. \quad (5)$$

**Proof.** By using Equation (2) when  $n$  is odd, we have

$$\begin{aligned} & (p^{n-1} + q^{n-1}) x \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (pq)^{j^2} \frac{[n-1]_{p,q}}{[n-j-1]_{p,q}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-2}}{((p, -q); (p, q))_j} s^j x^{n-2j-1} + (pq)^{n-1} s \\ & \quad \times \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} (pq)^{j^2} \frac{[n-2]_{p,q}}{[n-j-2]_{p,q}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-3}}{((p, -q); (p, q))_j} s^j x^{n-2j-2} \\ & = (p^{n-1} + q^{n-1}) ((p, -q); (p, q))_{n-2} x^n \\ & \quad + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{[n-1]_{p,q}}{[n-j-1]_{p,q}} \frac{1}{p^{n-j-1} + q^{n-j-1}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_{p,q} + (pq)^{-2j+n} \frac{[n-2]_{p,q}}{[n-j-1]_{p,q}} \left( \frac{p^j + q^j}{p^{n-j-1} + q^{n-j-1}} \right) \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix}_{p,q} \right) \\ & \quad \times (pq)^{j^2} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j x^{n-2j}. \end{aligned}$$

From Lemma 1, we get

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (pq)^{j^2} \frac{[n]_{p,q}}{[n-j]_{p,q}} \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j x^{n-2j}.$$

If  $n$  is even, the proof can be obtained similarly.  $\square$

The Fibonacci operator  $\eta_q$  was introduced by Andrews in [3], by  $\eta_q f(x) = f(qx)$ . Similarly, we define another operator  $\eta_{p,q} f(x) = f(pqx)$ . Now, we will give the generating function of the  $(p, q)$ -Chebyshev polynomials of the first kind.

**Theorem 2.** The generating function of the  $(p, q)$ -Chebyshev polynomials of the first kind is as follows:

$$S_{p,q}(z) = \frac{1}{1 - xz\eta_p - xz\eta_q - spqz^2\eta_{p,q}} \{1 - xz\}. \quad (6)$$

**Proof.** Let us consider the following equation

$$S_{p,q}(z) = \sum_{n=0}^{\infty} T_n(x, s, p, q) z^n.$$

For the proof of Theorem 2, we need to check the following equivalent relation:

$$(1 - xz\eta_p - xz\eta_q - spqz^2\eta_{p,q}) S_{p,q}(z) = 1 - xz.$$

Thus, we write

$$\begin{aligned}
 S_{p,q}(z) - xz\eta_p S_{p,q}(z) - xz\eta_q S_{p,q}(z) - spqz^2\eta_{p,q} S_{p,q}(z) \\
 = \sum_{n=0}^{\infty} T_n(x, s, p, q) z^n - x \sum_{n=0}^{\infty} T_n(x, s, p, q) p^n z^{n+1} - x \sum_{n=0}^{\infty} T_n(x, s, p, q) q^n z^{n+1} \\
 - s \sum_{n=0}^{\infty} p^{n+1} q^{n+1} T_n(x, s, p, q) z^{n+2} \\
 = \sum_{n=0}^{\infty} T_n(x, s, p, q) z^n - x \sum_{n=1}^{\infty} (p^{n-1} + q^{n-1}) T_{n-1}(x, s, p, q) z^n - s \sum_{n=2}^{\infty} (pq)^{n-1} T_{n-2}(x, s, p, q) z^n \\
 = T_0(x, s, p, q) + T_1(x, s, p, q) - 2xzT_0(x, s, p, q) \\
 + \sum_{n=2}^{\infty} (T_n(x, s, p, q) - x(p^{n-1} + q^{n-1})T_{n-1}(x, s, p, q) - s(pq)^{n-1}T_{n-2}(x, s, p, q)) z^n.
 \end{aligned}$$

From (2), we have

$$\begin{aligned}
 (1 - xz\eta_p - xz\eta_q - spqz^2\eta_{p,q}) S_{p,q}(z) &= T_0(x, s, p, q) + T_1(x, s, p, q) - 2xzT_0(x, s, p, q) \\
 &= 1 - xz.
 \end{aligned}$$

Finally, we obtain the desired relation

$$S_{p,q}(z) = \frac{1}{1 - xz\eta_p - xz\eta_q - spqz^2\eta_{p,q}} \{1 - xz\}.$$

□

**Definition 2.** For any integer  $n \geq 2$  and  $0 < q < p \leq 1$ , the  $(p, q)$ -Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$U_n(x, s, p, q) = (p^n + q^n)xU_{n-1}(x, s, p, q) + (pq)^{n-1}sU_{n-2}(x, s, p, q) \quad (7)$$

with the initial values  $U_0(x, s, p, q) = 1$  and  $U_1(x, s, p, q) = (p + q)x$  and  $s$  is a variable.

In the light of this recurrence relation, we will give the other following interesting table:

**Table 2.** Some special cases of the  $(p, q)$ -Chebyshev polynomials of the second kind.

$x$	$s$	$p$	$q$	$U_n(x, s, p, q)$	$(p, q)$ -Chebyshev Polynomials of Second Kind
$\frac{x}{2}$	$s$	$p$	$q$	$F_n(x, s, p, q)$	$(p, q)$ -Fibonacci Polynomials
$x$	$-1$	$1$	$1$	$U_n(x)$	Second kind of Chebyshev Polynomials
$\frac{x}{2}$	$1$	$1$	$1$	$F_{n+1}(x)$	Fibonacci Polynomials
$\frac{1}{2}$	$1$	$1$	$1$	$F_{n+1}$	Fibonacci Numbers
$x$	$1$	$1$	$1$	$P_{n+1}(x)$	Pell Polynomials
$1$	$1$	$1$	$1$	$P_{n+1}$	Pell Numbers
$\frac{1}{2}$	$2y$	$1$	$1$	$J_{n+1}(y)$	Jacobsthal Polynomials
$\frac{1}{2}$	$2$	$1$	$1$	$J_{n+1}$	Jacobsthal Numbers

**Theorem 3.** The explicit formula of  $(p, q)$ -Chebyshev polynomials of the second kind is as follows:

$$U_n(x, s, p, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (pq)^j \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j}}{((p, -q); (p, q))_j} s^j x^{n-2j}. \quad (8)$$

**Proof.** By using (7) when  $n$  is odd, we have

$$\begin{aligned}
 (p^n + q^n) x \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (pq)^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j x^{n-2j-1} + (pq)^{n-1} s \\
 \times \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} (pq)^{j^2} \begin{bmatrix} n-j-2 \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-2}}{((p, -q); (p, q))_j} s^j x^{n-2j-2} \\
 = (p^n + q^n) ((p, -q); (p, q))_{n-1} x^n \\
 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{p^n + q^n}{p^{n-j} + q^{n-j}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_{p,q} + (pq)^{-2j+n} \left( \frac{p^j + q^j}{p^{n-j} + q^{n-j}} \right) \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix}_{p,q} \right) \\
 \times (pq)^{j^2} \frac{((p, -q); (p, q))_{n-j}}{((p, -q); (p, q))_j} s^j x^{n-2j}.
 \end{aligned}$$

Then, using the Lemma 1, we have

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (pq)^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j}}{((p, -q); (p, q))_j} s^j x^{n-2j}.$$

If  $n$  is even, the proof can be obtained similarly.  $\square$

**Theorem 4.** The generating function of the  $(p, q)$ -Chebyshev polynomials of the second kind is as follows:

$$G_{p,q}(z) = \frac{1}{1 - x p z \eta_p - x q z \eta_q - s p q z^2 \eta_{p,q}}. \quad (9)$$

**Proof.** Let us consider the following equations:

$$G_{p,q}(z) = \sum_{n=0}^{\infty} U_n(x, s, p, q) z^n.$$

Similarly, for the proof of Theorem 4, we need to check the following equivalent relation:

$$(1 - x p z \eta_p - x q z \eta_q - s p q z^2 \eta_{p,q}) G_{p,q}(z) = 1.$$

Thus, we write

$$\begin{aligned}
 G_{p,q}(z) - x p z \eta_p G_{p,q}(z) - x q z \eta_q G_{p,q}(z) - s p q z^2 \eta_{p,q} G_{p,q}(z) \\
 = \sum_{n=0}^{\infty} U_n(x, s, p, q) z^n - x \sum_{n=0}^{\infty} U_n(x, s, p, q) p^{n+1} z^{n+1} - x \sum_{n=0}^{\infty} U_n(x, s, p, q) q^{n+1} z^{n+1} \\
 - s \sum_{n=0}^{\infty} U_n(x, s, p, q) (pq)^{n+1} z^{n+2} \\
 = U_0(x, s, p, q) + U_1(x, s, p, q) z - x p z U_0(x, s, p, q) - x q z U_0(x, s, p, q) \\
 + \sum_{n=2}^{\infty} (U_n(x, s, p, q) - x(p^n + q^n) U_{n-1}(x, s, p, q) - s(pq)^{n-1} U_{n-2}(x, s, p, q)).
 \end{aligned}$$

From (7), we obtain

$$(1 - x p z \eta_p - x q z \eta_q - s p q z^2 \eta_{p,q}) G_{p,q}(z) = 1.$$

Finally, we have

$$G_{p,q}(z) = \frac{1}{(1 - xpz\eta_p - xqz\eta_q - spqz^2\eta_{p,q})}.$$

□

The recurrence relations  $(p, q)$ -Chebyshev polynomials of the first and second kind can be expressed by the following determinant, respectively:

$$T_n(x, s, p, q) = \det \begin{pmatrix} x & (pq)s & 0 & \dots & 0 \\ -1 & (p+q)x & (pq)^2s & \ddots & \vdots \\ 0 & -1 & (p^2+q^2)x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (pq)^{n-1}s \\ 0 & 0 & 0 & \dots & (p^{n-1}+q^{n-1})x \end{pmatrix},$$

$$U_n(x, s, p, q) = \det \begin{pmatrix} (p+q)x & (pq)s & 0 & \dots & 0 \\ -1 & (p^2+q^2)x & (pq)^2s & \ddots & \vdots \\ 0 & -1 & (p^3+q^3)x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (pq)^{n-1}s \\ 0 & 0 & 0 & \dots & (p^n+q^n)x \end{pmatrix}.$$

**Theorem 5.** For these polynomials, we have an interesting relation

$$D_{(p,q)}T_n(x, s, p, q) = [n]_{p,q} U_{n-1}(x, s, p, q),$$

where  $D_{(p,q)}$  denoted by  $(p, q)$ -Jackson's derivative given by

$$D_{(p,q)}f(x) = \frac{f(px) - f(qx)}{(p-q)x}$$

in [14].

**Proof.** By using the (5) and (8), we have

$$\begin{aligned} D_{(p,q)}T_n(x, s, p, q) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (pq)^{j^2} \frac{[n]_{p,q}}{[n-j]_{p,q}} \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j D_{(p,q)}x^{n-2j} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (pq)^{j^2} \frac{[n]_{p,q}}{[n-j]_{p,q}} \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j \frac{(px)^{n-2j} - (qx)^{n-2j}}{(p-q)x} \\ &= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (pq)^{j^2} \frac{[n]_{p,q} [n-2j]_{p,q}}{[n-j]_{p,q}} \begin{bmatrix} n-j \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j x^{n-2j-1} \\ &= [n]_{p,q} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (pq)^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_{p,q} \frac{((p, -q); (p, q))_{n-j-1}}{((p, -q); (p, q))_j} s^j x^{n-2j-1} \\ &= [n]_{p,q} U_{n-1}(x, s, p, q). \end{aligned}$$

Thus, the proof is completed. □

### 3. Multilinear and Multilateral Generating Functions

In this section, we derive some multilinear and multilateral generating functions for  $(p, q)$ -Chebyshev polynomials of the first and the second kind which are generated by (6) and (9), and given explicitly by (5) and (8), respectively, with the help of similar methods in [22–25]. The presented results and their potential impacts seem to be relevant for a wider audience in the areas of mathematics including orthogonal polynomials, harmonic analysis and classical analysis [26].

**Theorem 6.** Corresponding to an identically non-vanishing function  $\Lambda_\mu(\mathbf{t})$  of  $m$  complex variables  $t_1, \dots, t_m$  ( $m \in \mathbb{N}$ ) and of complex order  $\mu$ , let

$$Y_{\mu,\nu}(\mathbf{t}; w) := \sum_{k=0}^{\infty} a_k \Lambda_{\mu+\nu k}(\mathbf{t}) w^k, \quad (10)$$

where  $(a_k \neq 0, \mu, \nu \in \mathbb{C})$ ;  $\mathbf{t} = (t_1, \dots, t_m)$  and

$$\Psi_{n,r,\mu,\nu}(x; \mathbf{t}; h) := \sum_{k=0}^{[n/r]} a_k T_{n-rk}(x, s, p, q) \Lambda_{\mu+\nu k}(\mathbf{t}) h^k, \quad (11)$$

where  $n, r \in \mathbb{N}$ . Then, we have

$$\sum_{n=0}^{\infty} \Psi_{n,r,\mu,\nu}\left(x; \mathbf{t}; \frac{\omega}{v^r}\right) v^n = Y_{\mu,\nu}(\mathbf{t}; \omega) \frac{1}{1 - xv\eta_p - xv\eta_q - spqv^2\eta_{p,q}} \{1 - xv\}. \quad (12)$$

**Proof.** We symbolize the left-hand side of the equality (12) of Theorem 6 as  $Q$ . Then, we can write

$$\sum_{k=0}^{[n/r]} a_k T_{n-rk}(x, s, p, q) \Lambda_{\mu+\nu k}(\mathbf{t}) \omega^k v^{n-rk} \quad (13)$$

instead of

$$\Psi_{n,r,\mu,\nu}\left(x; \mathbf{t}; \frac{\omega}{v^r}\right)$$

from the definition (11) into the left-hand side of (12), we have

$$Q = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/r]} a_k T_{n-rk}(x, s, p, q) \Lambda_{\mu+\nu k}(\mathbf{t}) \omega^k v^{n-rk}. \quad (14)$$

Writing  $n$  by  $n + rk$ , we can obtain

$$\begin{aligned} Q &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k T_n(x, s, p, q) \Lambda_{\mu+\nu k}(\mathbf{t}) \omega^k v^n \\ &= \left( \sum_{k=0}^{\infty} a_k \Lambda_{\mu+\nu k}(\mathbf{t}) \omega^k \right) \left( \sum_{n=0}^{\infty} T_n(x, s, p, q) v^n \right) \\ &= Y_{\mu,\nu}(\mathbf{t}; \omega) \frac{1}{1 - xv\eta_p - xv\eta_q - spqv^2\eta_{p,q}} \{1 - xv\}. \end{aligned}$$

□

Now, we can similarly derive the next result.



**Theorem 7.** Corresponding to an identically non-vanishing function  $\Lambda_\mu(\mathbf{t})$  of  $m$  complex variables  $t_1, \dots, t_m$  ( $m \in \mathbb{N}$ ) and of complex order  $\mu$ , let

$$Y_{\mu,\nu}(\mathbf{t}; w) := \sum_{k=0}^{\infty} a_k \Lambda_{\mu+\nu k}(\mathbf{t}) w^k, \quad (15)$$

where  $(a_k \neq 0, \mu, \nu \in \mathbb{C})$ ;  $\mathbf{t} = (t_1, \dots, t_m)$  and

$$\Psi_{n,r,\mu,\nu}(x; \mathbf{t}; h) := \sum_{k=0}^{[n/r]} a_k U_{n-rk}(x, s, p, q) \Lambda_{\mu+\nu k}(\mathbf{t}) h^k, \quad (16)$$

where  $n, r \in \mathbb{N}$ . Then, we have

$$\sum_{n=0}^{\infty} \Psi_{n,r,\mu,\nu}\left(x; \mathbf{t}; \frac{\omega}{v^r}\right) v^n = Y_{\mu,\nu}(\mathbf{t}; \omega) \frac{1}{1 - xpv\eta_p - xqv\eta_q - spqv^2\eta_{p,q}}. \quad (17)$$

#### 4. Some Examples for Generating Functions

Before obtaining new generating functions, we will recall  $(p, q)$ -Fibonacci and  $(p, q)$ -Lucas polynomials. In Ref. [27], for  $0 < q < p \leq 1$  and  $x, s$  are real variables, the authors define  $(p, q)$ -Fibonacci and  $(p, q)$ -Lucas polynomials as

$$F_{n+1}(x, s | p, q) = \sum_{k=0}^{[\frac{n}{2}]} (pq)^{k(k+1)/2} \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q} s^k x^{n-2k}, \quad (18)$$

$$L_n(x, s | p, q) = \sum_{k=0}^{[\frac{n}{2}]} (pq)^{\binom{k}{2}} \frac{[n]_{p,q}}{[n-k]_{p,q}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q} s^k x^{n-2k}, \quad (19)$$

and obtain a generating function as

$$\sum_{n=0}^{\infty} F_n(x, sp^{-n} | p, q) t^n = \frac{t}{1-xt} {}_2\phi_2 \left( \begin{matrix} (p, q), & 0 \\ (p, xtpq), & (p, 0) \end{matrix} \middle| (p, q); -qst^2 \right); |xt| < 1, \quad (20)$$

$$\sum_{n=0}^{\infty} L_n(x, sp^{-n} | p, q) t^n = \frac{1+sp t^2}{1-xpt} {}_2\phi_2 \left( \begin{matrix} (p, q), & 0 \\ (p, xtpq), & (p, 0) \end{matrix} \middle| (p, q); -qst^2 \right); |xt| < 1, \quad (21)$$

where

$${}_2\phi_2 \left( \begin{matrix} (a_1, b_1), & (a_2, b_2) \\ (c_1, d_1), & (c_2, d_2) \end{matrix} \middle| (p, q); x \right) = \sum_{n=0}^{\infty} \frac{((a_1, b_1), (a_2, b_2); (p, q))}{((c_1, d_1), (c_2, d_2); (p, q))} \frac{(-1)^n \left(\frac{q}{p}\right)^{\binom{n}{2}}}{((p, q); (p, q))_n} x^n,$$

respectively.

Now, we can give some examples for generating functions. For  $0 < q < p \leq 1$  and  $x, s$  are real variables, setting

$$m = 1 \text{ and } \Lambda_{\mu+\nu k}(z) = F_{\mu+\nu k}(z, sp^{-k} | p, q)$$

in Theorem 6, where the  $(p, q)$ -Fibonacci polynomials

$$F_n(z, sp^{-k} | p, q)$$

are generated by (20), and then we will derive the result, which provides bilateral generating functions for  $(p, q)$ -Fibonacci polynomials and  $(p, q)$ -Chebyshev polynomials of the first kind given by (5).

**Corollary 1.** If  $Y_{\mu,\nu}(z;w) := \sum_{k=0}^{\infty} a_k F_{\mu+\nu k}(z, sp^{-k} | p, q) w^k$ , ( $a_k \neq 0$ ,  $\mu, \nu \in \mathbb{C}$ ), and

$$W_{n,r,\mu,\nu}(x; z; \zeta) := \sum_{k=0}^{\lfloor n/r \rfloor} a_k T_{n-rk}(x, s, p, q) F_{\mu+\nu k}(z, sp^{-k} | p, q) \zeta^k,$$

where  $n \in \mathbb{N}_0$ ;  $r \in \mathbb{N}$ , and then

$$\sum_{n=0}^{\infty} W_{n,r,\mu,\nu} \left( x; z; \frac{u}{t^r} \right) t^n = Y_{\mu,\nu}(z; u) \frac{1}{1 - xt\eta_p - xt\eta_q - spqt^2\eta_{p,q}} \{1 - xt\},$$

where  $0 < q < p \leq 1$ .

**Remark 1.** From Equation (20), for  $(p, q)$ -Fibonacci polynomials and getting  $a_k = 1$ ,  $\mu = 0$ ,  $\nu = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/r \rfloor} T_{n-rk}(x, s, p, q) F_k(z, sp^{-k} | p, q) u^k t^{n-rk} &= \frac{u}{1 - zu} {}_2\phi_2 \left( \begin{matrix} (p, q), & 0 \\ (p, zuq), & (p, 0) \end{matrix} \middle| (p, q); -qsu^2 \right) \\ &\times \frac{1}{1 - xt\eta_p - xt\eta_q - spqt^2\eta_{p,q}} \{1 - xt\}, \end{aligned}$$

where  $|zu| < 1$ .

In addition, choosing  $m = 1$  and  $\Lambda_{\mu+\nu k}(z) = L_{\mu+\nu k}(z, sp^{-k} | p, q)$  for  $0 < q < p \leq 1$  in Theorem 7, we will derive the following bilateral generating functions for  $(p, q)$ -Lucas polynomials and  $(p, q)$ -Chebyshev polynomials of the second kind given by (8).

**Corollary 2.** If  $Y_{\mu,\nu}(z;w) := \sum_{k=0}^{\infty} a_k L_{\mu+\nu k}(z, sp^{-k} | p, q) w^k$ , ( $a_k \neq 0$ ,  $\mu, \nu \in \mathbb{C}$ ), and

$$W_{n,r,\mu,\nu}(x; z; \zeta) := \sum_{k=0}^{\lfloor n/r \rfloor} a_k U_{n-rk}(x, s, p, q) L_{\mu+\nu k}(z, sp^{-k} | p, q) \zeta^k,$$

where  $n \in \mathbb{N}_0$ ;  $r \in \mathbb{N}$ , and then

$$\sum_{n=0}^{\infty} W_{n,r,\mu,\nu} \left( x; z; \frac{u}{t^r} \right) t^n = Y_{\mu,\nu}(z; u) \frac{1}{1 - xpt\eta_p - xqt\eta_q - spqt^2\eta_{p,q}}.$$

**Remark 2.** From Equation (19), for  $(p, q)$ -Lucas polynomials and getting  $a_k = 1$ ,  $\mu = 0$ ,  $\nu = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/r \rfloor} U_{n-rk}(x, s, p, q) L_k(z, sp^{-k} | p, q) u^k t^{n-rk} &= \frac{1 + spu^2}{1 - zpu} {}_2\phi_2 \left( \begin{matrix} (p, q), & 0 \\ (p, zupq), & (p, 0) \end{matrix} \middle| (p, q); -qsu^2 \right) \\ &\times \frac{1}{1 - xpt\eta_p - xqt\eta_q - spqt^2\eta_{p,q}}, \end{aligned}$$

where  $|zu| < 1$ .

Finally, choosing  $m = 1$  and  $\Lambda_{\mu+\nu k}(t) = T_{\mu+\nu k}(x, s, p, q)$  in Theorem 6, we derive the following bilinear generating functions for  $(p, q)$ -Chebyshev polynomials of the first kind given by (8).

**Corollary 3.** If  $Y_{\mu,\nu}(z;w) := \sum_{k=0}^{\infty} a_k T_{\mu+\nu k}(z,s,p,q) w^k$ , ( $a_k \neq 0$ ,  $\mu, \nu \in \mathbb{C}$ ), and

$$W_{n,r,\mu,\nu}(x; z; \zeta) := \sum_{k=0}^{[n/r]} a_k T_{n-rk}(x,s,p,q) T_{\mu+\nu k}(z,s,p,q) \zeta^k,$$

where  $n \in \mathbb{N}_0$ ;  $r \in \mathbb{N}$ , and then

$$\sum_{n=0}^{\infty} W_{n,r,\mu,\nu} \left( x; z; \frac{u}{t^r} \right) t^n = Y_{\mu,\nu}(z;u) \frac{1}{1 - xt\eta_p - xt\eta_q - spqt^2\eta_{p,q}} (1 - xt).$$

**Remark 3.** From Equation (6), for  $(p,q)$ -Chebyshev polynomials of the first kind and getting  $a_k = 1$ ,  $\mu = 0$ ,  $\nu = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{[n/r]} T_{n-rk}(x,s,p,q) T_k(z,s,p,q) u^k t^{n-rk} &= \frac{1}{1 - xt\eta_p - xt\eta_q - spqt^2\eta_{p,q}} (1 - xt) \\ &\times \frac{1}{1 - zu\eta_p - zu\eta_q - spqu^2\eta_{p,q}} (1 - zu). \end{aligned}$$

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