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Estimates for the Commutators of p-Adic Hausdorff Operator on Herz-Morrey Spaces

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Abstract: In this paper, we investigate the boundedness of commutators of matrix Hausdorff operator on the weighted p-adic Herz-Morrey space with the symbol functions in weighted central bounded mean oscillations (BMO) and Lipschitz spaces. In addition, a result showing boundedness of Hausdorff operator on weighted p-adic λ-central BMO spaces is provided as well.

Keywords: p-adic Hausdorff operator; commutators; p-adic Herz-Morrey spaces; p-adic λ-central bounded mean oscillations (BMO) spaces

1. Introduction

Let p be a fixed prime and x be a nonzero rational number. If x can be represented in the form $x = p^{\gamma}m/n$, where the integers m, n and fixed prime p are coprime to each other then $|x|_p = p^{-\gamma}$, where $\gamma = \gamma(x) \in \mathbb{Z}$. If $x = 0$ then we have $|0|_p = 0$. The p-adic absolute value $|\cdot|_p$ satisfies all conditions of norm along with the following two extra properties:

$$|xy|_p = |x|_p|y|_p, \qquad |x+y|_p \leq \max\{|x|_p, |y|_p\}. \tag{1}$$

The symbol $\mathbb{Q}_p$ denotes the field of p-adic numbers and is the completion of the field of rational number $\mathbb{Q}$ with respect to ultrametric p-adic norm $|\cdot|_p$. In [1], it was shown that any $x \in \mathbb{Q}_p\backslash\{0\}$ can be expressed in the canonical form as:

$$x = p^{\gamma}\sum_{k=0}^{\infty}\beta_k p^k, \tag{2}$$

where $\beta_k, \gamma \in \mathbb{Z}, \beta_0 \neq 0 \leq \beta_k < p$. The series (2) converges in p-adic norm because one has $|p^{\gamma}\beta_k p^k|_p \leq p^{-\gamma-k}$.

The space $\mathbb{Q}_p^n = \mathbb{Q}_p \times ... \times \mathbb{Q}_p$ consists of points $\mathbf{x} = (x_1, x_2, ..., x_n)$, where $x_k \in \mathbb{Q}_p, k = 1, 2, ..., n$. If $\mathbf{x} \in \mathbb{Q}_p^n$ then the following definition of norm is used on $\mathbb{Q}_p^n$

$$|\mathbf{x}|_p = \max_{1\leq k\leq n}|x_k|_p.$$

Let us express

$$B_{\gamma}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x}-\mathbf{a}|_p \leq p^{\gamma}\},$$

the ball with center at $\mathbf{a} \in \mathbb{Q}_p^n$ and radius p^{γ}. In a same manner, express by

$$S_{\gamma}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x}-\mathbf{a}|_p = p^{\gamma}\},$$

the sphere with center at $\mathbf{a} \in \mathbb{Q}_p$ and radius p^{γ}. When $\mathbf{a} = \mathbf{0}$, we merely represent $B_{\gamma}(\mathbf{0}) = B_{\gamma}$ and $S_{\gamma}(\mathbf{0}) = S_{\gamma}$. Also, for every $\mathbf{a}_0 \in \mathbb{Q}_p^n$, $\mathbf{a}_0 + B_{\gamma} = B_{\gamma}(\mathbf{a}_0)$ and $\mathbf{a}_0 + S_{\gamma} = S_{\gamma}(\mathbf{a}_0)$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, therefore, there exists a Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^n$, such that

$$\int_{B_0(\mathbf{0})} d\mathbf{x} = |B_0(\mathbf{0})| = 1,$$

where $|B|$ denotes the Haar measure of a subset B of $\mathbb{Q}_p^n$, and B is measurable. In addition, an easy computation yields $|B_\gamma(\mathbf{a})| = p^{n\gamma}$, $|S_\gamma(\mathbf{a})| = p^{n\gamma}(1-p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

The p-adic analysis has gained a lot of attention in the past few decades due to its importance in mathematical physics and its usefulness in science and technology (see, for instance, [2–5]). It is a fact that the theory of function from $\mathbb{Q}_p^n$ into $\mathbb{C}$ play a vital role in p-adic quantum mechanics [1]. In the last few years, many researchers have taken interest in the study of harmonic and wavelet analysis over p-adic fields which resulted in a number of reseach items, for instance, see [6–8].

The Hausdorff operator is considered very important in harmonic analysis due to its relation with the summability of classical Fourier series (see e.g., [9,10]). The matrix Hausdorff operator with kernel function Φ in Euclidean space $\mathbb{R}^n, n \geq 2$ was studied by Lerner and Liflyand in [11] and is of the form

$$\mathcal{H}_{\Phi,A} f(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{t}) f(A(\mathbf{t})\mathbf{x}) d\mathbf{t}. \tag{3}$$

where $A(\mathbf{t})$ is $n \times n$ invertible matrix for almost everywhere $\mathbf{t}$ in the support of Φ. If the kernel function Φ is chosen wisely then the Hausdorff operator reduces to some classical operators like the Hardy operator, the adjoint Hardy operator, the Hardy-Littlewood averaging operator and the Cesàro operator. Here we would like to mention some important publications including [11–24] which discussed the boundedness of Hausdorff operator on function spaces.

On the other hand, the p-adic matrix Hausdorff operator was introduced by Volosivets [25], which is given by

$$H_{\Phi,A}(f)(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \Phi(\mathbf{t}) f(A(\mathbf{t})\mathbf{x}) d\mathbf{t},$$

where $\Phi(\mathbf{t})$ is locally integrable function on $\mathbb{Q}_p^n$ and $A(\mathbf{t})$ is $n \times n$ nonsingular matrix for almost everywhere in the support of Φ. In recent times, the boundedness of the Hardy operator and the Hausdorff operator on p-adic field has become point of discussion for many authors (see, for instance, [26–30]). In [29], the Hausdorff operator was studied on weighted p-adic Morrey and Herz type spaces where, by imposing special conditions on the norm of the matrix A, sharp estimates were also obtained.

The boundedness properties of commutator operators is also an important aspect of harmonic analysis as these are useful in the study of characterization of function spaces and regularity theory of partial differential equations. The commutator of Hausdorff operator $H_{\Phi,A}$ with locally integrable function b is given by

$$H^b_{\Phi,A} f = bH_{\Phi,A} f - H_{\Phi,A}(bf).$$

The boundedness of the analog of $H^b_{\Phi,A}$ on $\mathbb{R}^n$ and its special cases when $A(\mathbf{t}) = \mathrm{diag}[1/|\mathbf{t}|, 1/|\mathbf{t}|, ..., 1/|\mathbf{t}|]$ were discussed in [31–37]. However, this topic still needs further considerations in the sense of its boundedness on p-adic function spaces.

In this paper, we will mainly discuss the boundedness of $H^b_{\Phi,A}$ on p-adic weighted Herz type spaces when b is either from $CMO^{q_2}(w, \mathbb{Q}_p^n)$ or $\Lambda_\delta(\mathbb{Q}_p^n)$. In addition an intermediate result stating the boundedness of p-adic matrix Hausdorff operator on λ-central bounded mean oscillations (BMO) spaces will be given at first.

2. Preliminaries and the Main Results

Suppose $w(\mathbf{x})$ is a weight function on $\mathbb{Q}_p^n$, which is nonnegative and locally integrable function on $\mathbb{Q}_p^n$. Let $L^q(w; \mathbb{Q}_p^n)$, $(0 < q < \infty)$ be the space of all complex-valued functions f on $\mathbb{Q}_p^n$ such that:

$$\|f\|_{L^q(w;\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^q w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} < \infty.$$

If $f \in L^1(\mathbb{Q}_p^n)$, we will write

$$\int_{\mathbb{Q}_p^n} f(\mathbf{x}) w(\mathbf{x}) d\mathbf{x} = \sum_{\gamma\in\mathbb{Z}} \int_{S_\gamma} f(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}.$$

Definition 1. *Let $\lambda < \frac{1}{n}$ and $1 < q < \infty$. The space $CMO^{q,\lambda}(w, \mathbb{Q}_p^n)$ is defined as follows.*

$$\|f\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{w(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} |f(\mathbf{x}) - f_{B_\gamma}|^q w(\mathbf{x}) d\mathbf{x} \right)^{1/q} < \infty, \tag{4}$$

where

$$f_{B_\gamma} = \frac{1}{|B_\gamma|} \int_{B_\gamma} f(\mathbf{x}) d\mathbf{x}. \tag{5}$$

Remark 1. *When $\lambda = 0$, the space $CMO^{q,\lambda}(w, \mathbb{Q}_p^n)$ is just reduced to $CMO^q(w, \mathbb{Q}_p^n)$ with corresponding norm given as follows.*

$$\|f\|_{CMO^q(w,\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{w(B_\gamma)} \int_{B_\gamma} |f(\mathbf{x}) - f_{B_\gamma}|^q w(\mathbf{x}) d\mathbf{x} \right)^{1/q}.$$

Definition 2. *Suppose $\alpha \in \mathbb{R}$, $0 < l, q < \infty$, the weighted Herz space $K_q^{\alpha,l}(w, \mathbb{Q}_p^n)$ is defined by:*

$$K_q^{\alpha,l}(w, \mathbb{Q}_p^n) = \{f \in L^q_{\mathrm{loc}}(w, \mathbb{Q}_p^n \backslash \{0\}) : \|f\|_{K_q^{\alpha,l}(w,\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,l}(w,\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha l/n} \|f\chi_k\|^l_{L^q(w,\mathbb{Q}_p^n)} \right)^{1/l}, \tag{6}$$

and χ_k is the characteristic function of the sphere $S_k = B_k \setminus B_{k-1}$.

Remark 2. *Obviously $K_q^{0,q}(w, \mathbb{Q}_p^n) = L^q(w, \mathbb{Q}_p^n)$.*

Definition 3. *Suppose $\alpha \in \mathbb{R}$, $0 < l, q < \infty$, and λ be a non-negative real number. Then the weighted Morrey-Herz space $MK_{l,q}^{\alpha,\lambda}(w, \mathbb{Q}_p^n)$ is defined as follows.*

$$MK_{l,q}^{\alpha,\lambda}(w, \mathbb{Q}_p^n) = \{f \in L^q_{loc}(w, \mathbb{Q}_p^n \backslash \{0\}) : \|f\|_{MK_{l,q}^{\alpha,\lambda}(w,\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{MK_{l,q}^{\alpha,\lambda}(w,\mathbb{Q}_p^n)} = \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n} \left(\sum_{k=-\infty}^{k_0} w(B_k)^{\alpha l/n} \|f\chi_k\|^l_{L^q(w,\mathbb{Q}_p^n)} \right)^{1/l}. \tag{7}$$

Remark 3. *It is evident that $MK_{l,q}^{\alpha,0}(w, \mathbb{Q}_p^n) = K_q^{\alpha,l}(w, \mathbb{Q}_p^n)$.*

For the analog of Herz-Morrey space on Euclidean space $\mathbb{R}^n$, we refer the interested reader to the paper [38] by Lu and Xu. Recently, the study reported in [38] was extended to variable exponent Herz-Morrey spaces in [39,40].

Definition 4. *Suppose $\delta \in \mathbb{R}^+$. The Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$ is the space of all measurable functions f on $\mathbb{Q}_p^n$ such that:*

$$\|f\|_{\Lambda_\delta(\mathbb{Q}_p^n)} = \sup_{\mathbf{x},\mathbf{h}\in\mathbb{Q}_p^n,\mathbf{h}\neq\mathbf{0}} \frac{|f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})|}{|\mathbf{h}|_p^\delta} < \infty.$$

Lemma 1. *([30]) Let E be an $n \times n$ matrix with entries $e_{ij} \in \mathbb{Q}_p$. Then the norm of E, regarded as an operator from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^n$, is defined as:*

$$\|E\| = \max_{1\leq i\leq n}\max_{1\leq j\leq n} |e_{ij}|_p.$$

Definition 5. *([26]) Let $\beta \in \mathbb{R}$. The set $\mathbb{W}_\beta$ consist of all measurable function $w(\mathbf{x})$ on $\mathbb{Q}_p^n$, Satisfying:*

(a) $w(\mathbf{x}) > 0$ *a.e.,*
(b) $\int_{S_0} w(\mathbf{x})d\mathbf{x} < \infty$,
(c) $w(t\mathbf{x}) = |t|_p^\beta w(\mathbf{x})$ *for all* $t \in \mathbb{Q}_p\backslash\{0\}, \mathbf{x} \in \mathbb{Q}_p^n$.

It is not difficult to see that a weight $w(\mathbf{x}) \in \mathbb{W}_\beta$ needs not to be necessarily locally integrable function. Importantly, if $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$, then $w(\mathbf{x}) \in \mathbb{W}_\beta$ but $w(\mathbf{x}) \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$ if and only if $\beta > -n$.

Lemma 2. *([27]) Let $w \in \mathbb{W}_\beta, \beta > -n$. Then for any $\gamma \in \mathbb{Z}$, we have*

$$w(B_\gamma) = p^{(n+\beta)\gamma}w(B_0) \quad \textit{and} \quad w(S_\gamma) = p^{(n+\beta)\gamma}w(S_0).$$

Here and in the sequel, for the sake of easiness, we use the following notation:

$$G(E,\delta\beta) = \begin{cases} \|E\|^{\delta\beta} & \text{if } \beta > 0, \\ \|E^{-1}\|^{-\delta\beta} & \text{if } \beta \leq 0, \end{cases}$$

where E is any invertible matrix, $\beta \in \mathbb{R}$ and δ is a non-zero positive real number.

It is easy to see that:

$$G(E,\beta(1/q+1/p)) = G(E,\beta/q)G(E,\beta/p), \tag{8}$$

where $p,q \in \mathbb{Z}^+$.

Proposition 1. *([29]) Let $\beta > -n$, $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$, E is any nonsingular matrix and $\mathbf{x} \in \mathbb{Q}_p^n$, then*

$$\begin{aligned} w(E\mathbf{x}) &\leq \begin{cases} \|E\|^\beta w(\mathbf{x}) & \textit{if } \beta > 0, \\ \|E^{-1}\|^{-\beta} w(\mathbf{x}) & \textit{if } \beta \leq 0, \end{cases} \\ &= G(E,\beta)w(\mathbf{x}). \end{aligned}$$

Lemma 3. *([29]) Let $\beta > -n$, $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$ and E is any nonsingular matrix, then we have*

$$w(EB_\gamma(\mathbf{a})) \leq G(E,\beta)|\det E|_p w(B_\gamma(\mathbf{a})).$$

Now, we are in position to state our main results which are as under:

Main Results

Theorem 1. *Let $1 < q < \infty$, $0 \le \lambda < 1/n$, $\beta > -n$ and $w(\mathbf{x}) = |\mathbf{x}|_p^{\beta}$, then $H_{\Phi,A}$ is bounded on $CMO^{q,\lambda}(w,\mathbb{Q}_p^n)$ and satisfies the following inequality*

$$\|H_{\Phi,A}f\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)} \le K_1\|f\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)},$$

where

$$K_1 = \int_{\mathbb{Q}_p^n} |\Phi(\mathbf{t})||\det A(\mathbf{t})|_p^{\lambda} G(A^{-1}(\mathbf{t}),\beta/q)G(A(\mathbf{t}),\beta(\lambda+1/q))d\mathbf{t}.$$

Our first result regarding boundedness of $H^b_{\Phi,A}$ with $b \in CMO^q(w,\mathbb{Q}_p^n)$ can be stated as:

Theorem 2. *Let $1 \le l, q, q_1, q_2 < \infty$, $1/q_2 = 1/q + 1/q_1$, $\alpha_1 = \alpha_2 + n/q$, $0 \le \lambda > \alpha_1$, $\beta > -n$ and $w(\mathbf{x}) = |\mathbf{x}|_p^{\beta}$. Assume that $b \in CMO^q(w,\mathbb{Q}_p^n)$ and*

$$\varphi(\mathbf{t}) = |\Phi(\mathbf{t})||\det A^{-1}(\mathbf{t})|_p^{1/q_1} G(A^{-1}(\mathbf{t}),\beta/q_1)\max\left\{1, G(A^{-1}(\mathbf{t}),\beta/q)G(A(\mathbf{t}),\beta/q)\right\}.$$

Then the commutator operator $H^b_{\Phi,A}$ is bounded from $MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)$ to $MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)$ and satisfies the inequality:

$$\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)} \le CK_2\|b\|_{CMO^{q_2}(w,\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)},$$

where

$$\begin{aligned}K_2 = &\int_{\|A(\mathbf{t})\|\le 1} \varphi(\mathbf{t})\|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n}\left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} + \log_p \frac{p}{\|A(\mathbf{t})\|}\right)d\mathbf{t}\\ &+ \int_{\|A(\mathbf{t})\|>1} \varphi(\mathbf{t})\|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n}\left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} + \log_p(p\|A(\mathbf{t})\|)\right)d\mathbf{t}.\end{aligned}$$

In the following theorem we proved the boundedness of commutator of Hausdorff operator on Morrey Herz Space by taking $b \in \Lambda_\delta(\mathbb{Q}_p^n)$.

Theorem 3. *Let $1 \le q_2 \le q_1 < \infty$, $0 < l, \delta < \infty$, $1/r = 1/q_2 - 1/q_1$, $\beta > -n$ and $w(\mathbf{x}) = |\mathbf{x}|_p^{\beta}$, $\alpha_1 = \alpha_2 + n\delta/(n+\beta) + n(1/q_2 - 1/q_1)$, $0 \le \lambda > \alpha_1$ and $b \in \Lambda_\delta(\mathbb{Q}_p^n)$. Then the commutator operator $H^b_{\Phi,A}$ is bounded from $MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)$ to $MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)$ and satisfies the inequality:*

$$\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)} \le CK_3\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)},$$

where

$$K_3 = \int_{\mathbb{Q}_p^n} |\Phi(\mathbf{t})|\|A(\mathbf{t})\|^{(n+\beta)(\lambda/n - \alpha_1/n)}\max\{1, \|A(\mathbf{t})\|^{\delta}\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1} G(A^{-1}(\mathbf{t}),\beta/q_1)d\mathbf{t}.$$

In the rest of the article, the character C denote the constant free from essential values and variables whose value may change from line to line.

3. Proofs of Main Results

3.1. Proof of Theorem 1

Suppose $f \in CMO^{q,\lambda}(w, \mathbb{Q}_p^n)$. By means of Fubini theorem and p-adic change of variables we have

$$\begin{aligned}
\Big(H_{\Phi,A}f\Big)_{B_\gamma} &= \frac{1}{|B_\gamma|}\int_{B_\gamma}\left(\int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})f(A(\mathbf{t})\mathbf{x})d\mathbf{t}\right)d\mathbf{x}\\
&= \int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})\left(\frac{1}{|B_\gamma|}\int_{B_\gamma}f(A(\mathbf{t})\mathbf{x})d\mathbf{x}\right)d\mathbf{t}\\
&= \int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})\left(\frac{1}{|B_\gamma|}\int_{A(\mathbf{t})B_\gamma}f(\mathbf{x})d\mathbf{x}\right)|\det A^{-1}(\mathbf{t})|_p d\mathbf{t}\\
&= \int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})\left(\frac{1}{|A(\mathbf{t})B_\gamma|}\int_{A(\mathbf{t})B_\gamma}f(\mathbf{x})d\mathbf{x}\right)d\mathbf{t}\\
&= \int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})f_{A(\mathbf{t})B_\gamma}d\mathbf{t}.
\end{aligned}$$

Using Minkowski's inequality, Proposition 1 and Lemma 3 with $\mathbf{a} = 0$, we are down to

$$\begin{aligned}
&\left(\frac{1}{w(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}|H_{\Phi,A}f(\mathbf{x}) - (H_{\Phi,A})_{B_\gamma}|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&= \left(\frac{1}{w(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}\left|\int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})(f(A(\mathbf{t})\mathbf{x}) - f_{A(\mathbf{t})B_\gamma})d\mathbf{t}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&\leq \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\frac{1}{w(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}\left|f(A(\mathbf{t})\mathbf{x}) - f_{A(\mathbf{t})B_\gamma}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})||\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)\\
&\qquad\times\left(\frac{1}{w(B_\gamma)^{1+\lambda q}}\int_{A(\mathbf{t})B_\gamma}\left|f(\mathbf{x}) - f_{A(\mathbf{t})B_\gamma}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})||\det A(\mathbf{t})|_p^{\lambda}G(A^{-1}(\mathbf{t}),\beta/q)G(A(\mathbf{t}),\beta(\lambda+1/q))\\
&\qquad\times\left(\frac{1}{w(A(\mathbf{t})B_\gamma)^{1+\lambda q}}\int_{A(\mathbf{t})B_\gamma}\left|f(\mathbf{x}) - f_{A(\mathbf{t})B_\gamma}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}d\mathbf{t}\\
&\leq \|f\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})||\det A(\mathbf{t})|_p^{\lambda}G(A^{-1}(\mathbf{t}),\beta/q)G(A(\mathbf{t}),\beta(\lambda+1/q))d\mathbf{t}.
\end{aligned}$$

Thus, we completed the proof of Theorem 1.

3.2. Proof of Theorem 2

Let $f \in MK_{l,q_1}^{\alpha_1,\lambda}(w, \mathbb{Q}_p^n)$ and $b \in CMO^q(w, \mathbb{Q}_p^n)$,

$$
\begin{aligned}
\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)} =& \left(\int_{S_k}\left|\int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})(b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x}))f(A(\mathbf{t})\mathbf{x})d\mathbf{t}\right|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}\\
\leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\int_{S_k}|(b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x}))f(A(\mathbf{t})\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}d\mathbf{t}\\
\leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\int_{S_k}|(b(\mathbf{x})-b_{B_k})f(A(\mathbf{t})\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}d\mathbf{t}\\
&+\int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\int_{S_k}|(b_{B_k}-b_{A(\mathbf{t})B_k})f(A(\mathbf{t})\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}d\mathbf{t}\\
&+\int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\int_{S_k}|(b(A(\mathbf{t})\mathbf{x})-b_{A(\mathbf{t})B_k})f(A(\mathbf{t})\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}d\mathbf{t}\\
=& I+II+III.
\end{aligned}
$$

By Hölder's inequality, p-adic change of variables and Proposition 1, we estimate I as below:

$$
\begin{aligned}
I \leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\int_{S_k}|b(\mathbf{x})-b_{B_k}|^{q}w(\mathbf{x})d\mathbf{x}\right)^{1/q}\left(\int_{S_k}|f(A(\mathbf{t})\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}d\mathbf{t}\\
\leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\int_{B_k}|b(\mathbf{x})-b_{B_k}|^{q}w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&\times\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}|\det A^{-1}(\mathbf{t})|_pG(A^{-1}(\mathbf{t}),\beta)w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}d\mathbf{t}\\
\leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})||\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)\\
&\times\left(\int_{B_k}|b(\mathbf{x})-b_{B_k}|^{q}w(\mathbf{x})d\mathbf{x}\right)^{1/q}\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}d\mathbf{t}\\
\leq& w(B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
&\times\int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})||\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}d\mathbf{t}.
\end{aligned}
$$

Similarly for III, first making p-adic change of variables and then applying Hölder's inequality, we have

$$
\begin{aligned}
III \leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left(\int_{S_k}|(b(A(\mathbf{t})\mathbf{x})-b_{A(\mathbf{t})B_k})f(A(\mathbf{t})\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}d\mathbf{t}\\
=& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\det A^{-1}(\mathbf{t})|_p^{1/q_2}G(A^{-1}(\mathbf{t}),\beta/q_2)|\\
&\times\left(\int_{A(\mathbf{t})S_k}|(b(\mathbf{x})-b_{A(\mathbf{t})B_k})f(\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}d\mathbf{t}\\
\leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\det A^{-1}(\mathbf{t})|_p^{1/q_2}G(A^{-1}(\mathbf{t}),\beta/q_2)|\\
&\times\left(\int_{A(\mathbf{t})S_k}|b(\mathbf{x})-b_{A(\mathbf{t})B_k}|^{q}w(\mathbf{x})d\mathbf{x}\right)^{1/q}\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}d\mathbf{t}\\
\leq& \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\det A^{-1}(\mathbf{t})|_p^{1/q_2}G(A^{-1}(\mathbf{t}),\beta/q_2)|\\
&\times w(A(\mathbf{t})B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}d\mathbf{t}.
\end{aligned}
$$

Since $1/q_2 = 1/q + 1/q_1$, therefore, by virtue of the property (8) and Lemma 3, the above inequality becomes

$$
\begin{aligned}
III \leq & \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \int_{\mathbb{Q}_p^n} |\Phi(\mathbf{t})| \det A^{-1}(\mathbf{t})|_p^{1/q_2} G(A^{-1}(\mathbf{t}), \beta/q_1) G(A^{-1}(\mathbf{t}), \beta/q)| \\
& \times G(A(\mathbf{t}), \beta/q)| \det A(\mathbf{t})|_p^{1/q} w(B_k)^{1/q} \|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} d\mathbf{t} \\
= & w(B_k)^{1/q} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \int_{\mathbb{Q}_p^n} |\Phi(\mathbf{t})|| \det A^{-1}(\mathbf{t})|_p^{1/q_1} G(A^{-1}(\mathbf{t}), \beta/q_1) \\
& \times G(A^{-1}(\mathbf{t}), \beta/q) G(A(\mathbf{t}), \beta/q) \|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} d\mathbf{t}.
\end{aligned} \tag{9}
$$

The estimation of II is very much similar to that of I and III except that in this case, additionally, we have to bound the term $|b_{B_k} - b_{A(\mathbf{t})B_k}|$. Therefore, in this case, we will make use of the Hölder's inequality, Lemma 2 and p-adic change of variables to have

$$
\begin{aligned}
II \leq & \int_{\mathbb{Q}_p^n} |\Phi(\mathbf{t})| \left(\int_{S_k} |f(A(\mathbf{t})\mathbf{x})|^{q_1} w(\mathbf{x}) d\mathbf{x} \right)^{1/q_1} \left(\int_{S_k} w(\mathbf{x}) d\mathbf{x} \right)^{1/q} |b_{B_k} - b_{A(\mathbf{t})B_k}| d\mathbf{t} \\
\leq & w(S_0)^{1/q} w(B_k)^{1/q} \\
& \times \int_{\mathbb{Q}_p^n} |\Phi(\mathbf{t})|| \det A^{-1}(t)|_p^{1/q_1} G(A^{-1}(\mathbf{t}), \beta/q_1) \|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} |b_{B_k} - b_{A(\mathbf{t})B_k}| d\mathbf{t}.
\end{aligned}
$$

Next, if $\|A(\mathbf{t})\| > 1$, then there exists an integer $j \geq 0$, such that

$$p^j < \|A(\mathbf{t})\| \leq p^{j+1}.$$

Therefore,

$$|b_{B_k} - b_{A(\mathbf{t})B_k}| \leq \sum_{i=0}^{j} |b_{B_{k+i}} - b_{B_{k+i+1}}| + |b_{B_{k+j+1}} - b_{A(\mathbf{t})B_k}|.$$

A use of Hölder's Inequality and the definition of $CMO^q(w, \mathbb{Q}_p^n)$ yields

$$
\begin{aligned}
|b_{B_{k+i}} - b_{B_{k+i+1}}| \leq & \frac{1}{|B_{k+i}|} \int_{B_{k+i}} |b(\mathbf{x}) - b_{B_{k+i+1}}| d\mathbf{x} \\
\leq & \frac{C}{|B_{k+i+1}|} \int_{B_{k+i+1}} |b(\mathbf{x}) - b_{B_{k+i+1}}| d\mathbf{x} \\
\leq & \frac{C}{|B_{k+i+1}|} \left(\int_{B_{k+i+1}} |b(\mathbf{x}) - b_{B_{k+i+1}}|^q w(\mathbf{x}) d\mathbf{x} \right)^{1/q} \\
& \times \left(\int_{B_{k+i+1}} w(\mathbf{x})^{q'/q} \right)^{1/q'} \\
\leq & C \frac{w(B_{k+i+1})^{1/q}}{|B_{k+i+1}|} \left(\int_{B_{k+i+1}} |\mathbf{x}|_p^{-\beta q'/q} \right)^{1/q'} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \\
\leq & C \frac{p^{(n+\beta)(k+i+1)/q}}{p^{(k+i+1)n}} p^{(k+i+1)(-\beta/q + n/q')} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \\
= & C \|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.
\end{aligned}
$$

The other term can be treated in a similar way as below

$$
\begin{aligned}
|b_{B_{k+j+1}} - b_{A(\mathbf{t})B_k}| \leq & \frac{1}{|A(\mathbf{t})B_k|} \int_{A(\mathbf{t})B_k} |b(\mathbf{x}) - b_{B_{k+j+1}}| d\mathbf{x} \\
\leq & \frac{1}{|A(\mathbf{t})B_k|} \left(\int_{B_{k+j+1}} |b(\mathbf{x}) - b_{B_{k+j+1}}|^q w(\mathbf{x}) d\mathbf{x} \right)^{1/q} \\
& \times \left(\int_{B_{k+j+1}} w(\mathbf{x})^{-q'/q} \right)^{1/q'} \\
\leq & \frac{w(B_{k+j+1})^{1/q}}{|A(\mathbf{t})B_k|} \left(\int_{B_{k+j+1}} |\mathbf{x}|_p^{-\beta q'/q} d\mathbf{x} \right)^{1/q'} \\
& \times \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \\
\leq & \frac{p^{(n+\beta)(k+j+1)/q}}{|\det A(\mathbf{t})|_p p^{kn}} p^{(k+j+1)(-\beta/q+n/q')} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \\
= & \frac{p^{(j+1)n}}{|\det A(\mathbf{t})|_p} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \\
\leq & C \frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.
\end{aligned}
$$

Therefore, for $\|A(\mathbf{t})\| > 1$

$$
\begin{aligned}
|b_{B_k} - b_{A(\mathbf{t})B_k}| \leq & C \left(j + 1 + \frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} \right) \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \\
\leq & C \left\{ \log_p(\|A(\mathbf{t})\| p) + \frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} \right\} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.
\end{aligned}
$$

When $\|A(\mathbf{t})\| \leq 1$, a similar argument yields

$$
|b_{B_k} - b_{A(\mathbf{t})B_k}| \leq C \left\{ \log_p \frac{p}{\|A(\mathbf{t})\|} + \frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} \right\} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.
$$

Therefore,

$$
\begin{aligned}
II \leq & C w(B_k)^{1/q} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \int_{\mathbb{Q}_p^n} |\Phi(\mathbf{t})| \det A^{-1}(\mathbf{t})|_p^{1/q_1} G(A^{-1}(\mathbf{t}), \beta/q_1)| \\
& \times \left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} + 1 + \max \left\{ \log_p \|A(\mathbf{t})\|, \log_p \frac{1}{\|A(\mathbf{t})\|} \right\} \right) \|f \chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} d\mathbf{t}.
\end{aligned}
$$

Finally, we combine the estimates for I,II and III to have

$$
\begin{aligned}
& \|(H^b_{\Phi,A} f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)} \\
& \leq C w(B_k)^{1/q} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \\
& \times \int_{\mathbb{Q}_p^n} \varphi(\mathbf{t}) \left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} + 1 + \max \left\{ \log_p \|A(\mathbf{t})\|, \log_p \frac{1}{\|A(\mathbf{t})\|} \right\} \right) \|f \chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} d\mathbf{t}.
\end{aligned}
$$

In order to avoid repetition of the same factor in the subsequent calculations, we let

$$
\psi(\mathbf{t}) = \varphi(\mathbf{t}) \left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} + 1 + \max \left\{ \log_p(\|A\mathbf{t})\|, \log_p \frac{1}{\|A(\mathbf{t})\|} \right\} \right).
$$

Also, it is easy to see that (see [29], Theorem 3.1)

$$\begin{aligned}\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} &= \left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}d\mathbf{x}\right)^{1/q_1}\\ &\le \left(\int_{|\mathbf{x}|_p\le\|A(\mathbf{t})\|p^k}|f(\mathbf{x})|^{q_1}d\mathbf{x}\right)^{1/q_1}\\ &\le C\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}.\end{aligned}\tag{10}$$

Therefore,

$$\begin{aligned}&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\ &\le Cw(B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\psi(\mathbf{t})\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}d\mathbf{t}.\end{aligned}\tag{11}$$

Now, by the definition of Morrey-Herz space, the inequality (11), Minkowski's inequality and the condition $\alpha_1=\alpha_2+n/q$, we have

$$\begin{aligned}&\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)}\\ &=\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda(n+\beta)/n}\left(\sum_{k=-\infty}^{k_0}p^{k\alpha_2(n+\beta)l/n}\|(H^b_{\Phi,A}f)\chi_k\|^l_{L^q(w,\mathbb{Q}_p^n)}\right)^{1/l}\\ &\le C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\psi(\mathbf{t})\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda(n+\beta)/n}\\ &\quad\times\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{k(\alpha_2+n/q)(n+\beta)/n}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}d\mathbf{t}\\ &\le C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\psi(\mathbf{t})\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda(n+\beta)/n}\\ &\quad\times\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{k\alpha_1(n+\beta)/n}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}d\mathbf{t}\\ &\le C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\psi(\mathbf{t})\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{m(\lambda-\alpha_1)(n+\beta)/n}\\ &\quad\times\sup_{k_0\in\mathbb{Z}}p^{-(k_0+m)\lambda(n+\beta)/n}\left(\sum_{k=-\infty}^{k_0+m}p^{k\alpha_1(n+\beta)l/n}\|f\chi_k\|^l_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{1/l}d\mathbf{t}.\end{aligned}$$

Since $\alpha_1<\lambda$, as a consequence

$$\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{m(\lambda-\alpha_1)(n+\beta)/n}=\frac{\|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n}}{1-p^{(\alpha_1-\lambda)(n+\beta)/n}}.\tag{12}$$

Hence,

$$\begin{aligned}\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)}\le{}&C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\\ &\times\int_{\mathbb{Q}_p^n}\psi(\mathbf{t})\|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n}d\mathbf{t}.\end{aligned}$$

Thus the proof of the Theorem 2 is completed.

3.3. Proof of Theorem 3

Let $f \in MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)$, $b \in \Lambda_\delta(\mathbb{Q}_p^n)$. By applying the Minkowski's inequality and the Holder's inequality, we get

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\
&= \left[\int_{S_k}\left|\int_{\mathbb{Q}_p^n}\Phi(\mathbf{t})f(A(\mathbf{t})\mathbf{x})(b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x}))d\mathbf{t}\right|^{q_2}w(\mathbf{x})d\mathbf{x}\right]^{1/q_2}\\
&\leq \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left[\int_{S_k}\left|f(A(\mathbf{t})\mathbf{x})(b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x}))\right|^{q_2}w(\mathbf{x})d\mathbf{x}\right]^{1/q_2}d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|\left[\int_{S_k}|f(A(\mathbf{t})\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right]^{1/q_1}\\
&\quad\times\left[\int_{S_k}|b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x})|^{r}w(\mathbf{x})d\mathbf{x}\right]^{1/r}d\mathbf{t},
\end{aligned}\tag{13}$$

where $1/r = 1/q_2 - 1/q_1$. By the definition of Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$ we have

$$\begin{aligned}
|b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x})| &\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}|\mathbf{x}-A(\mathbf{t})\mathbf{x}|_p^\delta\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\max\{|\mathbf{x}|_p,|A(\mathbf{t})\mathbf{x}|_p\}^\delta\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\max\{|\mathbf{x}|_p,\|A(\mathbf{t})\||\mathbf{x}|_p\}^\delta\\
&\leq p^{k\delta}C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\max\{1,\|A(\mathbf{t})\|^\delta\},
\end{aligned}\tag{14}$$

for every $\mathbf{x} \in S_k$ and for almost everywhere $\mathbf{t} \in \mathbb{Q}_p^n$.

By p-adic change of variables, Proposition 1, inequality (14) together with $w(S_k)w(B_0) = w(B_k)w(S_0)$, inequality (13) assumes the following form

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}|\Phi(\mathbf{t})|w(B_k)^{\delta/(n+\beta)+1/r}\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\\
&\quad\times\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)d\mathbf{t}.
\end{aligned}\tag{15}$$

Furthermore, in view of inequality (10) and $1/r = 1/q_2 - 1/q_1$, we get

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}w(B_k)^{\delta/(n+\beta)+1/q_2-1/q_1}\int_{\mathbb{Q}_p^n}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\\
&\quad\times|\Phi(\mathbf{t})|\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)d\mathbf{t}.
\end{aligned}\tag{16}$$

The factor $|\Phi(\mathbf{t})|\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)$ repeats itself many times in the remaining proof of this theorem, so we let it be denoted by $\phi(\mathbf{t})$. With this we break our proof in the following two cases:

Case 1: $\lambda > 0$, in this case we first evaluate the inner norm $\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}$ as below:

$$\begin{aligned}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} &\leq w(B_{k+m})^{-\alpha_1/n}\left[\sum_{j=-\infty}^{k+m} w(B_j)^{\alpha_1 l/n}\|f\chi_j\|^l_{L^{q_1}(w,\mathbb{Q}_p^n)}\right]^{1/l}\\ &= w(B_{k+m})^{-\alpha_1/n}w(B_{k+m})^{\lambda/n}\\ &\quad\times w(B_{k+m})^{-\lambda/n}\left(\sum_{j=-\infty}^{k+m} w(B_j)^{\alpha_1 l/n}\|f\chi_j\|^l_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{1/l}\\ &= w(B_{k+m})^{(\lambda-\alpha_1)/n}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\\ &\leq p^{(k+m)(n+\beta)(\lambda-\alpha_1)/n}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}. \end{aligned} \tag{17}$$

Next, by virtue of Equation (12), the inequality (16) becomes

$$\begin{aligned}&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\ &\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1+(\lambda-\alpha_1)/n)}\\ &\quad\times\int_{\mathbb{Q}_p^n}\phi(\mathbf{t})|\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|} p^{m(n+\beta)(\lambda-\alpha_1)/n}d\mathbf{t}\\ &\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1+(\lambda-\alpha_1)/n)}\\ &\quad\times\int_{\mathbb{Q}_p^n}\phi(\mathbf{t})\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}d\mathbf{t}.\end{aligned}$$

Therefore, by definition of Morrey-Herz space and $\alpha_1 = \alpha_2 + n\delta/(n+\beta) + n(1/q_2 - 1/q_1)$, we have

$$\begin{aligned}&\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\\ &\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}p^{-k_0(n+\beta)\lambda/n}\\ &\quad\times\left[\sum_{k=-\infty}^{k_0}p^{kl(n+\beta)(\alpha_2/n+\delta/(n+\beta)+1/q_2-1/q_1+(\lambda-\alpha_1)/n)}\right.\\ &\quad\times\left.\left(\int_{\mathbb{Q}_p^n}\phi(\mathbf{t})\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}d\mathbf{t}\right)^l\right]^{1/l}\\ &\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\phi(\mathbf{t})\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}d\mathbf{t}\\ &\quad\times\sup_{k_0\in\mathbb{Z}}p^{-k_0(n+\beta)\lambda/n}\left[\sum_{k=-\infty}^{k_0}p^{kl(n+\beta)\lambda/n}\right]^{1/l}\\ &\leq C\frac{p^{(n+\beta)\lambda/n}}{(p^{l(n+\beta)\lambda/n}-1)^{1/l}}C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\phi(\mathbf{t})\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}d\mathbf{t}\\ &\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\phi(\mathbf{t})\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}d\mathbf{t},\end{aligned}$$

substituting back the value of $\phi(\mathbf{t})$ we get the desired result.

Case 2: $\lambda = 0$ and $l \in [1, \infty)$.
In this case Morrey-Herz spaces are reduced to the Herz spaces. It is clear that

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}^n_p)} \\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)} \int_{\mathbb{Q}^n_p} \phi(\mathbf{t}) p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1)} \sum_{m=-\infty}^{\log_p \|A(\mathbf{t})\|} \|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}^n_p)} d\mathbf{t}. \qquad (18)
\end{aligned}$$

Hence, by using the Minkowski's inequality and $\alpha_1 = \alpha_2 + n\delta/(n+\beta) + n(1/q_2 - 1/q_1)$, we obtain

$$\begin{aligned}
&\|H^b_{\Phi,A}f\|_{K^{\alpha_2,l}_{q_2}(w,\mathbb{Q}^n_p)} \\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)} \Bigg\{ \sum_{k=-\infty}^{+\infty} p^{k(n+\beta)\alpha_2 l/n} \\
&\quad \times \Bigg[\int_{\mathbb{Q}^n_p} \phi(\mathbf{t}) p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1)} \sum_{m=-\infty}^{\log_p \|A(\mathbf{t})\|} \|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}^n_p)} d\mathbf{t} \Bigg]^l \Bigg\}^{1/l} \\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)} \int_{\mathbb{Q}^n_p} \phi(\mathbf{t}) \Bigg\{ \sum_{k=-\infty}^{+\infty} \Bigg(\sum_{m=-\infty}^{\log_p \|A(\mathbf{t})\|} p^{k(n+\beta)\alpha_1/n} \|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}^n_p)} \Bigg)^l \Bigg\}^{1/l} d\mathbf{t} \\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)} \int_{\mathbb{Q}^n_p} \phi(\mathbf{t}) \sum_{m=-\infty}^{\log_p \|A(\mathbf{t})\|} p^{-m(n+\beta)\alpha_1/n} \Bigg(\sum_{k=-\infty}^{+\infty} p^{kl(n+\beta)\alpha_1/n} \|f\chi_k\|^l_{L^{q_1}(w,\mathbb{Q}^n_p)} \Bigg)^{1/l} d\mathbf{t} \\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)} \|f\|_{K^{\alpha_1,l}_{q_1}(w,\mathbb{Q}^n_p)} \\
&\quad \times \int_{\mathbb{Q}^n_p} |\Phi(\mathbf{t})| \max\{1, \|A(\mathbf{t})\|^\delta\} |\det A^{-1}(\mathbf{t})|_p^{1/q_1} G(A^{-1}(\mathbf{t}), \beta/q_1) \|A(\mathbf{t})\|^{-(n+\beta)\alpha_1/n} d\mathbf{t}. \qquad (19)
\end{aligned}$$

From the inequalities (18) and (19), we get the proof.

4. Conclusions

Here we employed some conditions on the norm of matrix $A(\mathbf{t})$ to ensure the boundedness of the commutators of Hausdorff operator on p-adic Herz-Morrey spaces. An idea that can be employed on various situations to obtain boundedness results for the p-adic matrix Hausdorff operator and its commutators on different function spaces.

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