


## Article

# Change of Basis Transformation from the Bernstein Polynomials to the Chebyshev Polynomials of the Fourth Kind

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**Abstract:** In this paper, the change of bases transformations between the Bernstein polynomial basis and the Chebyshev polynomial basis of the fourth kind are studied and the matrices of transformation among these bases are constructed. Some examples are given.

**Keywords:** Bernstein polynomials; Chebyshev polynomials of fourth kind; computer aided design; basis transformation

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## 1. Introduction

This paper discusses and finds the matrices of transformation between Bernstein polynomials and Chebyshev polynomials of the fourth kind. Both bases have some excellent advantages and there is a need to convert polynomial forms between them. The Bernstein polynomials form the basis for the Bézier curves and surfaces that are used in modeling and design and in many other applications. The Bézier curves have many geometric properties like the fact that they are modeled by their control points, lying in the convex hull of their control points, and the control polygon is tangent to the Bézier curve. Calculus and degree raising and reduction of the Bézier curves are essential operations in applications. In some cases, the orthogonality property of the Chebyshev polynomials simplifies calculations and enables us to get the solution in explicit form. Thereafter, there is a need to change bases transformations to get the solution in the Bézier form.

An example of a specific essential operation is the Bernstein polynomials [1–3], which are not orthogonal while possessing interesting geometric properties. On the other hand, the Chebyshev polynomials of fourth kind [4,5] are orthogonal, but do not have geometric properties that are in Bernstein polynomials. When the geometric properties are needed, but the curve is given in Chebyshev polynomials of fourth kind, change of bases transformations should be carried out. Some cases are given in [6–8]. These change of bases transformations can also be studied for the cases between the Chebyshev polynomials of fourth kind and the q-Bernstein polynomials in [9], degenerate Bernstein polynomials in [10], and the multidimensional Bernstein polynomials in [11]. This paper is organized as follows. In Section 2, materials, methods, and some preliminary results are stated and proved. In Section 3, the main result of the matrices of change of bases transformations are derived. In Section 4, the proof of the main theorem together with two examples are presented.

In this section, materials related to this paper are stated. This includes defining the main concepts, giving their properties, and related notations. The Chebyshev polynomials of fourth kind satisfy the following linear homogeneous differential equation of the second order

$$(1-x^2)y'' - (1+2x)y' + n(n+1)y = 0.$$

The Chebyshev polynomials of the fourth kind,  $W_n(x)$ , are orthogonal polynomials on  $[-1, 1]$  with respect to the weight function  $w(x) = \sqrt{\frac{1-x}{1+x}}$ . The following recurrence relation is used to acquire the Chebyshev polynomials of the fourth kind:

$$W_n(u) = 2(2u-1)W_{n-1}(u) - W_{n-2}(u), \quad n = 2, 3, 4, \dots,$$

where  $W_0(u) = 1$  and  $W_1(u) = 4u - 1$ ,  $u \in [0, 1]$ . The Chebyshev polynomials have many applications in science and engineering; the best uniform approximation is characterized by the Chebyshev polynomials of first kind, the weighted least squares approximations are characterized by the relevant weight of Chebyshev polynomials of first, second, third, and fourth kinds. In Computer Aided Design (CAD), the weighted degree reduction of the Bézier curves is handled by the Chebyshev polynomials. For more details about the Chebyshev polynomials of the fourth kind and their applications, see [4,5,12]. Other related applications and properties are found in [13–21].

The following notation

$$B_i^n(u) = \binom{n}{i} (1-u)^{n-i} u^i, \quad u \in [0, 1], \quad i = 0, \dots, n,$$

is used for the Bernstein polynomials of degree  $n$  over the interval  $[0, 1]$ , where  $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ .

The Bernstein polynomials are the basis for the Bézier curves and surfaces. The Bernstein polynomials are non-negative  $B_i^n(u) \geq 0$ ,  $\forall u \in [0, 1]$  and are symmetric in the sense  $B_i^n(u) = B_{n-i}^n(1-u)$ ,  $\forall u \in [0, 1]$ ,  $\forall i = 0, 1, \dots, n$ . The definite integrals of the Bernstein polynomials  $B_i^n(u)$ ,  $u \in [0, 1]$ ,  $i = 0, 1, \dots, n$ , are given by

$$\int_0^1 B_i^n(u) du = \frac{1}{n+1}, \quad i = 0, 1, \dots, n.$$

For more on Bernstein polynomials, see [2, 5]. The factorial of an integer  $n$  is defined by

$$n! = \begin{cases} n(n-1)(n-2) \cdots (2)(1), & n > 0 \\ 0, & o.w \end{cases}$$

The half-integer factorial is defined as follows:

$$(n - \frac{1}{2})! = (n - \frac{1}{2})(n - \frac{3}{2}) \cdots (\frac{3}{2})(\frac{1}{2}).$$

The double factorial of an integer  $n$  is defined by:

$$n!! = \begin{cases} n(n-2) \cdots (4)(2), & \text{if } n \text{ even} \\ n(n-2) \cdots (3)(1), & \text{if } n \text{ odd} \end{cases}.$$

## 2. Materials and Methods

The following results from [6,8] are used in the proof of the results in this paper.

For every  $n > 0$  and  $k = 0, 1, \dots, n$ , we have

$$\frac{\binom{2n-1}{2k-1}}{\binom{n-1}{k-1}} = \frac{(2n-1)!!}{(2k-1)!!(2n-2k-1)!!}. \quad (1)$$

The combinatorials with an integer plus and minus one half forms satisfy the following relation:

$$\binom{n-\frac{1}{2}}{n-k} \binom{n+\frac{1}{2}}{k} = \frac{\binom{2n-1}{n} \binom{2n+1}{2k}}{2^{2n-1}}. \quad (2)$$

The beta function with integer plus one half as parameters has the following equality:

$$\beta\left(z + \frac{1}{2}, k + \frac{1}{2}\right) = \frac{\pi}{2^{z+k}} \frac{(2z-1)!!(2k-1)!!}{(z+k)!}. \quad (3)$$

The Chebyshev polynomial of the fourth kind  $W_n(u)$  of degree  $n$  is expressed in the Bernstein basis  $B_0^n(u), B_1^n(u), \dots, B_n^n(u)$  as follows:

$$W_n(u) = (2n+1)!! \sum_{k=0}^n \frac{(-1)^{n-k}}{(2k+1)!!(2n-2k-1)!!} B_k^n(u). \quad (4)$$

In the next lemma the integral of the Bernstein and the Chebyshev polynomials of the fourth kind with respect to the weight function  $w(u) = \sqrt{\frac{1-u}{u}}$  are found.

**Lemma 1.** *The integral of the product of the Bernstein polynomial of degree  $n$  and the Chebyshev polynomials of the fourth kind of degree  $j$  is given by:*

$$\begin{aligned} I_{kj} &= \int_0^1 \sqrt{\frac{1-u}{u}} B_k^n(u) W_j(u) du \\ &= \frac{\binom{n}{k} \pi}{2^{2n+2j+1}} \sum_{i=0}^j \frac{(-1)^{j-1} \binom{2j+1}{2i+1} \binom{2k+2i}{k+i} \binom{2n+2j-k-2i+1}{n+j-k-i}}{\binom{n+j+1}{k+i+1}}. \end{aligned}$$

**Proof of Lemma 1.** The proof of this lemma is accomplished using Equations (2)–(4) as follows:

$$I_{kj} = \int_0^1 B_k^n(u) W_j(u) (u)^{-\frac{1}{2}} (1-u)^{\frac{1}{2}} du.$$

Using Equation (4) yields

$$\begin{aligned} I_{kj} &= \int_0^1 B_k^n(u) (1-u)^{\frac{1}{2}} (u)^{-\frac{1}{2}} (2j+1)!! \sum_{i=0}^j \frac{(-1)^{j-i}}{(2i+1)!!(2j-2i-1)!!} B_i^j(u) du \\ &= \int_0^1 \binom{n}{k} (1-u)^{n-k} (u)^k (1-u)^{\frac{1}{2}} (u)^{-\frac{1}{2}} (2j+1)!! \times \end{aligned}$$

$$\begin{aligned}
& \sum_{i=0}^j \frac{(-1)^{j-i}}{(2i+1)!!(2j-2i-1)!!} \binom{j}{i} (1-u)^{j-i} (u)^i du \\
&= \int_0^1 \binom{n}{k} (1-u)^{n-k+\frac{1}{2}} (u)^{k-\frac{1}{2}} (2j+1)!! \times \\
& \quad \sum_{i=0}^j \frac{(-1)^{j-i}}{(2i+1)!!(2j-2i-1)!!} \binom{j}{i} (1-u)^{j-i} (u)^i du \\
&= \binom{n}{k} (2j+1)!! \sum_{i=0}^j \frac{(-1)^{j-i}}{(2i+1)!!(2j-2i-1)!!} \binom{j}{i} \int_0^1 (1-u)^{n+j-(k+i)+\frac{1}{2}} u^{k+i-\frac{1}{2}} du.
\end{aligned}$$

By Equation (3), we get

$$I_{kj} = \binom{n}{k} (2j+1)!! \sum_{i=0}^j \frac{(-1)^{j-i}}{(2i+1)!!(2j-2i-1)!!} \binom{j}{i} \beta\left(n+j-k-i+\frac{1}{2}+1, k+i+\frac{1}{2}\right).$$

Using Equation (3) yields

$$\begin{aligned}
I_{kj} &= \binom{n}{k} (2j+1)!! \sum_{i=0}^j \frac{(-1)^{j-i}}{(2i+1)!!(2j-2i-1)!!} \times \\
& \quad \frac{\pi(2(n+j-k-i+1)-1)!! (2(k+i)-1)!!}{2^{n+j+1} (n+j+1)!} \\
&= \frac{\binom{n}{k} \pi}{2^{n+j+1}} \sum_{i=0}^j \frac{(-1)^{j-i} \binom{j}{i} (2n+2j-2k-2i+1)!! (2k+2i-1)!! (2j+1)!!}{(2i+1)!! (2j-2i-1)!! (n+j+1)!}.
\end{aligned}$$

Using Equation (1) gives

$$\begin{aligned}
I_{kj} &= \frac{\binom{n}{k} \pi}{2^{n+j+1}} \sum_{i=0}^j \frac{(-1)^{j-i} \binom{j}{i} (2n+2j-2k-2i+1)!! (2k+2i-1) \binom{2j+1}{2i+1}}{(n+j+1)! \binom{j}{i}} \\
&= \frac{\binom{n}{k} \pi}{2^{n+j+1}} \sum_{i=0}^j \frac{(-1)^{j-i} (2n+2j-2k-2i+1)!! (2k+2i-1) \binom{2j+1}{2i+1}}{(n+j+1)!} \\
&= \frac{\binom{n}{k} \pi}{2^{3n+3j+1}} \sum_{i=0}^j \frac{(-1)^{j-i} \binom{2j+1}{2i+1} \binom{2n+2j-2k-2i}{n+j-k-i}}{1} \times \\
& \quad \frac{\binom{2k+2i}{k+i} (2n+2j-2k-2i)!! (2k+2i)!! (2n+2j-2k-2i+1)}{(n+j+1)!}.
\end{aligned}$$

From the definition of the double factorial of the even integer we get:

$$\begin{aligned}
 I_{kj} &= \frac{\binom{n}{k}}{2^{3n+3j+1}} \sum_{i=0}^j \frac{(-1)^{j-i} \binom{2j+1}{2i+1} \binom{2n+2j-2k-2i}{n+j-k-i}}{(n+j+1)} \times \\
 &\quad \frac{\binom{2k+2i}{k+i} (2n+2j-2k-2i+1)(n+j-k-i)!(k+i)!}{(n+j)!} \\
 &= \frac{\binom{n}{k}}{2^{2n+2j+1}} \sum_{i=0}^j \frac{(-1)^{j-i} \binom{2j+1}{2i+1} \binom{2k+2i}{k+i}}{\binom{n+j}{k+i}} \times \\
 &\quad \frac{\binom{2n+2j-2k-2i}{n+j-i-k} (2n+2j-2k-2i+1)}{(n+j+1)}.
 \end{aligned}$$

With some more simple calculations, the result is confirmed.  $\square$

### 3. Results

The matrices of converting the bases between the Chebyshev polynomial of the fourth kind and the Bernstein polynomials are presented in this section. Given a polynomial  $P_n(u)$ ,  $u \in [0, 1]$  written in the Bernstein basis form and in the Chebyshev polynomials of the fourth kind form as follows:

$$P_n(u) = \sum_{j=0}^n c_j B_j^n(u) = B_n \mathbf{c}_n, \quad (5)$$

and

$$P_n(u) = \sum_{k=0}^n t_k W_k(u) = W_n \mathbf{t}_n, \quad (6)$$

where

$$\begin{aligned}
 B_n &= (B_0^n(u), B_1^n(u), \dots, B_n^n(u)), \quad \mathbf{c}_n = (c_0, c_1, \dots, c_n)^T, \\
 W_n &= (W_0^n(u), W_1^n(u), \dots, W_n^n(u)), \quad \mathbf{t}_n = (t_0, t_1, \dots, t_n)^T.
 \end{aligned}$$

The  $(n+1) \times (n+1)$  matrices of transformation  $M$  and its inverse  $M^{-1}$  are to be found. They fulfill the following relations:

$$c_j = \sum_{k=0}^n M_{jk} t_k \text{ and } t_j = \sum_{k=0}^n M_{jk}^{-1} c_k.$$

In matrix form, we have

$$\mathbf{c}_n = M \mathbf{t}_n \text{ and } \mathbf{t}_n = M^{-1} \mathbf{c}_n.$$

The Chebyshev polynomial of the fourth kind  $W_k(u)$  is presented using the Bernstein polynomials in the following form:

$$W_k(u) = \sum_{j=0}^n N_{kj} B_j^n(u), \quad (7)$$

where  $N$  is the matrix of change of bases transformation of dimension  $(n+1) \times (n+1)$ . Multiply with  $t_k$  and take the summation over  $k$  to get

$$\begin{aligned}\sum_{k=0}^n t_k W_k(u) &= \sum_{k=0}^n t_k \sum_{j=0}^n N_{kj} B_j^n(u) \\ &= \sum_{j=0}^n \sum_{k=0}^n t_k N_{kj} B_j^n(u).\end{aligned}$$

Compare it with Equation (5) to get

$$c_j = \sum_{k=0}^n t_k N_{kj}. \quad (8)$$

Using the following relations  $\mathbf{c}_n = M \mathbf{t}_n$  and  $\mathbf{t}_n = M^{-1} \mathbf{c}_n$  yields

$$c_j = \sum_{k=0}^n M_{jk} t_k, \quad j = 0, 1, \dots, n,$$

and

$$t_k = \sum_{j=0}^n M_{jk}^{-1} c_j, \quad k = 0, 1, \dots, n.$$

By comparing with Equation (8), we get  $M_{jk} = N_{kj}$ ; therefore,  $M = N^T$ . The following examples should motivate finding general formulas for these change of basis transformations.

**Example 1.** For  $n = 1$ : we have  $P_1(u) = 1B_0 + 1B_1 = 1 = 1W_0 + 0W_1$ . Thus,

$$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since  $B_0^1(u) = 1 - u$ ,  $B_1^1(u) = u$ , and  $W_0(u) = 1$ ,  $W_1(u) = 4u - 1$ , thus

$$M_{00}^{-1} = \frac{1}{4}, \quad M_{01}^{-1} = \frac{3}{4}, \quad M_{10}^{-1} = \frac{-1}{4}, \quad M_{11}^{-1} = \frac{1}{4}.$$

Hence

$$M^{-1}C = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{-1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T.$$

**Example 2.** For  $n = 2$ , we have  $P_2(u) = 1B_0 + 1B_1 + 1B_2 = 1 = 1W_0 + 0W_1 + 0W_2$ . Thus

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since  $B_0^2(u) = u^2 - 2u + 1$ ,  $B_1^2(u) = 2u - 2u^2$ ,  $B_2^2(u) = u^2$ ,  $W_0(u) = 1$ ,  $W_1(u) = 4u - 1$ ,  $W_2(u) = 16u^2 - 12u + 1$ , thus

$$M_{00}^{-1} = \frac{1}{8}, \quad M_{01}^{-1} = \frac{1}{4}, \quad M_{02}^{-1} = \frac{5}{8}, \quad M_{10}^{-1} = \frac{-12}{64}, \quad M_{11}^{-1} = \frac{-8}{64},$$

$$M_{12}^{-1} = \frac{20}{64}, \quad M_{20}^{-1} = \frac{2}{8}, \quad M_{21}^{-1} = \frac{-1}{8}, \quad M_{22}^{-1} = \frac{-1}{8}.$$

Thus

$$M^{-1}C = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \\ \frac{-12}{64} & \frac{-8}{64} & \frac{20}{64} \\ \frac{2}{8} & \frac{-1}{8} & \frac{-1}{8} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = T.$$

**Theorem 1.** The elements of the Matrix  $M^{-1}$  that satisfies  $B_n = W_n M^{-1}$  which transforms from the Bernstein polynomial basis into the Chebyshev polynomial basis of the fourth kind for  $0 \leq j, k \leq n$  are given by:

$$M_{jk}^{-1} = \frac{\binom{n}{k}}{4^{n+j}} \sum_{i=0}^j \frac{(-1)^{j-i} \binom{2j+1}{2i+1} \binom{2k+2i}{k+i} \binom{2n+2j-2k-2i+1}{n+j-k-i}}{\binom{n+j+1}{k+i+1}}.$$

#### 4. Discussion

In this section, the proof of the main theorem is given.

**Proof of Theorem 1.** In order to get the elements  $M_{jk}^{-1}$ , we want to find the elements  $N_{ki}^{-1}$  first then we get the elements  $M_{jk}^{-1}$  by transposing the elements  $N_{kj}^{-1}$ . We know that

$$B_k^n(u) = \sum_{i=0}^n N_{ki}^{-1} W_i(u).$$

Multiply the previous equation by  $W_j(u) \left(\frac{1-u}{u}\right)^{\frac{1}{2}}$  and then integrate from 0 to 1 to get

$$\int_0^1 B_k^n(u) W_j(u) \left(\frac{1-u}{u}\right)^{\frac{1}{2}} du = \sum_{i=0}^n \int_0^1 \left(\frac{1-u}{u}\right)^{\frac{1}{2}} N_{ki}^{-1} W_i(u) W_j(u) du.$$

By the orthogonality property of the Chebyshev polynomials of the fourth kind, we get

$$\int_0^1 B_k^n(u) W_j(u) \left(\frac{1-u}{u}\right)^{\frac{1}{2}} du = N_{kj}^{-1} \int_0^1 \left(\frac{1-u}{u}\right)^{\frac{1}{2}} W_i(u) W_j(u) du.$$

Thus,

$$\int_0^1 B_k^n(u) W_j(u) \left(\frac{1-u}{u}\right)^{\frac{1}{2}} du = N_{kj}^{-1} \frac{\pi}{2}.$$

Therefore,

$$N_{kj}^{-1} = \frac{2}{\pi} \int_0^1 B_k^n(u) W_j(u) \left(\frac{1-u}{u}\right)^{\frac{1}{2}} du.$$

After applying the Lemma we have

$$N_{kj}^{-1} = \frac{\binom{n}{k}}{2^{n+j}} \sum_{i=0}^n \frac{(-1)^{j-i} \binom{j}{i} (2n+2j-2k-2i+1)!! (2k+2i-1)!! (2j-1)!!}{(2i-1)!! (2j-2i+1)!! (n+j+1)!}$$

and

$$N_{kj}^{-1} = \frac{2}{\pi} \frac{\binom{n}{k}}{2 \times 4^{n+j}} \sum_{i=0}^n \frac{(-1)^{j-i} \binom{2j+1}{2i+1} \binom{2k+2i}{k+i} \binom{2n+2j-2k-2i+1}{n+j-k-i}}{\binom{n+j+1}{k+i+1}}.$$

By transposing  $N_{kj}^{-1}$  we get  $M_{jk}^{-1}$ .  $\square$

Examples 1 and 2 can be verified using Theorem 1.

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