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Existence and Stability Analysis for Fractional Differential Equations with Mixed Nonlocal Conditions

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Abstract: In this paper, we study the existence and uniqueness of solution for fractional differential equations with mixed fractional derivatives, integrals and multi-point conditions. After that, we also establish different kinds of Ulam stability for the problem at hand. Examples illustrating our results are also presented.

Keywords: Caputo fractional derivative; nonlocal boundary conditions; integral conditions; existence; uniqueness; Ulam–Hyers stability

1. Introduction

Fractional differential equations has proved to be an important tool in the modelling of dynamical systems associated with phenomena such as fractal and chaos. In fact, fractional differential equations has found its applications in many real world phenomena and process of dynamics, biology, signal and image processing, cosmology, physics, chemistry, etc. For more details, see the monographs [1–4] and references therein. For theoretical development of the topic, we refer the reader to papers [5–18] and references cited therein.

One important and interesting area of research of fractional differential equations is devoted to the stability analysis. The notion of Ulam stability, which can be considered as a special type of data dependence was initiated by Ulam [19,20]. Hyers, Aoki, Rassias and Obloza contributed in the development of this field (see [21–25] and the references therein). Meanwhile, there have been few works considering the Ulam stability of variety of classes of fractional differential equations [26–30].

In this paper, motivated by the papers [26–30] we investigate the existence, uniqueness and Ulam stability such as Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability for fractional differential equations with more general nonlocal boundary conditions. More precisely we study the following problem

$$\begin{cases} {}^c D^\alpha u(t) = f(t, u(t)), & t \in [0, T], \\ \sum_{i=1}^m \gamma_i u(\eta_i) + \sum_{j=1}^n \lambda_j {}^c D^{\beta_j} u(\xi_j) + \sum_{r=1}^k \sigma_r I^{\delta_r} u(\phi_r) = A, \end{cases} \quad (1)$$

where $u \in C^1([0, T], \mathbb{R})$ is a continuous function, ${}^c D^\alpha$, ${}^c D^{\beta_j}$ denote the Caputo fractional derivative of orders α and β_j , respectively, $0 < \beta_j < \alpha \leq 1$ for $j = 1, 2, \dots, n$, the notation I^{δ_r} is the Riemann–Liouville fractional integral operator of order $\delta_r > 0$ for $r = 1, 2, \dots, k$, the given constants $\gamma_i, \lambda_j, \sigma_r, A \in \mathbb{R}$, the points $\eta_i, \xi_j, \phi_r \in [0, T]$, $i = 1, 2, \dots, m$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. We emphasize that (1) is a multi-point, fractional derivative multi-order and fractional integral multi-order problem. In addition, we observe that if $\lambda_j = \sigma_r = 0$, then (1) is reduced to multi-point problem, if $\gamma_i = \sigma_r = 0$ the (1) is reduced to fractional derivative multi-order problem and if $\gamma_i = \lambda_j = 0$ then it also reduced to fractional integral multi-order problem, $i = 1, \dots, m$, $j = 1, \dots, n$ and $r = 1, \dots, k$. If $\alpha = 1$, problem (1) arises in nonlocal equations of real world phenomena, see [3]. Fixed point theorems are used to investigate existence results. After that we also study different types of Ulam stability for the proposed problem.

The rest of this paper is organized as follows. Some definitions from fractional calculus theory are recalled in Section 2. In Section 3, we will prove the existence and uniqueness of solutions for Problem (1). In Section 4, we discuss the Ulam stability results. Finally, examples are given in Section 5 to illustrate the usefulness of our main results.

2. Preliminaries and Background Materials

Now we recall in this section some preliminary concepts of fractional calculus [1]. Let $C([0, T], \mathbb{R})$ and $L([0, T], \mathbb{R})$ denote the spaces of continuous real-valued and integrable real-valued functions respectively.

Definition 1. The Riemann–Liouville fractional integral of order $\rho > 0$ starting at a point 0 of the function $f \in L([0, T], \mathbb{R})$ is defined by

$$I^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s) ds,$$

where the right hand side exists and $\Gamma(\cdot)$ is the classical gamma function defined by

$$\Gamma(\rho) = \int_0^\infty t^{\rho-1} e^{-t} dt, \quad \rho > 0.$$

Definition 2. The Caputo fractional derivative of order $\rho > 0$ starting at a point 0 for the n -times differentiable function f is defined by

$${}^c D^\rho f(t) = \frac{1}{\Gamma(n-\rho)} \int_0^t (t-s)^{n-\rho-1} f^{(n)}(s) ds,$$

where $n = [\rho] + 1$ and $[\rho]$ denotes the integer part of the real number ρ .

Lemma 1 ([1]). The following formula holds

$$I^\rho [{}^c D^\rho f](t) = f(t) + k_0 + k_1 t + \dots + k_{n-1} t^{n-1},$$

where $k_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\rho] + 1$ and $\rho > 0$.

Next we give the definitions of Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability for the fractional-order differential Equation (1), see [31].

Definition 3. For every $\varepsilon > 0$, the function $z \in C^1([0, T], \mathbb{R})$ satisfies

$$|{}^c D^\rho z(t) - f(t, z(t))| \leq \varepsilon, \quad t \in [0, T], \quad (2)$$

where the function f is defined in (1). Let $x \in C^1([0, T], \mathbb{R})$ be a solution of the Problem (1). If there is a non zero positive constant κ such that

$$|z(t) - x(t)| \leq \kappa \varepsilon, \quad t \in [0, T].$$

Then the Problem (1) is said to be Ulam–Hyers (UH) stable.

Definition 4. Assume that $z \in C^1([0, T], \mathbb{R})$ satisfies the inequality in (2) and $x \in C^1([0, T], \mathbb{R})$ is a solution of the Problem (1). If there is a function $\Phi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Phi_f(0) = 0$ satisfying

$$|z(t) - x(t)| \leq \Phi_f(\varepsilon), \quad t \in [0, T].$$

Then the Problem (1) is said to be generalized Ulam–Hyers (GUH) stable.

Definition 5. For every $\varepsilon > 0$, the function $z \in C^1([0, T], \mathbb{R})$ satisfies

$$|{}^c D^\rho z(t) - f(t, z(t))| \leq \varepsilon \Phi_f(t), \quad t \in [0, T], \quad (3)$$

where $\Phi_f \in C([0, T], \mathbb{R})$. Let $x \in C^1([0, T], \mathbb{R})$ be a solution of the Problem (1). If there exists a non zero positive constant $\kappa_{\Phi, f}$ such that

$$|z(t) - x(t)| \leq \kappa_{\Phi, f} \varepsilon \Phi_f(t), \quad t \in [0, T].$$

Then the problem (1) is said to be Ulam–Hyers–Rassias (UHR) stable.

Definition 6. Let $z \in C^1([0, T], \mathbb{R})$ be a solution of the inequality (3) and $x \in C^1([0, T], \mathbb{R})$ be a solution of Problem (1). If there exists a non zero positive constant $\kappa_{\Phi, f}$ such that

$$|z(t) - x(t)| \leq \kappa_{\Phi, f} \Phi_f(t), \quad t \in [0, T].$$

Then the problem (1) is said to be generalized Ulam–Hyers–Rassias (GUHR) stable.

Remark 1. If there is a function $\psi \in C([0, T], \mathbb{R})$ (dependent on z), such that

- (I) $|\psi(t)| \leq \varepsilon$, for all $t \in [0, T]$;
- (II) ${}^c D^\rho z(t) = f(t, z(t)) + \psi(t)$, $t \in [0, T]$.

Then a function $z \in C^1([0, T], \mathbb{R})$ is a solution of inequality (2).

Lemma 2 ((Schaefer fixed point theorem) [32]). Suppose that E is a Banach space. Let $\mathbb{T} : E \rightarrow E$ be a completely continuous operator and

$$Q = \{z \in E : z = \lambda \mathbb{T}z, \quad 0 < \lambda < 1\}$$

be a bounded set. Then $\mathbb{T}$ has a fixed point in E .

3. Existence Results for the Problem

Let $E = C^1([0, T], \mathbb{R})$ is the Banach space of all continuous functions $x : [0, T] \rightarrow \mathbb{R}$ endowed with the sup-norm $\|x\| = \sup_{t \in [0, T]} \{|x(t)| : t \in [0, T]\}$.

The following lemma concern a linear variant of Problem (1).

Lemma 3. Suppose $\mu \in C^1([0, T], \mathbb{R})$ and

$$\Omega = \sum_{i=1}^m \gamma_i + \sum_{r=1}^k \frac{\sigma_r \phi_r^{\delta_r}}{\Gamma(\delta_r + 1)} \neq 0.$$

Then, the unique solution of the linear problem

$$\begin{cases} {}^c D^\alpha x(t) = \mu(t), & t \in [0, T], \\ \sum_{i=1}^m \gamma_i x(\eta_i) + \sum_{j=1}^n \lambda_j {}^c D^{\beta_j} x(\xi_j) + \sum_{r=1}^k \sigma_r I^{\delta_r} x(\phi_r) = A, \end{cases} \quad (4)$$

is given by the integral equation

$$x(t) = \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha \mu(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} \mu(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} \mu(\phi_r) \right) + I^\alpha \mu(t). \quad (5)$$

Proof. By Lemma 1 and the first Equation (4), we obtain

$$x(t) = c_0 + I^\alpha \mu(t), \quad (6)$$

where c_0 an arbitrary constant. Using condition the boundary condition, we get

$$c_0 \sum_{i=1}^m \gamma_i + \sum_{i=1}^m \gamma_i I^\alpha \mu(\eta_i) + \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} \mu(\xi_j) + c_0 \sum_{r=1}^k \sigma_r \frac{\phi_r^{\delta_r}}{\Gamma(\delta_r + 1)} + \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} \mu(\phi_r) = A,$$

from which we have

$$c_0 = \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha \mu(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} \mu(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} \mu(\phi_r) \right).$$

Substituting the constant c_0 into (6), we get the integral Equation (5). The converse can be proven by direct computation. The proof is completed. $\square$

In the following, we set an abbreviated notation for the Riemann–Liouville fractional integral of order $\rho > 0$, for a function with two variables as

$$I^\rho f_x(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s, x(s)) ds.$$

Moreover, for computational convenience we put

$$K^* = \sum_{i=1}^m |\gamma_i| \frac{\eta_i^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^n |\lambda_j| \frac{\xi_j^{\alpha-\beta_j}}{\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^k |\sigma_r| \frac{\phi_r^{\alpha+\delta_r}}{\Gamma(\alpha+\delta_r+1)} + \frac{|\Omega| T^\alpha}{\Gamma(\alpha+1)}. \quad (7)$$

Using Lemma 3 we define the operator $\mathcal{A} : E \rightarrow E$ by

$$\begin{aligned} \mathcal{A}x(t) &= \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha f_x(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} f_x(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} f_x(\phi_r) \right) \\ &\quad + I^\alpha f_x(t). \end{aligned} \quad (8)$$

Theorem 1. Let $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Suppose that:

(H1) $f(t, x)$ is Lipschitz continuous (in x) i.e., there exists a constant $L > 0$ such that

$$|f(t, x) - f(t, y)| \leq L|x - y| \text{ for all } t \in [0, T], x, y \in \mathbb{R}.$$

If $LK^* < |\Omega|$, then the Problem (1) has the unique solution ($x \in E$) on $[0, T]$.

Proof. Let a ball B_r be defined as $B_r = \{x \in E : \|x\| \leq r\}$, where r is a positive constant with $r \geq \frac{|A| + MK^*}{|\Omega| - LK^*}$, when $M = \sup\{|f(t, 0)| : t \in [0, T]\}$. Then we have

$$\begin{aligned} |\mathcal{A}x(t)| &\leq \frac{1}{|\Omega|} \left(|A| + \sum_{i=1}^m |\gamma_i| I^\alpha |f_x|(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} |f_x|(\xi_j) + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} |f_x|(\phi_r) \right) \\ &\quad + I^\alpha |f_x|(t) \\ &\leq \frac{1}{|\Omega|} \left(|A| + \sum_{i=1}^m |\gamma_i| I^\alpha (|f_x - f_0| + |f_0|)(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} (|f_x - f_0| + |f_0|)(\xi_j) \right. \\ &\quad \left. + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} (|f_x - f_0| + |f_0|)(\phi_r) \right) + I^\alpha (|f_x - f_0| + |f_0|)(t) \\ &\leq \frac{|A|}{|\Omega|} + \frac{(Lr + M)}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha (\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} (\xi_j) + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} (\phi_r) \right) + (Lr + M) I^\alpha(T) \\ &= \frac{|A|}{|\Omega|} + \frac{1}{|\Omega|} (Lr + M) K^* \leq r, \end{aligned}$$

which, by taking the norm on $[0, T]$, yields $\|\mathcal{A}x\| \leq r$. This shows that $\mathcal{A}B_r \subset B_r$. In order to show that $\mathcal{A}$ is a contraction, we put $x, y \in B_r$, we obtain

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq \frac{1}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha |f_x - f_y|(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} |f_x - f_y|(\xi_j) \right. \\ &\quad \left. + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} |f_x - f_y|(\phi_r) \right) + I^\alpha |f_x - f_y|(t) \\ &\leq \frac{L\|x - y\|}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha(1)(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j}(1)(\xi_j) \right. \\ &\quad \left. + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r}(1)(\phi_r) \right) + L\|x - y\| I^\alpha(1)(T) \\ &= \frac{K^* L \|x - y\|}{|\Omega|}. \end{aligned}$$

Therefore

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \frac{K^* L}{|\Omega|} \|x - y\|,$$

which implies that $\mathcal{A}$ is a contraction, since $K^* L / |\Omega| < 1$. By Banach contraction mapping principle the operator $\mathcal{A}$ defined in (8), has the unique fixed point, which implies that Problem (1) has the unique solution on $[0, T]$. The proof is completed. $\square$

Theorem 2. Let f be a continuous function on $[0, T] \times \mathbb{R}$. Assume that

$$(H2) \quad |f(t, x(t))| \leq \varphi(t), \quad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Then Problem (1) has at least one solution $(x \in E)$ on $[0, T]$.

Proof. Now, we need to show that the operator $\mathcal{A}$ is compact by applying the well known Arzelà-Ascoli theorem. So we will show that the operator $\mathcal{A}B_r$ is a uniformly bounded set, where $B_r = \{x \in E : \|x\| \leq r, r > 0\}$ and equicontinuous set. Let $\varphi^* = \sup\{\varphi(t) : t \in [0, T]\}$. For $x \in B_r$, it follows that

$$\begin{aligned} |\mathcal{A}x(t)| &\leq \frac{|A|}{|\Omega|} + \frac{1}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha |f_x|(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} |f_x|(\xi_j) \right. \\ &\quad \left. + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} |f_x|(\phi_r) \right) + I^\alpha |f_x|(t) \\ &\leq \frac{|A|}{|\Omega|} + \varphi^* \frac{1}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha(1)(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j}(1)(\xi_j) \right. \\ &\quad \left. + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r}(1)(\phi_r) \right) + \varphi^* I^\alpha(1)(T) \\ &\leq \frac{1}{|\Omega|} (|A| + \varphi^* K^*), \end{aligned}$$

and consequently

$$\|\mathcal{A}x\| \leq \frac{1}{|\Omega|} (|A| + \varphi^* K^*),$$

which implies that the set $\mathcal{A}B_r$ is uniformly bounded. Next, we are going to prove that $\mathcal{A}B_r$ is equicontinuous set. For $\tau_1, \tau_2 \in [0, T]$ such that $\tau_1 < \tau_2$ and for $x \in B_r$, we obtain

$$\begin{aligned} |\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)| &= |I^\alpha(f_x)(\tau_2) - I^\alpha(f_x)(\tau_1)| \\ &\leq \frac{\varphi^*}{\Gamma(\alpha)} \left| \int_0^{\tau_2} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] ds \right| \\ &\quad + \frac{\varphi^*}{\Gamma(\alpha)} \left| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} ds \right| \\ &\leq \frac{\varphi^*}{\Gamma(\alpha+1)} \{|\tau_2^\alpha - \tau_1^\alpha| + 2(\tau_2 - \tau_1)^\alpha\}. \end{aligned}$$

The right-hand side of the above inequality tends to zero as $\tau_1 \rightarrow \tau_2$ independently of x which implies that $\mathcal{A}B_r$ is equicontinuous set. By using Arzelà-Ascoli theorem, the set $\mathcal{A}B_r$ is relative compact, that is, the operator $\mathcal{A}$ is completely continuous.

Finally we will show that $\mathcal{W} = \{x \in E : x = \lambda \mathcal{A}x, 0 < \lambda < 1\}$ is a bounded set. Let $x \in \mathcal{W}$. Then we have

$$\begin{aligned} |x(t)| &\leq \lambda \|\mathcal{A}x\| \\ &\leq \frac{1}{|\Omega|} (|A| + \varphi^* K^*), \end{aligned}$$

which yields $\|x\| \leq (|A| + \varphi^* K^*)/|\Omega|$. Therefore $\mathcal{W}$ is bounded and the proof is completed by using Schaefer fixed point theorem (Lemma 2). $\square$

4. Ulam Stability Analysis Results

Lemma 4. If $z \in C^1([0, T], \mathbb{R})$ satisfies the inequality in (2), then, for $\varepsilon \in (0, 1]$, z is a solution of the inequality

$$|z(t) - \mathcal{A}z(t)| \leq \frac{\varepsilon K^*}{|\Omega|}, \quad (9)$$

where K^* is defined in (7).

Proof. Form Remark 1 (II) and Lemma 3, we have

$$z(t) = \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha (f_z + \psi)(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} (f_z + \psi)(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} (f_z + \psi)(\phi_r) \right) + I^\alpha (f_z + \psi)(t).$$

Then, by Remark 1 (I), we obtain

$$\begin{aligned} |z(t) - \mathcal{A}z(t)| &= \left| \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha (f_z + \psi)(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} (f_z + \psi)(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} (f_z + \psi)(\phi_r) \right) + I^\alpha (f_z + \psi)(t) \right. \\ &\quad \left. - \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha f_z(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} f_z(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} f_z(\phi_r) \right) - I^\alpha f_z(t) \right| \\ &\leq \frac{1}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha |\psi|(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} |\psi|(\xi_j) + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} |\psi|(\phi_r) \right) + I^\alpha |\psi|(t) \\ &\leq \frac{\varepsilon K^*}{|\Omega|}, \end{aligned}$$

which is satisfied inequality in (9). This completes the proof. $\square$

Theorem 3. If the conditions (H1), (H2) are fulfilled and $|\Omega| - LK^* \neq 0$ holds, then the problem (1) is UH stable.

Proof. Suppose $z \in C^1([0, T], \mathbb{R})$ is the solution of the inequality in (2) and let $x(t)$ be the unique solution of problem (1). Consider

$$\begin{aligned} |z(t) - x(t)| &= \left| z(t) - \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha f_x(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} f_x(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} f_x(\phi_r) \right) - I^\alpha f_x(t) \right| \\ &= |z(t) - \mathcal{A}z(t) + \mathcal{A}z(t) - \mathcal{A}x(t)| \\ &\leq |z(t) - \mathcal{A}z(t)| + |\mathcal{A}z(t) - \mathcal{A}x(t)| \\ &\leq \frac{\varepsilon K^*}{|\Omega|} + \frac{LK^* |x - z|}{|\Omega|}, \end{aligned}$$

from which we obtain

$$|z(t) - x(t)| \leq \frac{\varepsilon K^*}{|\Omega| - LK^*}.$$

By setting

$$\kappa = \frac{K^*}{|\Omega| - LK^*},$$

we obtain

$$|z(t) - x(t)| \leq \kappa \varepsilon.$$

Therefore, the Problem (1) is UH stable. Next, by setting $\Phi_f(\varepsilon) = \kappa \varepsilon$, $\Phi_f(0) = 0$, the Problem (1) is GUH stable. The proof is completed. $\square$

Lemma 5. Let $z \in C^1([0, T], \mathbb{R})$ be a solution of the inequality in (3) and assume that

$$(H3) \quad \frac{1}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha |\psi|(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} |\psi|(\xi_j) + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} |\psi|(\phi_r) \right) + I^\alpha |\psi|(t) \leq \frac{\varepsilon \Phi_f(t) K^*}{|\Omega|},$$

then z is satisfied the inequality

$$|z(t) - \mathcal{A}z(t)| \leq \frac{\varepsilon \Phi_f(t) K^*}{|\Omega|}. \quad (10)$$

Proof. Form Remark 1 (II) and Lemma 3 we have

$$z(t) = \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha (f_z + \psi)(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} (f_z + \psi)(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} (f_z + \psi)(\phi_r) \right) + I^\alpha (f_z + \psi)(t).$$

Therefore,

$$\begin{aligned} |z(t) - \mathcal{A}z(t)| &= \left| \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha (f_z + \psi)(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} (f_z + \psi)(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} (f_z + \psi)(\phi_r) \right) + I^\alpha (f_z + \psi)(t) \right. \\ &\quad \left. - \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha f_z(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} f_z(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} f_z(\phi_r) \right) + I^\alpha f_z(t) \right| \\ &\leq \frac{1}{|\Omega|} \left(\sum_{i=1}^m |\gamma_i| I^\alpha |\psi|(\eta_i) + \sum_{j=1}^n |\lambda_j| I^{\alpha-\beta_j} |\psi|(\xi_j) + \sum_{r=1}^k |\sigma_r| I^{\alpha+\delta_r} |\psi|(\phi_r) \right) + I^\alpha |\psi|(t) \\ &\leq \frac{\varepsilon \Phi_f(t) K^*}{|\Omega|}, \end{aligned}$$

which leads to inequality in (10). $\square$

Theorem 4. If the assumptions (H1), (H2), (H3) and $LK^* \neq |\Omega|$ are satisfied, then the problem (1) is UHR stable.

Proof. Let $z \in C^1([0, T], \mathbb{R})$ be a solution of the inequality in (3) and let $x(t)$ be the unique solution of Problem (1). Next we consider

$$\begin{aligned} |z(t) - x(t)| &= \left| z(t) - \frac{1}{\Omega} \left(A - \sum_{i=1}^m \gamma_i I^\alpha f_x(\eta_i) - \sum_{j=1}^n \lambda_j I^{\alpha-\beta_j} f_x(\xi_j) - \sum_{r=1}^k \sigma_r I^{\alpha+\delta_r} f_x(\phi_r) \right) + I^\alpha f_x(t) \right| \\ &= |z(t) - \mathcal{A}z(t) + \mathcal{A}z(t) - \mathcal{A}x(t)| \\ &\leq |z(t) - \mathcal{A}z(t)| + |\mathcal{A}z(t) - \mathcal{A}x(t)| \\ &\leq \frac{\varepsilon \Phi_f(t) K^*}{|\Omega|} + \frac{LK^* |x - z|}{|\Omega|}, \end{aligned}$$

from which we have

$$|z(t) - x(t)| \leq \frac{\varepsilon \Phi_f(t) K^*}{||\Omega| - LK^*|}.$$

By taking a constant

$$\kappa_{\Phi, f} = \frac{K^*}{||\Omega| - LK^*|},$$

we get the following inequality

$$|z(t) - x(t)| \leq \kappa_{\Phi, f} \varepsilon \Phi_f(t).$$

Therefore, the Problem (1) is UHR stable. Next by putting $\Phi_f^*(\varepsilon) = \varepsilon \Phi_f(t)$ with $\Phi_f^*(0) = 0$, we deduce that the Problem (1) is GUHR stable. This completes the proof. $\square$

5. Examples

In this section, we would like to show the applicability of our theoretical results to specific numerical examples.

Example 1. Consider the following problem

$$\begin{cases} {}^c D^{0.75} u(t) = \frac{te^{\cos t} |u(t)|}{40(1 + |u(t)|)}, & t \in [0, 2], \\ u(1) + 0.5u(2) + 0.25 {}^c D^{0.25} u(1.5) - 0.5 I^{0.7} u(1.75) = 3.64. \end{cases} \quad (11)$$

Here $\alpha = 0.75$, $\gamma_1 = 1$, $\gamma_2 = 0.5$, $\eta_1 = 1$, $\eta_2 = 2$, $\lambda_1 = 0.25$, $\beta_1 = 0.25$, $\xi_1 = 1.5$, $\sigma_1 = -0.5$, $\delta_1 = 0.7$, $\phi_1 = 1.75$, $A = 3.46$, $T = 2$ and $f(t, u) = te^{\cos t} |u| / [40(1 + |u|)]$. From given information, we find that $\Omega = 0.6858$ and $K^* = 4.2756$.

Since

$$f(t, u) = \frac{te^{\cos t} |u|}{40(1 + |u|)},$$

we obtain that $|f(t, x) - f(t, y)| \leq (1/20)|x - y|$. Then, we get $L = (1/20)$ and $f(t, u) \leq (1/20)$. In view of Theorem 1, $LK^* = (4.2756/20) = 0.2138 < 0.6858 = \Omega$. Hence the Problem (11) has the unique solution.

In view of Theorem 3, $\kappa = (K^*) / (|\Omega| - LK^*) = 9.0571$. Therefore Problem (11) is UH stable and hence GUH stable. By setting $\Phi_f(t) = 0.1745t^{0.75} + 0.7532$, it satisfies (H3) of Lemma 5. Therefore Problem (11) is UHR stable and also GUHR stable.

Example 2. Consider the following problem

$$\begin{cases} {}^c D^{0.5} u(t) = f(t, u(t)), & t \in [0, 1], \\ u(0) + 2u(1) + 0.1 {}^c D^{0.25} u(0.5) + 0.25 {}^c D^{0.45} u(1) + I^{0.75} u(1) = 1. \end{cases} \quad (12)$$

Here $\alpha = 0.55$, $\gamma_1 = 1$, $\gamma_2 = 2$, $\eta_1 = 0$, $\eta_2 = 1$, $\lambda_1 = 0.1$, $\lambda_2 = 0.25$, $\beta_1 = 0.25$, $\beta_2 = 0.45$, $\xi_1 = 0.5$, $\xi_2 = 1$, $\sigma_1 = 1$, $\delta_1 = 0.75$, $\phi_1 = 1$, $A = 1$, $T = 1$. From these, we can find constants as $\Omega = 4.0881$ and $K^* = 8.1018$.

(i) Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t, u) = \frac{e^{-2t} \sin^2(|u|)}{t^4 + 2}. \quad (13)$$

Now we can obtain $|f(t, x) - f(t, y)| \leq 0.5|x - y|$ and then we set $L = 0.5$. Since $LK^* = (0.5)(8.1018) = 4.0509 < 4.0881 = \Omega$, Problem (12) with the function f is given by (13) has the unique solution on $[0, 1]$. Since f satisfies (H1) and (H2), Problem (12) is UH stable and also GUH stable. By direct computation, all conditions of Theorem 4 are satisfied. Therefore this problem is UHR stable and also GUHR stable.

(ii) Consider now the function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(t, u) = \frac{3te^{-\cos t}|u|}{5(1 + |u|)}. \quad (14)$$

Indeed, we obtain $|f(t, x) - f(t, y)| \leq 0.6|x - y|$ and so $L = 0.6$. Since $LK^* = (0.6)(8.1018) = 4.8611 > 4.0881 = \Omega$, Theorem 1 can not be applied to Problem (12) with f given by (14). However the function f satisfies (H2) and consequently by Theorem 2 Problem (12) with f given by (14) has at least one solution on $[0, 1]$. Since f satisfies (H1) and (H2) and $\kappa = 10.4810$, Problem (12) is UH stable and also GUH stable. By setting $\Phi_f(t) = 0.5694t^{0.5} + 0.4306$, then all conditions of Theorem 4 are satisfied. Therefore this problem is UHR stable and GUHR stable.

6. Conclusions

We have proved the existence and uniqueness of solutions for fractional differential equations with mixed fractional derivatives, integrals and multi-point conditions. We applied Banach and Schaefer fixed point theorems. Different kinds of Ulam stability, such as, Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability are also investigated. The obtained results are illustrated by numerical examples. It seems that the results of this paper can be extended to cover the case $1 < \alpha \leq 2$.

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References

- Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
- Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2014.
- Bucur, C.; Valdinoci, E. *Nonlocal Diffusion and Applications*; Lecture Notes of the Unione Matematica Italiana; Springer: Cham, Switzerland, 2016; Volume 20.
- Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer International Publishing AG: Cham, Switzerland, 2017.
- Petras, I.; Magin, R.L. Simulation of drug uptake in a two compartmental fractional model for a biological system. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 4588–4595. [[CrossRef](#)] [[PubMed](#)]
- Dipierro, S.; Valdinoci, E. A simple mathematical model inspired by the Purkinje cells: from delayed travelling waves to fractional diffusion. *Bull. Math. Biol.* **2018**, *80*, 1849–1870. [[CrossRef](#)] [[PubMed](#)]
- Ding, Y.; Wang, Z.; Ye, H. Optimal control of a fractional-order HIV-immune system with memory. *IEEE Trans. Control Syst. Technol.* **2012**, *20*, 763–769. [[CrossRef](#)]
- Graef, J.R.; Kong, L. Existence of positive solutions to a higher order singular boundary value problem with fractional q -derivatives. *Fract. Calc. Appl. Anal.* **2013**, *16*, 695–708. [[CrossRef](#)]
- Alsaedi, A.; Ntouyas, S.K.; Agarwal, R.P.; Ahmad, B. A nonlocal multi-point multi-term fractional boundary value problem with Riemann–Liouville type integral boundary conditions involving two indices. *Adv. Differ. Equ.* **2013**, *2013*, 369. [[CrossRef](#)]
- Zhai, C.; Xu, L. Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 2820–2827. [[CrossRef](#)]
- Li, B.; Sun, S.; Li, Y.; Zhao, P. Multi-point boundary value problems for a class of Riemann–Liouville fractional differential equations. *Adv. Differ. Equ.* **2014**, *2014*, 151. [[CrossRef](#)]
- Zhang, L.; Ahmad, B.; Wang, G. Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half line. *Bull. Aust. Math. Soc.* **2015**, *91*, 116–128. [[CrossRef](#)]

13. Ntouyas, S.K.; Etemad, S. On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions. *Appl. Math. Comput.* **2015**, *266*, 235–243. [[CrossRef](#)]
14. Qarout, D.; Ahmad, B.; Alsaedi, A. Existence theorems for semilinear Caputo fractional differential equations with nonlocal discrete and integral boundary conditions. *Fract. Calc. Appl. Anal.* **2016**, *19*, 463–479. [[CrossRef](#)]
15. Ahmad, B.; Ntouyas, S.K.; Agarwal, R.P.; Alsaedi, A. Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions. *Bound. Value Probl.* **2016**, *2016*, 205. [[CrossRef](#)]
16. Agarwal, R.P.; Ahmad, B.; Garout, D.; Alsaedi, A. Existence results for coupled nonlinear fractional differential equations equipped with nonlocal coupled flux and multi-point boundary conditions. *Chaos Solit. Fract.* **2017**, *102*, 149–161. [[CrossRef](#)]
17. Xu, M.; Han, Z. Positive solutions for integral boundary value problem of two-term fractional differential equations. *Bound. Value Probl.* **2018**, *2018*, 100. [[CrossRef](#)]
18. Wang, G.; Pei, K.; Agarwal, R.P.; Zhang, L.; Ahmad, B. Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line. *J. Comput. Appl. Math.* **2018**, *343*, 230–239. [[CrossRef](#)]
19. Ulam, S.M. *Problems in Modern Mathematics*; John Wiley and Sons: New York, NY, USA, 1940.
20. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1968.
21. Hyers, D.H. On the stability of linear functional equations. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [[CrossRef](#)] [[PubMed](#)]
22. Aoki, T. On the stability of linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66. [[CrossRef](#)]
23. Rassias, T.M. On the stability of linear mapping in Banach Spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [[CrossRef](#)]
24. Rassias, T.M. On a modified Hyers-Ulam sequence. *J. Math. Anal. Appl.* **2003**, *158*, 106–113. [[CrossRef](#)]
25. Oblaza, M. Hyers stability of linear differential equation. *Rocznik Nauk-dydakt. Prace Mater.* **1993**, *13*, 259–270.
26. Benchohra, M.; Lazreg, J.E. On stability of nonlinear implicit fractional differential equations. *Matematicheskoe* **2015**, *70*, 49–61.
27. Benchohra, M.; Lazreg, J.E. Existence and Ulam stability for non-linear implicit fractional differential equations with Hadamard derivative. *Stud. Univ. Babeş-Bolyai Math.* **2017**, *62*, 27–38. [[CrossRef](#)]
28. Wang, J.; Lv, L.; Zhou, Y. Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electron. J. Qual. Theory Differ. Equ.* **2011**, *63*, 1–10. [[CrossRef](#)]
29. Wang, J.; Zhou, Y. New concepts and results in stability of fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *17*, 2530–2538. [[CrossRef](#)]
30. Wang, J.R.; Zhou, Y.; Medved, M. Existence and stability of fractional differential equations with Hadamard derivative. *Topol. Methods Nonlinear Anal.* **2013**, *41*, 113–133.
31. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpathian J. Math.* **2010**, *26*, 103–107.
32. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.