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Article

New Generalized Mizoguchi-Takahashi's Fixed Point Theorems for Essential Distances and e^0-Metrics

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Abstract: In this paper, we present some new generalizations of Mizoguchi-Takahashi's fixed point theorem which also improve and extend Du-Hung's fixed point theorem. Some new examples illustrating our results are also given. By applying our new results, some new fixed point theorems for essential distances and e^0-metrics were established.

Keywords: $\mathcal{MT}$-function; $\mathcal{MT}(\lambda)$-function; τ-function; essential distance (e-distance); e^0-metric; Du-Hung's fixed point theorem; Mizoguchi-Takahashi's fixed point theorem; Nadler's fixed point theorem; Banach contraction principle

MSC: 47H10; 54H25

1. Introduction

Let (W, ρ) be a metric space. For each $a \in W$ and any nonempty subset M of W, let

$$\rho(a, M) = \inf_{b \in M} \rho(a, b).$$

Denote by $\mathcal{N}(W)$, the family of all nonempty subsets of W, and by $\mathcal{CB}(W)$, the class of all nonempty closed and bounded subsets of W. A function $\mathcal{H} : \mathcal{CB}(W) \times \mathcal{CB}(W) \to [0, +\infty)$ defined by

$$\mathcal{H}(C, D) = \max \left\{ \sup_{a \in D} \rho(a, C), \sup_{a \in C} \rho(a, D) \right\}$$

is said to be the *Hausdorff metric* on $\mathcal{CB}(W)$ induced by the metric ρ on W. A point z in W is a fixed point of a mapping T if $z = Tz$ (when $T : W \to W$ is a single-valued mapping) or $z \in Tz$ (when $T : W \to \mathcal{N}(W)$ is a multivalued mapping). The set of fixed points of T is denoted by $\mathcal{F}(T)$.

Fixed point theory is a fascinating mathematical theory that has a wide range of applications in many areas of mathematics, including nonlinear analysis, optimization, variational inequality problems, integral and differential equations and inclusions, critical point theory, nonsmooth analysis, dynamic system theory, control theory, economics, game theory, finance mathematics and so on. The famous Banach contraction principle [1] is undoubtedly one of the most important and applicable fixed point theorems which has played a significant role in nonlinear analysis and applied mathematical analysis. Many authors have devoted their attentions to study generalizations in various different directions of the Banach contraction principle; see, e.g., [2–23] and references therein.

Theorem 1. *(Banach [1]) Let (W, ρ) be a complete metric space and T:W$\to$ W be a selfmapping. Assume that there exists a nonnegative number $\lambda < 1$ such that*

$$\rho(Ta, Tb) \leq \lambda\rho(a, b) \text{ for all } a, b \in W.$$

Then T has a unique fixed point in W.

Nadler's fixed point theorem [21] was established in 1969 to extend the Banach contraction principle for multivalued mappings.

Theorem 2. *(Nadler [21]) Let (W, ρ) be a complete metric space and $T : W \to \mathcal{CB}(W)$ be a multivalued mapping. Suppose that there exists a nonnegative number $\lambda < 1$ such that*

$$\mathcal{H}(Ta, Tb) \leq \lambda\rho(a, b) \quad \text{for all } a, b \in W.$$

Then T has a fixed point in W.

Later, in 1989, Mizoguchi and Takahashi [20] presented a celebrated generalization of Nadler's fixed point theorem. In 2008, Suzuki gave an example [22] (Example 1) to show that Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's fixed point theorem.

Theorem 3. *(MizoguchiandTakahashi [20]) Let (W, ρ) be a complete metric space and $T : W \to \mathcal{CB}(W)$ be a multivalued mapping. Assume that*

$$\mathcal{H}(Ta, Tb) \leq \mu(\rho(a, b))\rho(a, b) \quad \text{for all } a, b \in W,$$

where μ: $[0, +\infty) \to [0, 1)$ is an $\mathcal{MT}$-function; that is, μ satisfies $\limsup_{x \to s^+} \mu(x) < 1$ for all $s \in [0, +\infty)$. Then T has a fixed point in W.

A number of generalizations of Mizoguchi-Takahashi's fixed point theorem were studied; see [2,4,8–13,15,16] and references therein. In 2016, Du and Hung [10] established the following generalized Mizoguchi-Takahashi's fixed point theorem.

Theorem 4. *(Du and Hung [10]) Let (W, ρ) be a complete metric space, $T : W \to CB(W)$ be a multivalued mapping and $\mu : [0, +\infty) \to [0, 1)$ be an $\mathcal{MT}$-function. Suppose that*

$$\min\{\mathcal{H}(Ta, Tb), \rho(a, Ta)\} \leq \mu(\rho(a, b))\rho(a, b) \quad \text{for all } a, b \in W \text{ with } a \neq b.$$

Then T admits a fixed point in W.

Theorem 4 is different from known generalizations in the existing literature and was illustrated by [7] (Example A) in which Mizoguchi-Takahashi's fixed point theorem is not applicable.

In this paper, we establish some new generalizations of Mizoguchi-Takahashi's fixed point theorem which also improve and extend Du-Hung's fixed point theorem. Some new examples illustrating our results are also given. By applying our new results, we obtained some new fixed point theorems for essential distances and e^0-metrics.

2. Preliminaries

Let (W, ρ) be a metric space. A real valued function $f : W \to \mathbb{R}$ is called *lower semicontinuous* if $\{x \in W : f(x) \leq r\}$ is *closed* for any $r \in \mathbb{R}$. Recall that a function $\kappa : W \times W \to [0, +\infty)$ is called a *w-distance* [14,18], if the following are satisfied:

($w1$) $\kappa(a,c) \leq \kappa(a,b) + \kappa(b,c)$ for any $a,b,c \in W$;
($w2$) For any $a \in W$, $\kappa(a,\cdot) : W \to [0,+\infty)$ is lower semicontinuous;
($w3$) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\kappa(c,a) \leq \delta$ and $\kappa(c,b) \leq \delta$ imply $\rho(a,b) \leq \varepsilon$.

A function $\kappa : W \times W \to [0,+\infty)$ is said to be a τ*-function* [2,3,6,8,9,17,19], if the following conditions hold:

($\tau1$) $\kappa(a,c) \leq \kappa(a,b) + \kappa(b,c)$ for any $a,b,c \in W$;
($\tau2$) If $a \in W$ and $\{b_n\}$ in W with $\lim_{n\to\infty} b_n = b$ such that $\kappa(a,b_n) \leq \beta$ for some $\beta = \beta(a) > 0$, then $\kappa(a,b) \leq \beta$;
($\tau3$) For any sequence $\{a_n\}$ in W with $\limsup_{n\to\infty}\{\kappa(a_n,a_m) : m > n\} = 0$, if there exists a sequence $\{b_n\}$ in X such that $\lim_{n\to\infty} \kappa(a_n,b_n) = 0$, then $\lim_{n\to\infty} \rho(a_n,b_n) = 0$;
($\tau4$) For $a,b,c \in W$, $\kappa(a,b) = 0$ and $\kappa(a,c) = 0$ imply $b = c$.

It is obvious that the metric ρ is a w-distance and any w-distance is a τ-function, but the converse is not true; see [2,17] for more details.

The following result is useful in our proofs.

Lemma 1. *(See [6, Lemma 1.1].) If condition* $(\tau4)$ *is weakened to the following condition* $(\tau4)'$:

$(\tau4)'$ *for any* $a \in W$ *with* $\kappa(a,a) = 0$, *if* $\kappa(a,b) = 0$ *and* $\kappa(a,c) = 0$, *then* $b = c$,

then $(\tau3)$ *implies* $(\tau4)'$.

In 2016, Du [6] introduced the concept of essential distance; see also [8].

Definition 1. *(See [6] (Definition 1.2).) Let* (W,d) *be a metric space. A function* $\kappa : W \times W \to [0,+\infty)$ *is called an essential distance (abbreviated as "e-distance") if conditions* $(\tau1)$, $(\tau2)$ *and* $(\tau3)$ *hold.*

Remark 1.

(i) Clearly, any τ*-function is an e-distance.*
(ii) By Lemma 1, we know that if κ *is an e-distance, then condition* $(\tau4)'$ *holds.*

The following known result is crucial in this paper.

Lemma 2. *(See [3] (Lemma 2.1).) Let* (W,ρ) *be a metric space and* $\kappa : W \times W \to [0,+\infty)$ *be a function. Assume that* κ *satisfies the condition* $(\tau3)$. *If a sequence* $\{a_n\}$ *in* W *with* $\lim\limits_{n\to\infty} \sup\{\kappa(a_n,a_m) : m > n\} = 0$, *then* $\{a_n\}$ *is a Cauchy sequence in* W.

In 2016, Du introduced the concept of $\mathcal{MT}(\lambda)$-function [5] as follows (see also [7]).

Definition 2. *Let* $\lambda > 0$. *A function* $\tau : [0,+\infty) \to [0,\lambda)$ *is said to be an* $\mathcal{MT}(\lambda)$*-function [5] if* $\limsup\limits_{x\to\gamma^+} \tau(x) < \lambda$ *for all* $\gamma \in [0,+\infty)$. *As usual, we simply write "*$\mathcal{MT}$*-function" instead of "*$\mathcal{MT}(1)$*-function".*

A useful characterization theorem for $\mathcal{MT}(\lambda)$-functions was established by Du [5] in 2016 as follows.

Theorem 5. *(See [5] (Theorem 2.4).) Let* $\lambda > 0$ *and let* $\tau : [0,+\infty) \to [0,\lambda)$ *be a function. Then the following statements are equivalent.*

(1) τ *is an* $\mathcal{MT}(\lambda)$*-function.*

(2) *$\lambda^{-1}\tau$ is an $\mathcal{MT}$-function.*

(3) *For each $\gamma \in [0,+\infty)$, there exists $\xi_t^{(1)} \in [0,\lambda)$ and $\epsilon_t^{(1)} > 0$ such that $\tau(x) \leq \xi_t^{(1)}$ for all $x \in (\gamma, \gamma+\epsilon_t^{(1)})$.*

(4) *For each $\gamma \in [0,+\infty)$, there exists $\xi_t^{(2)} \in [0,\lambda)$ and $\epsilon_t^{(2)} > 0$ such that $\tau(x) \leq \xi_t^{(2)}$ for all $x \in [\gamma, \gamma+\epsilon_t^{(2)}]$.*

(5) *For each $\gamma \in [0,+\infty)$, there exists $\xi_t^{(3)} \in [0,\lambda)$ and $\epsilon_t^{(3)} > 0$ such that $\tau(x) \leq \xi_t^{(3)}$ for all $x \in (\gamma, \gamma+\epsilon_t^{(3)}]$.*

(6) *For each $\gamma \in [0,+\infty)$, there exists $\xi_t^{(4)} \in [0,\lambda)$ and $\epsilon_t^{(4)} > 0$ such that $\tau(x) \leq \xi_t^{(4)}$ for all $x \in [\gamma, \gamma+\epsilon_t^{(4)})$.*

(7) *For any nonincreasing sequence $\{\beta_n\}_{n\in\mathbb{N}}$ in $[0,+\infty)$, we have $0 \leq \sup\limits_{n\in\mathbb{N}} \tau(\beta_n) < \lambda$.*

(8) *For any strictly decreasing sequence $\{\beta_n\}_{n\in\mathbb{N}}$ in $[0,+\infty)$, we have $0 \leq \sup\limits_{n\in\mathbb{N}} \tau(\beta_n) < \lambda$.*

(9) *For any eventually nonincreasing sequence $\{\beta_n\}_{n\in\mathbb{N}}$ (i.e., there exists $\alpha \in \mathbb{N}$ such that $\beta_{n+1} \leq \beta_n$ for all $n \in \mathbb{N}$ with $n \geq \alpha$) in $[0,+\infty)$, we have $0 \leq \sup\limits_{n\in\mathbb{N}} \tau(\beta_n) < \lambda$.*

(10) *For any eventually strictly decreasing sequence $\{\beta_n\}_{n\in\mathbb{N}}$ (i.e., there exists $\alpha \in \mathbb{N}$ such that $\beta_{n+1} < \beta_n$ for all $n \in \mathbb{N}$ with $n \geq \alpha$) in $[0,+\infty)$, we have $0 \leq \sup\limits_{n\in\mathbb{N}} \tau(\beta_n) < \lambda$.*

Let κ be an e-distance on a metric space (W,ρ). For each $a \in W$ and any nonempty subset G of W, we define $\kappa(a,G)$ by

$$\kappa(a,G) = \inf_{b\in G} \kappa(a,b).$$

The following Lemma is essentially proved in [2].

Lemma 3. *(See [2] (Lemma 1.2).) Let G be a closed subset of a metric space (W,ρ) and κ be a function satisfying the condition $(\tau3)$. Suppose that there exists $c \in W$ such that $\kappa(c,c) = 0$. Then $\kappa(c,G) = 0$ if and only if $c \in G$.*

Very recently, Du introduced and studied the concept of e^0-distance [9].

Definition 3. *(See [9] (Definition 1.3).) Let (W,ρ) be a metric space. A function $\kappa : W \times W \to [0,+\infty)$ is called an e^0-distance if it is an e-distance on W with $\kappa(a,a) = 0$ for all $a \in W$.*

Remark 2. *By applying Lemma 1, if κ is an e^0-distance on W, then for $a,b \in W$, $\kappa(a,b) = 0 \Longleftrightarrow a = b$.*

Example 1. *Let $W = \mathbb{R}$ with the metric $\rho(a,b) = |a-b|$. Then (W,ρ) is a metric space. Define the function $\kappa : W \times W \to [0,+\infty)$ by*

$$\kappa(x,y) = \max\{9(x-y), 5(y-x)\}.$$

Therefore κ is not a metric due to its asymmetry. It is easy to see that κ is an e^0-distance on W.

The following concept of e^0-metric was studied by Du in [9] which generalizes the concept of Hausdorff metric.

Definition 4. *(See [9] (Definition 1.4).) Let (W,ρ) be a metric space and κ be an e^0-distance. For any E, $F \in \mathcal{CB}(W)$, define a function $\mathcal{D}_\kappa : \mathcal{CB}(W) \times \mathcal{CB}(W) \to [0,+\infty)$ by*

$$\mathcal{D}_\kappa(E,F) = max\{\xi_\kappa(E,F), \xi_\kappa(F,E)\},$$

where $\xi_\kappa(E,F) = \sup_{x\in E} \kappa(x,F)$, and then $\mathcal{D}_\kappa$ is said to be the e^0-metric on $\mathcal{CB}(W)$ induced by κ.

The following result presented in [9] (Theorem 1.3) is quite important in our proofs. Although its proof is similar to the proof of [2] (Theorem 1.2), we give it here for the sake of completeness and the readers convenience.

Theorem 6. *(See [9] (Theorem 1.3).) Let (W, ρ) be a metric space and $\mathcal{D}_\kappa$ be an e^0-metric defined as in Definition 4 on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Then, for $E, F, G \in \mathcal{CB}(W)$, the following hold:*

(i) $\xi_\kappa(E, F) = 0 \Longleftrightarrow E \subseteq F$;
(ii) $\xi_\kappa(E, F) \leq \xi_\kappa(E, G) + \xi_\kappa(G, F)$;
(iii) *Every e^0-metric $\mathcal{D}_\kappa$ is a metric on $\mathcal{CB}(W)$.*

Proof. To see (i), if $\xi_\kappa(E, F) = 0$, then $\kappa(a, F) = 0$ for all $a \in E$. By Lemma 3, we get $E \subseteq F$. Conversely, if $E \subseteq F$, by Lemma 3 again, we obtain $\xi_\kappa(E, F) = 0$ and (i) is proven. Fix $a \in E$ and $c \in G$. Then we have

$$\kappa(a, F) \leq \kappa(a, b) \leq \kappa(a, c) + \kappa(c, b) \quad \text{for all } b \in F,$$

which deduces

$$\kappa(a, F) \leq \kappa(a, c) + \kappa(c, F).$$

So, for any $a \in E$, we obtain

$$\kappa(a, F) \leq \inf\{\kappa(a, c) + \kappa(c, F) : c \in G\} \leq \kappa(a, G) + \xi_\kappa(G, F).$$

Taking the supremum on both sides of the last inequality over all $a \in E$, we can obtain (ii). Finally, we verify (iii). Obviously, $\mathcal{D}_\kappa(E, F) \geq 0$ and $\mathcal{D}_\kappa(E, F) = \mathcal{D}_\kappa(F, E)$. By using (i), we have $\mathcal{D}_\kappa(E, F) = 0$ $\Longleftrightarrow E = F$. Applying (ii), we have

$$\begin{aligned}\mathcal{D}_\kappa(E, F) &= \max\{\xi_\kappa(E, F), \xi_\kappa(F, E)\} \\ &\leq \max\{\xi_\kappa(E, G) + \xi_\kappa(G, F), \xi_\kappa(F, G) + \xi_\kappa(G, E)\} \\ &\leq \mathcal{D}_\kappa(E, G) + \mathcal{D}_\kappa(G, F).\end{aligned}$$

These arguments show that $\mathcal{D}_\kappa$ is a metric on $\mathcal{CB}(W)$. □

Definition 5. *Let U be a nonempty subset of a metric space (W, ρ) and κ be an e-distance on W. A multivalued mapping $T{:}U \to \mathcal{N}(W)$ is said to have the κ-approximate fixed point property in U provided $\inf_{a \in U} \kappa(a, Ta) = 0$. In particular, if $\kappa \equiv \rho$, then T is said to have the approximate fixed point property in U.*

Remark 3. *Let U be a nonempty subset of a metric space (W, ρ) and $T : U \to \mathcal{N}(W)$ be a multivalued mapping. Clearly, $\mathcal{F}(T) \cap U \neq \emptyset$ implies that T has the approximate fixed point property in U.*

3. Main Results

In this section, we first prove a new generalized Mizoguchi-Takahashi's fixed point theorem with a new nonlinear condition.

Theorem 7. *Let (W, ρ) be a metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $T : W \to \mathcal{CB}(W)$ be a multivalued mapping and $\varphi : [0, +\infty) \to [0, 1)$ be an $\mathcal{MT}$-function. Assume that*

$$\kappa(a, x) \leq \kappa(x, a) \qquad \text{for all } a \in Tx \tag{1}$$

and

$$\min\{\mathcal{D}_\kappa(Tu, Tv), \kappa(u, Tu)\} \leq \varphi(\kappa(u, v))\kappa(u, v) \quad \text{for all } u, v \in W \text{ with } u \neq v. \tag{2}$$

Then, the following statements hold:

(a) *For any $z_0 \in W$, there exists a Cauchy sequence $\{z_n\}_{n=0}^{\infty}$ in W started at z_0 satisfying $z_n \in Tz_{n-1}$ for each $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \kappa(z_n, z_{n-1}) = \lim_{n\to\infty} \kappa(z_{n-1}, z_n) = \inf_{n\in\mathbb{N}} \kappa(z_n, z_{n-1}) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0;$$

(b) *T has the κ-approximate fixed point property in W.*

Moreover, if W is complete and T further satisfies one of the following conditions:

(D1) *T is closed; that is, $GrT = \{(a,b) \in W \times W : b \in Ta\}$, the graph of T, is closed in $W \times W$;*
(D2) *The function $f : W \to \mathbb{R}$ defined by $f(a) = \kappa(a, Ta)$ is lower semicontinuous;*
(D3) *The function $g : W \to \mathbb{R}$ defined by $g(a) = \rho(a, Ta)$ is lower semicontinuous;*
(D4) *For each sequence $\{z_n\}$ in W with $z_{n+1} \in Tz_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = w$, we have $\lim_{n\to\infty} \kappa(z_n, Tw) = 0$;*
(D5) *$\inf\{\kappa(a,v) + \kappa(a,Ta) : a \in W\} > 0$ for every $v \notin \mathcal{F}(T)$,*

then T admits a fixed point in W.

Proof. Let $\tau : [0,+\infty) \to [0,1)$ be defined by

$$\tau(x) = \frac{1}{2}(\varphi(x) + 1) \quad \text{for all } x \in [0,+\infty).$$

Hence $0 \leq \varphi(x) < \tau(x) < 1$ for all $x \in [0,\infty)$. Given $b \in W$. Take $z_0 = b \in W$ and choose $z_1 \in Tz_0$. If $z_1 = z_0$, then $z_0 \in \mathcal{F}(T)$ and we are done. Otherwise, if $z_1 \neq z_0$, then $\kappa(z_1, z_0) > 0$ and we obtain from (2) that

$$\min\{\mathcal{D}_\kappa(Tz_1, Tz_0), \kappa(z_1, Tz_1)\} \leq \varphi(\kappa(z_1,z_0))\kappa(z_1,z_0) < \tau(\kappa(z_1,z_0))\kappa(z_1,z_0). \tag{3}$$

Since

$$\kappa(z_1, Tz_1) \leq \sup_{w\in Tx_0} \kappa(w, Tz_1) \leq \mathcal{D}_\kappa(Tz_0, Tz_1) = \mathcal{D}_\kappa(Tz_1, Tz_0),$$

we get

$$\min\{\mathcal{D}_\kappa(Tz_1, Tz_0), \kappa(z_1, Tz_1)\} = \kappa(z_1, Tz_1). \tag{4}$$

Hence, by (3) and (4), we obtain

$$\kappa(z_1, Tz_1) < \tau(\kappa(z_1,z_0))\kappa(z_1,z_0),$$

which deduces that there exists $z_2 \in Tz_1$ such that

$$\kappa(z_1, z_2) < \tau(\kappa(z_1,z_0))\kappa(z_1,z_0).$$

Since $z_2 \in Tz_1$, by (1), we have

$$\kappa(z_2, z_1) < \tau(\kappa(z_1,z_0))\kappa(z_1,z_0).$$

Next, if $z_2 = z_1$, then $z_1 \in \mathcal{F}(T)$ and we finish the proof. Otherwise, since

$$\kappa(z_2, Tz_2) = \min\{\mathcal{D}_\kappa(Tz_2, Tz_1), \kappa(z_2, Tz_2)\} < \tau(\kappa(z_2,z_1))\kappa(z_2,z_1),$$

there exists $z_3 \in Tz_2$ such that

$$\kappa(z_2, z_3) < \tau(\kappa(z_2,z_1))\kappa(z_2,z_1).$$

By (1), we have

$$\kappa(z_3, z_2) < \tau(\kappa(z_2,z_1))\kappa(z_2,z_1).$$

So, by induction, we can obtain a sequence $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ in W satisfying the following: for each $n \in \mathbb{N}$,

(i) $z_n \in Tz_{n-1}$ with $z_n \neq z_{n-1}$;
(ii) $\kappa(z_n, z_{n+1}) < \tau(\kappa(z_n, z_{n-1}))\kappa(z_n, z_{n-1})$;
(iii) $\kappa(z_{n+1}, z_n) < \tau(\kappa(z_n, z_{n-1}))\kappa(z_n, z_{n-1})$.

By (iii), the sequence $\{\kappa(z_n, z_{n-1})\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0, +\infty)$. Hence

$$\lim_{n\to\infty} \kappa(z_n, z_{n-1}) = \inf_{n\in\mathbb{N}} \kappa(z_n, z_{n-1}) \quad \text{exists.} \tag{5}$$

Since φ is an $\mathcal{MT}$-function, by applying (8) of Theorem 5 with $\lambda = 1$, we obtain

$$0 \leq \sup_{n\in\mathbb{N}} \varphi(\kappa(z_n, z_{n-1})) < 1.$$

So we get

$$0 < \sup_{n\in\mathbb{N}} \tau(\kappa(z_n, z_{n-1})) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \varphi(\kappa(z_n, z_{n-1}))\right] < 1.$$

Put $\gamma := \sup_{n\in\mathbb{N}} \tau(\kappa(z_n, z_{n-1}))$. Thus $\gamma \in (0, 1)$. For any $n \in \mathbb{N}$, by (iii) again, we have

$$\kappa(z_{n+1}, z_n) < \tau(\kappa(z_n, z_{n-1}))\kappa(z_n, z_{n-1}) \leq \gamma\kappa(z_n, z_{n-1}). \tag{6}$$

By (6), we get

$$\kappa(z_{n+1}, z_n) < \gamma\kappa(z_n, z_{n-1}) < \cdots < \gamma^n\kappa(z_1, z_0) \quad \text{for each } n \in \mathbb{N}. \tag{7}$$

Since $0 < \gamma < 1$, by taking the limit as $n \to \infty$ in (7), we obtain

$$\lim_{n\to\infty} \kappa(z_n, z_{n-1}) = 0. \tag{8}$$

Taking into account (5) and (8), we obtain

$$\lim_{n\to\infty} \kappa(z_n, z_{n-1}) = \inf_{n\in\mathbb{N}} \kappa(z_n, z_{n-1}) = 0.$$

On the other hand, from (ii) and using (1), we have

$$\kappa(z_n, z_{n+1}) < \gamma\kappa(z_n, z_{n-1}) \leq \gamma\kappa(z_{n-1}, z_n) \quad \text{for each } n \in \mathbb{N}.$$

which shows that the sequence $\{\kappa(z_{n-1}, z_n)\}_{n\in\mathbb{N}}$ is also strictly decreasing in $[0, +\infty)$, and hence, we can deduce

$$\kappa(z_n, z_{n+1}) < \gamma^n\kappa(z_0, z_1) \quad \text{for each } n \in \mathbb{N}. \tag{9}$$

So, by (9), we get

$$\lim_{n\to\infty} \kappa(z_{n-1}, z_n) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0. \tag{10}$$

Since $z_n \in Tz_{n-1}$ for all $n \in \mathbb{N}$, by (10), we prove

$$\inf_{a\in W} \kappa(a, Ta) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0;$$

that is, T has the κ-approximate fixed point property in W. Next, we claim that $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in W. For $m,n \in \mathbb{N}$ with $m > n$, we have from (9) that

$$\kappa(z_n, z_m) \leq \sum_{j=n}^{m-1} \kappa(z_j, z_{j+1}) < \frac{\gamma^n}{1-\gamma}\kappa(z_0, z_1). \tag{11}$$

Since $0 < \gamma < 1$, the last inequality implies

$$\lim_{n\to\infty} \sup\{\kappa(z_n, z_m) : m > n\} = 0. \tag{12}$$

Applying Lemma 2, we prove that $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in W.

Now, we assume that W is complete. We want to show $\mathcal{F}(T) \neq \emptyset$ if T further satisfies one of conditions (D1)–(D5). Since $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is Cauchy in W and W is complete, there exists $w \in W$ such that $z_m \to w$ as $m \to \infty$. From $(\tau 2)$ and (11), we have

$$\kappa(z_n, w) \leq \frac{\gamma^n}{1-\gamma}\kappa(z_0, z_1) \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

In order to finish the proof, it is sufficient to show $w \in \mathcal{F}(T)$. If (D1) holds, since T is closed and $z_n \in Tz_{n-1}$ and $z_n \to w$ as $n \to \infty$, we get $w \in Tw$. If (D2) holds, by the lower semicontinuity of f, $z_n \to w$ as $n \to \infty$ and (10), we obtain

$$\begin{aligned}
\kappa(w, Tw) &= f(w) \\
&\leq \liminf_{n\to\infty} \kappa(z_n, Tz_n) \\
&\leq \lim_{n\to\infty} k(z_n, z_{n+1}) = 0.
\end{aligned}$$

By Lemma 3, $w \in \mathcal{F}(T)$. Suppose that (D3) is satisfied. Since $\{z_n\}$ is Cauchy, we have $\lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0$. So, by the lower semicontinuity of g and $z_n \to w$ as $n \to \infty$, we get

$$\rho(w, Tw) = g(w) \leq \lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0.$$

By the closedness of Tw, we show $w \in \mathcal{F}(T)$. Assume that (D4) holds. By (12), there exists $\{u_n\} \subset \{z_n\}$ with $\limsup_{n\to\infty}\{\kappa(u_n, u_m) : m > n\} = 0$ and $\{v_n\} \subset Tw$ such that $\lim_{n\to\infty} \kappa(u_n, v_n) = 0$. By $(\tau 3)$, $\lim_{n\to\infty} \rho(u_n, v_n) = 0$. Since $\rho(v_n, w) \leq \rho(v_n, u_n) + \rho(u_n, w)$, we have $v_n \to w$ as $n \to \infty$. By the closedness of Tw, we obtain $w \in Tw$. Finally, suppose that (D5) holds. If $w \notin Tw$, then, by (11) and (13), we obtain

$$\begin{aligned}
0 &< \inf_{a\in W}\{k(a, w) + k(a, Ta)\} \\
&\leq \inf_{n\in\mathbb{N}}\{k(z_n, w) + k(z_n, Tz_n)\} \\
&\leq \inf_{n\in\mathbb{N}}\{k(z_n, w) + k(z_n, z_{n+1})\} \\
&\leq \lim_{n\to\infty} \frac{2\gamma^n}{1-\gamma}\kappa(z_0, z_1) \\
&= 0,
\end{aligned}$$

which leads to a contradiction. Therefore, it must be $w \in \mathcal{F}(T)$. The proof is completed. $\square$

Here, we give a simple example illustrating Theorem 7.

Example 2. *Let $W = [0, +\infty)$ with the metric $\rho(x, y) = |x - y|$ for $x, y \in W$. Let $Tx = [0, x]$ for $x \in W$. It is obvious that each $x \in W$ is a fixed point of T. Let φ be any $\mathcal{MT}$-function. Let $\kappa : W \times W \to [0, +\infty)$ be defined by*

$$\kappa(u, v) = max\{9(u - v), 5(v - u)\} \quad \text{for } u, v \in W.$$

Then, κ is an e^0-metric on W. Given $x \in W$. For any $a \in Tx = [0, x]$, we have

$$\kappa(a, x) = 5(x - a) \leq 9(x - a) = \kappa(x, a),$$

which shows that (1) holds. Clearly, the function $x \mapsto \rho(x, Tx)$ is a zero function on W, so it is lower semicontinuous. Hence (D3) holds. We now claim

$$\min\{\mathcal{D}_\kappa(Tu, Tv), \kappa(u, Tu)\} \leq \varphi(\kappa(u, v))\kappa(u, v) \quad \text{for all } u, v \in W \text{ with } u \neq v.$$

We consider the following two possible cases:

Case 1. *If $0 \leq u < v$, we have*

$$\kappa(u, Tu) = 0,$$

$$\mathcal{D}_\kappa(Tu, Tv) = max\left\{\sup_{z \in Tu} \kappa(z, Tv), \sup_{z \in Tv} \kappa(z, Tu)\right\} = 9(v - u)$$

and

$$\kappa(u, v) = 5(v - u).$$

So, $\min\{\mathcal{D}_\kappa(Tu, Tv), \kappa(u, Tu)\} = 0 \leq \varphi(\kappa(u, v))\kappa(u, v)$.

Case 2. *If $0 \leq v < u$, we obtain*

$$\kappa(u, Tu) = 0,$$

$$\mathcal{D}_\kappa(Tu, Tv) = 9(u - v)$$

and

$$\kappa(u, v) = 9(u - v).$$

Hence, $\min\{\mathcal{D}_\kappa(Tu, Tv), \kappa(u, Tu)\} = 0 \leq \varphi(\kappa(u, v))\kappa(u, v)$.

By Cases 1 and 2, our claim is verified, and hence, (2) holds. Therefore, all the assumptions of Theorem 7 are satisfied and we also show that T has a fixed point in W from Theorem 7. Notice that

$$\mathcal{H}(T(5), T(9)) = 4 > \varphi(\rho(5, 9))\rho(5, 9),$$

so Mizoguchi-Takahashi's fixed point theorem is not applicable here. This example shows that Theorem 7 is a real generalization of Mizoguchi-Takahashi's fixed point theorem.

Remark 4. *Du-Hung's fixed point theorem (i.e., Theorem 4) can be proven immediately from Theorem 7. Indeed, let $\kappa \equiv \rho$. Then, (1) and (2), as in Theorem 7, are satisfied. We claim that (D4) as in Theorem 7 holds. Let $\{z_n\}$ in X with $z_{n+1} \in Tz_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = w$. We obtain*

$$\begin{aligned} \lim_{n\to\infty} \rho(z_{n+1}, Tw) &\leq \lim_{n\to\infty} \mathcal{H}(Tz_n, Tw) \\ &\leq \lim_{n\to\infty} \{\varphi(\rho(z_n, w))\rho(z_n, w)\} = 0, \end{aligned}$$

which shows that (D4) holds. Therefore, all the assumptions of Theorem 7 are satisfied. By applying Theorem 7, we prove $\mathcal{F}(T) \neq \emptyset$.

In Theorem 7, if $T : W \to W$ is a self-mapping, then we obtain the following new fixed point theorem which generalizes Banach contraction principle.

Corollary 1. *Let (W,ρ) be a metric space, $T : W \to W$ be a self-mapping and $\varphi : [0,+\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Assume that*

$$\kappa(a,x) \leq \kappa(x,a) \quad \text{for all } a \in Tx$$

and

$$\min\{\kappa(Tu,Tv),\kappa(u,Tu)\} \leq \varphi(\kappa(u,v))\kappa(u,v) \quad \text{for all } u,v \in W \text{ with } u \neq v.$$

Then the following statements hold:

(a) *For any $z_0 \in W$, there exists a Cauchy sequence $\{z_n\}_{n=0}^{\infty}$ in W started at z_0 satisfying $z_n = Tz_{n-1}$ for each $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \kappa(z_n, z_{n-1}) = \lim_{n\to\infty} \kappa(z_{n-1}, z_n) = \inf_{n\in\mathbb{N}} \kappa(z_n, z_{n-1}) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0;$$

(b) *T has the κ-approximate fixed point property in W.*

Moreover, if W is complete and T further satisfies one of conditions (D1)-(D5) as in Theorem 7, then T admits a fixed point in W.

By applying Theorem 7, we establish some new fixed point theorems for e^0-metrics and e^0-distances.

Corollary 2. *Let (W,ρ) be a complete metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $\varphi : [0,+\infty) \to [0,1)$ be an $\mathcal{MT}$-function and $T : W \to \mathcal{CB}(W)$ be a multivalued mapping satisfying one of conditions (D1)-(D5) as in Theorem 7. Assume that*

$$\kappa(a,x) \leq \kappa(x,a) \quad \text{for all } a \in Tx$$

and

$$\mathcal{D}_\kappa(Tu,Tu) + \kappa(u,Tu) \leq 2\varphi(\kappa(u,v))\kappa(u,v) \quad \text{for all } u,v \in W \text{ with } u \neq v. \tag{14}$$

Then T admits a fixed point in W.

Proof. For any $u,v \in W$ with $u \neq v$, by (14), we have

$$\min\{\mathcal{D}_\kappa(Tu,Tv),\kappa(u,Tu)\} \leq \frac{1}{2}\left(\mathcal{D}_\kappa(Tu,Tu) + \kappa(u,Tu)\right) \leq \varphi(\kappa(u,v))\kappa(u,v).$$

Hence the condition (2) in Theorem 7 holds. Therefore, the conclusion is immediate from Theorem 7. □

Corollary 3. *Let (W,ρ) be a complete metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $\varphi : [0,+\infty) \to [0,1)$ be an $\mathcal{MT}$-function and $T : W \to \mathcal{CB}(W)$ be a multivalued mapping satisfying one of conditions (D1)-(D5) as in Theorem 7. Assume that*

$$\kappa(a,x) \leq \kappa(x,a) \quad \text{for all } a \in Tx$$

and

$$\sqrt{\mathcal{D}_\kappa(Tu,Tv)\kappa(u,Tu)} \leq \varphi(\kappa(u,v))\kappa(u,v) \quad \text{for all } u,v \in W \text{ with } u \neq v. \tag{15}$$

Then T admits a fixed point in W.

Proof. For any $u,v \in W$ with $u \neq v$, from (15), we obtain

$$\min\{\mathcal{D}_\kappa(Tu,Tv),\kappa(u,Tu)\} \leq \sqrt{\mathcal{D}_\kappa(Tu,Tv)\kappa(u,Tu)} \leq \varphi(\kappa(u,v))\kappa(u,v).$$

So the condition (2) in Theorem 7 holds. Hence, the conclusion is immediate from Theorem 7. □

In fact, we can establish a wide generalization of Corollary 2 as follows.

Corollary 4. *Let (W,ρ) be a complete metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $\varphi : [0,+\infty) \to [0,1)$ be an $\mathcal{MT}$-function and $T : X \to \mathcal{CB}(W)$ be a multivalued mapping satisfying one of conditions (D1)-(D5) as in Theorem 7. Assume that*

$$\kappa(a,x) \leq \kappa(x,a) \quad \text{for all } a \in Tx$$

and

$$\frac{s\mathcal{D}_\kappa(Tu,Tv) + t\kappa(u,Tv)}{s+t} \leq \varphi(\kappa(u,v))\kappa(u,v) \quad \text{for all } u,v \in W \text{ with } u \neq v, \tag{16}$$

where $s,t \geq 0$ with $s+t > 0$. Then T admits a fixed point in W.

Proof. For any $u,v \in W$ with $u \neq v$, by (16), we get

$$\min\{\mathcal{D}_\kappa(Tu,Tv), \kappa(u,Tu)\} \leq \frac{s\mathcal{D}_\kappa(Tu,Tv) + t\kappa(u,Tv)}{s+t} \leq \varphi(\kappa(u,v))\kappa(u,v),$$

and hence the condition (2) in Theorem 7 is satisfied. So the desired conclusion follows from Theorem 7 immediately. □

Now, we focus the following new fixed point theorem without the assumption (1) and satisfy another new condition

$$\min\{\mathcal{D}_\kappa(Tu,Tv), \kappa(v,Tv)\} \leq \varphi(\kappa(u,v))\kappa(u,v) \quad \text{for all } u,v \in W \text{ with } u \neq v,$$

which is different from (2) as in Theorem 7. It is worth mentioning that this new fixed point theorem is meaningful because an e^0-*distance* is asymmetric in general.

Theorem 8. *Let (W,ρ) be a metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $T : W \to \mathcal{CB}(W)$ be a multivalued mapping and $\varphi : [0,+\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Assume that*

$$\min\{\mathcal{D}_\kappa(Tu,Tv), \kappa(v,Tv)\} \leq \varphi(\kappa(u,v))\kappa(u,v) \quad \text{for all } u,v \in W \text{ with } u \neq v. \tag{17}$$

Then the following statements hold:

(a) For any $z_0 \in W$, there exists a Cauchy sequence $\{z_n\}_{n=0}^{\infty}$ in W started at z_0 satisfying $z_n \in Tz_{n-1}$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \kappa(z_{n-1}, z_n) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0;$$

(b) T has the κ-approximate fixed point property in W.

Moreover, if W is complete and T further satisfies one of conditions (D1)-(D5) as in Theorem 7, then $\mathcal{F}(T) \neq \emptyset$.

Proof. Define $\tau(x) = \frac{1}{2}(\varphi(x)+1)$ for all $x \in [0,+\infty)$. Then $0 \leq \varphi(x) < \tau(x) < 1$ for all $x \in [0,+\infty)$. Let $b \in W$. Take $z_0 = b \in W$ and choose $z_1 \in Tz_0$. If $z_1 = z_0$, then $z_0 \in \mathcal{F}(T)$ and we are done. Otherwise, if $z_1 \neq z_0$, then $\kappa(z_0,z_1) > 0$. By (17), we have

$$\begin{aligned}\kappa(z_1, Tz_1) &= \min\{\mathcal{D}_\kappa(Tz_0, Tz_1), \kappa(z_1, Tz_1)\} \\ &\leq \varphi(\kappa(z_0,z_1))\kappa(z_0,z_1) \\ &< \tau(\kappa(z_0,z_1))\kappa(z_0,z_1),\end{aligned}$$

from which one can deduce that there exists $z_2 \in Tz_1$ such that

$$\kappa(z_1, z_2) < \tau(\kappa(z_0, z_1))\kappa(z_0, z_1).$$

Next, if $z_2 = z_1$, then $z_1 \in \mathcal{F}(T)$, and we finish the proof. Otherwise, since

$$\kappa(z_2, Tz_2) = \min\{\mathcal{D}_\kappa(Tz_1, Tz_2), \kappa(z_2, Tz_2)\} < \tau(\kappa(z_1, z_2))\kappa(z_1, z_2),$$

then there exists $z_3 \in Tz_2$ such that

$$\kappa(z_2, z_3) < \tau(\kappa(z_1, z_2))\kappa(z_1, z_2).$$

Hence, by induction, we can obtain a sequence $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ satisfying the following: for each $n \in \mathbb{N}$,

(iv) $z_n \in Tz_{n-1}$ with $z_n \neq z_{n-1}$;
(v) $\kappa(z_n, z_{n+1}) < \tau(\kappa(z_{n-1}, z_n))\kappa(z_{n-1}, z_n)$.

By (v), the sequence $\{\kappa(z_{n-1}, z_n)\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0, +\infty)$. So

$$\lim_{n\to\infty} \kappa(z_{n-1}, z_n) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) \quad \text{exists.} \tag{18}$$

Since φ is an $\mathcal{MT}$-function, by applying (8) of Theorem 5 with $\lambda = 1$, we obtain

$$0 \leq \sup_{n\in\mathbb{N}} \varphi(\kappa(z_{n-1}, z_n)) < 1.$$

So we get

$$0 < \sup_{n\in\mathbb{N}} \tau(\kappa(z_{n-1}, z_n)) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \varphi(\kappa(z_{n-1}, z_n))\right] < 1.$$

Hence $c := \sup_{n\in\mathbb{N}} \tau(\kappa(z_{n-1}, z_n)) \in (0, 1)$. For any $n \in \mathbb{N}$, by (v) again, we obtain

$$\kappa(z_n, z_{n+1}) < \tau(\kappa(z_{n-1}, z_n))\kappa(z_{n-1}, z_n) \leq c\kappa(z_{n-1}, z_n).$$

which implies

$$\kappa(z_n, z_{n+1}) < c^n \kappa(z_0, z_1) \quad \text{for each } n \in \mathbb{N}. \tag{19}$$

Since $0 < c < 1$, by taking the limit as $n \to \infty$ in (19), we have

$$\lim_{n\to\infty} \kappa(z_n, z_{n+1}) = 0. \tag{20}$$

Combining (18) and (20), we obtain

$$\lim_{n\to\infty} \kappa(z_{n-1}, z_n) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0 \tag{21}$$

and hence (a) is proven. To see (b), since $z_n \in Tz_{n-1}$ for all $n \in \mathbb{N}$, by (21), we show that

$$\inf_{a\in W} \kappa(a, Ta) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0.$$

Using a similar argument as in the proof of Theorem 7, one can verify that $\mathcal{F}(T) \neq \emptyset$ and finish this proof. □

The following example not only illustrates Theorem 8 but also shows that Theorem 8 is different from Theorem 7.

Example 3. *Let $W = [0, +\infty)$ with the metric $\rho(x, y) = |x - y|$ for $x, y \in W$. Let $Tx = [0, x]$ for $x \in W$. So each $x \in W$ is a fixed point of T. Let φ be any $\mathcal{MT}$-function. Let $\kappa : W \times W \to [0, +\infty)$ be defined by*

$$\kappa(u, v) = max\{4(u - v), 7(v - u)\} \quad \text{for } u, v \in W.$$

Then κ is an e^0-metric on W. Clearly, the function $x \mapsto \rho(x, Tx)$ is a zero function on W, so it is lower and semicontinuous. Hence, (D3) holds. Using a similar argument as in Example 2, we can prove that

$$\min\{\mathcal{D}_\kappa(Tu, Tv), \kappa(v, Tv)\} \leq \varphi(\kappa(u, v))\kappa(u, v) \quad \text{for all } u, v \in W \text{ with } u \neq v.$$

Hence, all the assumptions of Theorem 8 are satisfied. Applying Theorem 8, we also prove that T has a fixed point in W. Notice that $1 \in T(2) = [0, 2]$ and

$$\kappa(1, 2) = 7 > 4 = \kappa(2, 1),$$

so (1) does not hold and hence Theorem 7 is not applicable here. Moreover, since

$$\mathcal{H}(T(3), T(8)) = 5 > \varphi(\rho(3, 8))\rho(3, 8),$$

Mizoguchi-Takahashi's fixed point theorem is also not applicable.

Some new fixed point theorems are established by Theorem 8 immediately.

Corollary 5. *Let (W, ρ) be a metric space, $T : W \to W$ be a selfmapping and $\varphi : [0, +\infty) \to [0, 1)$ be an $\mathcal{MT}$-function. Assume that*

$$\min\{\kappa(Tu, Tv), \kappa(v, Tv)\} \leq \varphi(\kappa(u, v))\kappa(u, v) \quad \text{for all } u, v \in W \text{ with } u \neq v.$$

Then the following statements hold:

(a) *For any $z_0 \in W$, there exists a Cauchy sequence $\{z_n\}_{n=0}^{\infty}$ in W started at z_0 satisfying $z_n = Tz_{n-1}$ for each $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \kappa(z_{n-1}, z_n) = \inf_{n\in\mathbb{N}} \kappa(z_{n-1}, z_n) = 0;$$

(b) *T has the κ-approximate fixed point property in W.*

Moreover, if W is complete and T further satisfies one of conditions (D1)-(D5) as in Theorem 7, then T admits a fixed point in W.

Corollary 6. *Let (W, ρ) be a complete metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $\varphi : [0, +\infty) \to [0, 1)$ be an $\mathcal{MT}$-function and $T : W \to \mathcal{CB}(W)$ be a multivalued mapping satisfying one of conditions (D1)–(D5) as in Theorem 7. Assume that*

$$\mathcal{D}_\kappa(Tu, Tv) + \kappa(v, Tv) \leq 2\varphi(\kappa(u, v))\kappa(u, v) \quad \text{for all } u, v \in W \text{ with } u \neq v.$$

Then $\mathcal{F}(T) \neq \emptyset$.

Corollary 7. *Let (W, ρ) be a complete metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $\varphi : [0, +\infty) \to [0, 1)$ be an $\mathcal{MT}$-function and $T : W \to \mathcal{CB}(W)$ be a multivalued mapping satisfying one of conditions (D1)-(D5) as in Theorem 7. Assume that*

$$\sqrt{\mathcal{D}_\kappa(Tu, Tv)\kappa(v, Tv)} \leq \varphi(\kappa(u, v))\kappa(u, v) \quad \text{for all } u, v \in W \text{ with } u \neq v.$$

Then $\mathcal{F}(T) \neq \emptyset$.

Corollary 8. *Let (W, ρ) be a complete metric space and $\mathcal{D}_\kappa$ be an e^0-metric on $\mathcal{CB}(W)$ induced by an e^0-distance κ. Let $\varphi : [0, +\infty) \to [0, 1)$ be an $\mathcal{MT}$-function and $T : W \to \mathcal{CB}(W)$ be a multivalued mapping satisfying one of conditions (D1)-(D5) as in Theorem 7. Assume that*

$$\frac{s\mathcal{D}_\kappa(Tu, Tv) + t\kappa(v, Tv)}{s + t} \leq \varphi(\kappa(u, v))\kappa(u, v) \quad \textit{for all } u, v \in W \textit{ with } u \neq v,$$

where $s, t \geq 0$ with $s + t > 0$. Then $\mathcal{F}(T) \neq \emptyset$.

Remark 5.

(a) *Theorem 7, Corollary 4, Theorem 8 and Corollary 8 all generalize and extend Mizoguchi-Takahashi's fixed point theorem;*
(b) *All results in [10] are special cases of our results established in this paper.*
(c) *Theorems 7 and 8 improve and generalize some of the existence results on the topic in the literature; see, e.g., [1,2,4,7,8,10,11,13–16,20–23] and references therein.*

4. Conclusions

Our main purpose in this paper is to establish new generalizations of Mizoguchi-Takahashi's fixed point theorem for essential distances and e^0-metrics satisfying the following new conditions:

- $\min\{\mathcal{D}_\kappa(Tu, Tv), \kappa(u, Tu)\} \leq \varphi(\kappa(u, v))\kappa(u, v)$ for all $u, v \in W$ with $u \neq v$ (see Theorem 7 for details),
- $\min\{\mathcal{D}_\kappa(Tu, Tv), \kappa(v, Tv)\} \leq \varphi(\kappa(u, v))\kappa(u, v)$ for all $u, v \in W$ with $u \neq v$ (see Theorem 8 for details).

We give new examples to illustrate our results. As applications, some new fixed point theorems for essential distances and e^0-metrics are also established by applying these new generalized Mizoguchi-Takahashi's fixed point theorems. Our new results generalize and improve some of known results on the topic in the literature.

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