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Article

Modulation Equation for the Stochastic Swift–Hohenberg Equation with Cubic and Quintic Nonlinearities on the Real Line

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Abstract: The purpose of this paper is to rigorously derive the cubic–quintic Ginzburg–Landau equation as a modulation equation for the stochastic Swift–Hohenberg equation with cubic–quintic nonlinearity on an unbounded domain near a change of stability, where a band of dominant pattern is changing stability. Also, we show the influence of degenerate additive noise on the stabilization of the modulation equation.

Keywords: Swift–Hohenberg equation; stabilization by noise; Ginzburg–Landau equation; additive noise; multi-scale analysis

1. Introduction

The Swift–Hohenberg equation, which is a nonlinear parabolic equation containing fourth-order space-derivatives, takes the form

$$dU = [-(1+\partial_x^2)^2 U + \nu_\varepsilon U - U^3]dt. \tag{1}$$

Equation (1) was first proposed in 1977 by Swift and Hohenberg [1] as a simple model for the Rayleigh–Benard instability of roll waves. The equation also plays a key role in the studies of pattern formation [2].

Near the change of stability, Kirrmann et al. [3] approximated the solution of the Swift–Hohenberg Equation (1) on an unbounded domain via the Ginzburg–Landau equation

$$\partial_T A = 4\partial_X^2 A + \nu A - 3|A|^2 A.$$

This method of approximation depends on high regularity of the modulation equation, as they needed $A \in C_b^{1,4}([0,T] \times \mathbb{R})$.

In this paper, we consider the stochastic Swift–Hohenberg equation with cubic–quintic nonlinearity defined on an unbounded domain, which reads as follows:

$$dU = [-(1+\partial_x^2)^2 U + \nu_\varepsilon U + \eta_\varepsilon U^3 - \gamma U^5]dt + \sigma_\varepsilon d\beta(t), \tag{2}$$

where U is a real-valued scalar function and ν_ε, η_ε, γ are real coefficients. The coefficients η_ε and γ are chosen to be positive so that $\eta_\varepsilon > 0$ is responsible for the subcritical bifurcation of periodic states and $\gamma > 0$ is responsible for saturating the growth of the instability. The linear differential operator $\mathcal{L} = -(1+\partial_x^2)^2$ has eigenvalues $-\lambda_k = -(1-k^2)^2$ for $k \in \mathbb{R}$ corresponding to eigenfunctions e^{ikx}. We have for ν_ε changing sign a band of uncountably many eigenvalues changing sign around $k = \pm 1$.

We perturb the equation for simplicity by a space-independent noise shaking the system uniformly, which is given by the derivative of a standard real-valued Brownian motion $\{\beta(t)\}_{t\geq 0}$.

The deterministic (i.e., $\sigma_\varepsilon = 0$), Swift–Hohenberg Equation (2) has been used to model convective systems with mid-plane and left–right reflection symmetries and yields qualitatively reliable predictions of the properties of localized convection in binary fluids [4–7] and doubly diffusive convection [8].

Burke et al. [9], Dawes [10] and Hiraoka et al. [11] have studied the bifurcation behavior of the deterministic Equation (2) with the former focusing on the spatially localized states on an extended domain, and the latter focusing on the modulated structures formed on a finite domain. Sakaguchi and Brand [12] have found by computer simulations that the deterministic Equation (2) may have many types of stable localized stationary solutions in suitable parameter regions.

The deterministic Swift–Hohenberg Equation (2) with cubic and quintic nonlinearities is a model equation for describing pattern forming systems near instabilities with non-trivial spatial wavelength [13]. Since many dynamical systems are subject to the influence of noise, it is desirable to incorporate this influence into the mathematical model. The analysis of the effect of the noise on the properties of the solution and its approximation by a corresponding modulation equation become very relevant in this context.

Our aim in this paper is to rigorously derive the modulation Equation (3) which is called the cubic–quintic Ginzburg–Landau equation

$$dA = [4A'' + (\nu + \frac{3}{2}\eta\sigma^2 - \frac{15}{4}\gamma\sigma^4)A + (3\eta - 15\gamma\sigma^2)\left|A\right|^2 A - 10\gamma\left|A\right|^4 A]dT, \tag{3}$$

and to show that the approximation of the mild solutions U of Equation (2) by a modulated wave-train of the form

$$U(t,x) = \varepsilon A(\varepsilon^4 t, \varepsilon^2 x)e^{ix} + \varepsilon \bar{A}(\varepsilon^4 t, \varepsilon^2 x)e^{-ix} + \varepsilon \mathcal{Z}_\varepsilon(\varepsilon^4 t) + \text{error}, \tag{4}$$

where $A(T,X)$ is the solution of the Ginzburg–Landau Equation (3) which is defined on the slow time $T = \varepsilon^4 t$ and "slow" space $X = \varepsilon^2 x$. The perturbation $\mathcal{Z}_\varepsilon$ is a fast Ornstein–Uhlenbeck process, defined as

$$\mathcal{Z}_\varepsilon(T) = \varepsilon^{-2}\sigma \int_0^T e^{-\varepsilon^{-4}(T-\tau)} d\tilde{\beta}(\tau)\,, \tag{5}$$

where $\tilde{\beta}(T) := \varepsilon^2\beta(\varepsilon^{-4}T)$ is a rescaled version of the Brownian motion.

Also, our purpose is to treat the modulation equation with space–time white noise with least possible regularity. Although in our situation the solution of the modulation Equation (3) is spatially smooth, but is only Hölder in time.

In our previous paper [14] (written in collaboration with D. Blömker and K. Klepel), we studied the simpler case of a stochastic Swift–Hohenberg Equation (2), when $\gamma = 0$ and $\eta_\varepsilon = -1$, on unbounded domain and we established rigorously the following modulation equation

$$\partial_T A = 4\partial_X^2 A + (\nu - \tfrac{3}{2}\sigma^2)A - 3|A|^2 A.$$

For more results on the stochastic Swift–Hohenberg equation, see for instance [15–17]. Moreover, the generalized Swift–Hohenberg equation with simpler quadratic nonlinearity of the type u^2 is treated in [18].

The paper is divided into the following sections. In the next section we define the space $\mathcal{H}^r$ and mild solution of Equation (2). In Section 3 we give a formal derivation of the modulation equation. In Section 4 we give general bounds on the Ornstein–Uhlenbeck process $\mathcal{Z}_\varepsilon(T)$. In Section 5 we state and prove the main result of this paper. In Section 6, we show the effect of the degenerate additive noise on the stabilization of the modulation equation of the Swift–Hohenberg Equation (2). Finally, we give conclusions of this paper.

2. Space and Mild Solution

In this paper we will work in the following well known Sobolev space $\mathcal{H}^r$, see [19].

Definition 1. *For $r \in \mathbb{R}$ we define the space $\mathcal{H}^r$ by*

$$\mathcal{H}^r = \left\{ u : L^2(\mathbb{R}) \to L^2(\mathbb{R}) : \int_{-\infty}^{\infty} \left(1+y^2\right)^r |\mathcal{F}(u)(y)|^2 dy < \infty \right\},$$

with norm

$$\|u\|_r^2 = \int_{-\infty}^{\infty} \left(1+y^2\right)^r |\mathcal{F}(u)(y)|^2 \, dy,$$

where $\mathcal{F}(u)$ is the Fourier transform of u, defined by

$$\mathcal{F}(u)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(y) e^{-iky} dy.$$

Note that in the space $\mathcal{H}^r$ functions still decay to 0 at ∞. Thus if $A \in \mathcal{H}^r$ we are still in a setting, where the solutions of Equation (2) and the amplitude A decay to 0 for $|x| \to \infty$.

Definition 2. *(Mild solution) Let $T_0 > 0$ be a time. The continuous stochastic process $U \in C^0([0, T_0], \mathcal{H}^r)$ is called a mild solution of Equation (2) if*

$$U(t) = e^{t\mathcal{L}} U(0) + \int_0^t e^{(t-s)\mathcal{L}} [\nu_\varepsilon U(s) + \eta_\varepsilon U^3(s) - \gamma U^5(s)] ds + \sigma_\varepsilon \int_0^t e^{(t-s)\mathcal{L}} d\beta(s), \tag{6}$$

where $U(0)$ is initial function.

The existence of the mild solution of Equation (6) is standard by using a fixed point arguments for example in the space $\mathcal{H}^1$ for sufficiently smooth noise.

In next definition we state what we mean exactly when we write "order of" or its abbreviation $\mathcal{O}()$.

Definition 3. *For a family of real-valued stochastic processes $\{X_\varepsilon(t)\}_{t \geq 0}$ we say $X_\varepsilon = \mathcal{O}(g_\varepsilon)$, if for every $p \geq 1$ there exists a constant C_p such that*

$$\mathbb{E} \left| \sup_{t \in [0, T_0]} \|X_\varepsilon(t)\|_\infty \right|^p \leq C_p \, |g_\varepsilon|^p \,, \tag{7}$$

where $\|\cdot\|_\infty$ is the norm in $C^0(\mathbb{R})$. The similar notation is also used for time-independent random variables.

3. Derivation of Cubic–Quintic Ginzburg–Landau Equation

Before we discuss a formal derivation of the amplitude or modulation equation corresponding to Equation (2). Let us state the following lemma without proof (see the proof of Lemma 5.1 in [20]) on averaging over the fast OU-process $\mathcal{Z}_\varepsilon$, where $\mathcal{Z}_\varepsilon = \mathcal{O}(\varepsilon^{-\kappa})$ for $\kappa > 0$ (see the proof of Lemma 4.2 in [20]). This lemma displays that the integrals over the OU-process $\mathcal{Z}_\varepsilon$ contains even powers like $\mathcal{Z}_\varepsilon^2$ or $\mathcal{Z}_\varepsilon^4$ and have a contribution, which is a constant of order one. While the integrals contain odd powers like $\mathcal{Z}_\varepsilon$ or $\mathcal{Z}_\varepsilon^3$ are small.

Lemma 1. *Let $Y(T)$ be a complex valued stochastic process and $Y(0) = \mathcal{O}(\varepsilon^{-\alpha})$ for $\alpha \geq 0$. If $dY = GdT$, with $G = \mathcal{O}(\varepsilon^{-\alpha})$, then for $\kappa \in (0,1)$ and $n \in \mathbb{N}$*

$$\int_0^T Y\mathcal{Z}_\varepsilon^n d\tau = \begin{cases} \mathcal{O}(\varepsilon^{2-\alpha-\kappa}) & \text{if } n \text{ is odd} \\ \frac{(n-1)\sigma^n}{n}\int_0^T Y\mathcal{Z}_\varepsilon^{n-2}d\tau + \mathcal{O}(\varepsilon^{2-\alpha-\kappa}) & \text{if } n \text{ is even} \end{cases}. \tag{8}$$

Now, we assume that a solution of the modulation Equation (3) is sufficiently smooth. First, let us introduce a small parameter $\varepsilon \ll 1$ and define the parameters ν_ε, η_ε and σ_ε in Equation (2) as

$$\nu_\varepsilon = \varepsilon^4\nu, \eta_\varepsilon = \varepsilon^2\eta \text{ and } \sigma_\varepsilon = \varepsilon\sigma.$$

This means that one actually considers the equation

$$dU = [-(1+\partial_x^2)^2U + \varepsilon^4\nu U + \varepsilon^2\eta U^3 - \gamma U^5]dt + \varepsilon\sigma d\beta(t). \tag{9}$$

If we rescale to the slow time-scale $T = \varepsilon^4 t$ and slow spatial scale $X = \varepsilon^2 x$ via

$$U(t,x) = \varepsilon u(\varepsilon^4 t, \varepsilon^2 x),$$

then Equation (9) takes the form

$$du = [\mathcal{L}_\varepsilon u + \nu u + \eta u^3 - \gamma u^5]dT + \varepsilon^{-2}\sigma d\tilde{\beta}(T), \tag{10}$$

where $\mathcal{L}_\varepsilon = -\varepsilon^{-4}(1+\varepsilon^4\partial_X^2)^2$, $T = \varepsilon^4 t$ and $X = \varepsilon^2 x$. Now define w as

$$u(T,X) = w(T,X) + \mathcal{Z}_\varepsilon(T). \tag{11}$$

Plugging Equation (11) into Equation (10), to obtain

$$\begin{aligned} \partial_T w &= \mathcal{L}_\varepsilon w + \nu w + \eta w^3 + 3\eta w^2\mathcal{Z}_\varepsilon + 3\eta w\mathcal{Z}_\varepsilon^2 \\ &\quad -\gamma w^5 - 5\gamma w^4\mathcal{Z}_\varepsilon - 10\gamma w^3\mathcal{Z}_\varepsilon^2 - 10\gamma w^2\mathcal{Z}_\varepsilon^3 \\ &\quad -5\gamma w\mathcal{Z}_\varepsilon^4 + \nu\mathcal{Z}_\varepsilon + \eta\mathcal{Z}_\varepsilon^3 - \gamma\mathcal{Z}_\varepsilon^5. \end{aligned} \tag{12}$$

Let us make the following ansatz:

$$\begin{aligned} w_A(T,X) &= A_1(T,X)e^{ix} + \varepsilon^4 A_2(T,X)e^{2ix} + \varepsilon^4 A_3(T,X)e^{3ix} \\ &\quad +\varepsilon^4 A_4(T,X)e^{3ix} + \varepsilon^4 A_5(T,X)e^{5ix} + c.c. + \varepsilon^4 A_0(T,X), \end{aligned} \tag{13}$$

where *c.c.* denotes the complex conjugate. The additional higher order terms do not improve the approximation result, but they are necessary to eliminate large error terms.

Plugging Equation (13) into Equation (12), assuming that all terms are sufficiently smooth, and using the relation

$$\begin{aligned} \mathcal{L}_\varepsilon\left(f(X)e^{in\varepsilon^{-2}X}\right) &= -[\varepsilon^{-4}(1-n^2)^2 f + 4i\varepsilon^{-2}n(1-n^2)f' + (2-6n^2)f'' \\ &\quad + 4i\varepsilon^2 n f''' + \varepsilon^4 f'''']e^{in\varepsilon^{-2}X}, \end{aligned} \tag{14}$$

we obtain

$$\begin{aligned}\partial_T A_1 e^{ix} + c.c. = {} & 4A_1'' e^{ix} - 9A_2 e^{2ix} - 64A_3 e^{3ix} - 225A_4 e^{4ix} \\ & - 576A_5 e^{5ix} + c.c. - A_0 + \nu[A_1 e^{ix} + c.c.] \\ & + \eta[A_1^3 e^{3ix} + 3|A_1|^2 A_1 e^{ix} + c.c.] \\ & + 3\eta \mathcal{Z}_\varepsilon [A_1^2 e^{2ix} + c.c. + 2|A_1|^2] + 3\eta \mathcal{Z}_\varepsilon^2 [A_1 e^{ix} + c.c.] \\ & - \gamma[A_1^5 e^{5ix} + 5|A_1|^2 A_1^3 e^{3ix} + 10|A_1|^4 A_1 e^{ix} + c.c.] \\ & - 5\gamma \mathcal{Z}_\varepsilon [A_1^4 e^{4ix} + 4|A_1|^2 A_1^2 e^{2ix} + c.c. + 6|A_1|^4] \\ & - 10\gamma \mathcal{Z}_\varepsilon^2 [A_1^3 e^{3ix} + 3|A_1|^2 A_1 e^{ix} + c.c.] \\ & - 10\gamma \mathcal{Z}_\varepsilon^3 [A_1^2 e^{2ix} + c.c. + 2|A_1|^2] + \nu \mathcal{Z}_\varepsilon \\ & - 5\gamma \mathcal{Z}_\varepsilon^4 [A_1 e^{ix} + c.c.] + \eta \mathcal{Z}_\varepsilon^3 - \gamma \mathcal{Z}_\varepsilon^5 + \mathcal{O}(\varepsilon^2)\,.\end{aligned}$$

In order to remove all unwanted terms, define

$$\begin{aligned} A_2 &= \frac{1}{9}\mathcal{Z}_\varepsilon A_1^2[3\eta - 20\gamma|A_1|^2 - 10\gamma \mathcal{Z}_\varepsilon^2], \\ A_3 &= \frac{1}{64} A_1^3 [\eta - 5\gamma|A_1|^2 - 10\gamma \mathcal{Z}_\varepsilon^2], \\ A_4 &= \frac{-\gamma}{75}\mathcal{Z}_\varepsilon A_1^4, \quad A_5 = \frac{-\gamma}{576} A_1^5, \\ \text{and } A_0 &= \mathcal{Z}_\varepsilon |A_1|^2 [6\eta - 30\gamma|A_1|^2 - 20\gamma \mathcal{Z}_\varepsilon^2] + \nu \mathcal{Z}_\varepsilon + \eta \mathcal{Z}_\varepsilon^3 - \gamma \mathcal{Z}_\varepsilon^5\,. \end{aligned} \tag{15}$$

Hence

$$\begin{aligned}\partial_T A_1 e^{ix} + c.c. = {} & [4A_1'' + \nu A_1 + 3\eta|A_1|^2 A_1 + 3\eta \mathcal{Z}_\varepsilon^2 A_1 \\ & - 10\gamma|A_1|^4 A_1 - 30\gamma \mathcal{Z}_\varepsilon^2 |A_1|^2 A_1 \\ & - 5\gamma \mathcal{Z}_\varepsilon^4 A_1] e^{ix} + c.c. + \mathcal{O}(\varepsilon^2)\,.\end{aligned}$$

Collecting all terms in front of e^{ix}, yields

$$\begin{aligned}\partial_T A_1 &= 4A_1'' + \nu A_1 + 3\eta|A_1|^2 A + 3\eta \mathcal{Z}_\varepsilon^2 A_1 - 10\gamma|A_1|^4 A_1 \\ &\quad - 30\gamma \mathcal{Z}_\varepsilon^2 |A_1|^2 A_1 - 5\gamma \mathcal{Z}_\varepsilon^4 A_1 + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{16}$$

In order to remove $\mathcal{Z}_\varepsilon$ from Equation (16), we apply Lemma 1 to Equation (16)

$$\partial_T A_1 = 4A_1'' + (\nu + \frac{3}{2}\eta\sigma^2 - \frac{15}{4}\gamma\sigma^4)A_1 + (3\eta - 15\gamma\sigma^2)|A_1|^2 A_1 - 10\gamma|A_1|^4 A_1 + \mathcal{O}(\varepsilon^{2-\kappa}). \tag{17}$$

To obtain the modulation Equation (3), we ignore all small terms in ε.

4. General Bounds on OU Process $\mathcal{Z}_\varepsilon$

In this section we give a general bound on Ornstein–Uhlenbeck process $\mathcal{Z}_\varepsilon$.

Lemma 2. *If A is the solution of the mild formulation of Equation (3) with $\|A\|_r = \mathcal{O}(1)$, then for $\kappa \in (0,1)$*

$$\int_0^T e^{4(T-s)\partial_X^2} A(s)\{\mathcal{Z}_\varepsilon^2(s) - \frac{\sigma^2}{2}\}ds = \mathcal{O}(\varepsilon^{2-\kappa}), \tag{18}$$

and

$$\int_0^T e^{4(T-s)\partial_X^2} A(s)\{\mathcal{Z}_\varepsilon^4(s) - \frac{3\sigma^4}{4}\}ds = \mathcal{O}(\varepsilon^{2-\kappa}). \tag{19}$$

Proof. Let for $s \in [0,T]$
$$Y(s) = e^{4(T-s)\partial_X^2} A(s).$$

Hence
$$dY = (-4\partial_X^2) e^{4(T-s)\partial_X^2} A(s)ds + e^{4(T-s)\partial_X^2} dA.$$

Using Equation (3), we obtain
$$dY = e^{4(T-s)\partial_X^2} \big[(\nu + \frac{3}{2}\eta\sigma^2 - \frac{15}{4}\gamma\sigma^4)A + (3\eta - 15\gamma\sigma^2)|A|^2 A - 10\gamma |A|^4 A\big] ds.$$

Define
$$G(s) = e^{4(T-s)\partial_X^2} \big[(\nu + \frac{3}{2}\eta\sigma^2 - \frac{15}{4}\gamma\sigma^4)A + (3\eta - 15\gamma\sigma^2)|A|^2 A - 10\gamma |A|^4 A\big] \tag{20}$$

Taking $\|.\|_\infty$ for both sides and using the following two inequalities $\|u\|_\infty \le \|u\|_r$ (Theorem 5.4 in [19]) and $\left\| e^{4T\partial_X^2} u \right\|_r \le \|u\|_r$ (Corollary 4.6 in [14]), yields
$$\|G\|_\infty \le \|G\|_r \le C\|A\|_r + C\|A\|_r^3 + C\|A\|_r^5 .$$

Hence
$$\mathbb{E} \sup_{[0,T_0]} \|G\|_\infty \le C .$$

Now applying the averaging result of Lemma 1, yields Equations (18) and (19). □

Lemma 3. *If A is the solution of the mild formulation of Equation (3) with $\|A\|_{r+\delta} = \mathcal{O}(1)$, for $r > \frac{1}{2}$, $\kappa \in (0,1)$ and for $\delta \in \{0,1\}$, then*
$$\int_0^T e^{4(T-s)\partial_X^2} |A(s)|^2 A(s)\{\mathcal{Z}_\varepsilon^2(s) - \frac{\sigma^2}{2}\} ds = \mathcal{O}(\varepsilon^{2-\kappa}), \tag{21}$$

Proof. Let for $s \in [0,T]$
$$Y(s) = e^{4(T-s)\partial_X^2} |A(s)|^2 A(s).$$

Hence
$$dY = (-4\partial_X^2) e^{4(T-s)\partial_X^2} |A|^2 A ds + e^{4(T-s)\partial_X^2} d(|A|^2 A).$$

Using Equation (3), we obtain
$$dY = e^{4(T-s)\partial_X^2} \big[-8\partial_X A^2 \partial_X \bar{A} - 8\partial_X A \partial_X |A|^2 + 3c_1 |A|^2 A + 3c_2 |A|^4 A + 3c_3 |A|^6 A\big] ds.$$

where
$$c_1 = (\nu + \frac{3}{2}\eta\sigma^2 - \frac{15}{4}\gamma\sigma^4),\ \ c_2 = (3\eta - 15\gamma\sigma^2), \text{ and } c_3 = -10\gamma.$$

Define
$$G(s) = e^{4(T-s)\partial_X^2} \big[-8\partial_X A^2 \partial_X \bar{A} - 8\partial_X A \partial_X |A|^2 + 3c_1 |A|^2 A + 3c_2 |A|^4 A + 3c_3 |A|^6 A\big] \tag{22}$$

Taking $\|.\|_\infty$ for both sides and using the inequalities $\|u\|_\infty \le \|u\|_r$ and $\left\| e^{4T\partial_X^2} u \right\|_r \le \|u\|_r$, yields
$$\|G\|_\infty \le \|G\|_r \le C\|A\|_{r+1}^3 + C\|A\|_r^3 + C\|A\|_r^5 + C\|A\|_r^7 .$$

Hence
$$\mathbb{E} \sup_{[0,T_0]} \|G\|_\infty \le C .$$

Applying the averaging result of Lemma 1, yields Equation (21). □

5. Main Results

In this section, we state and prove the main result of this paper. First, let us state without proof two lemma's from [14]. The purpose of the first one is to change the semigroup $e^{T\mathcal{L}_\varepsilon}$ by the semigroup $e^{4T\partial_X^2}$, when they are applied to a modulated wave $Ae^{i\varepsilon^{-2}X}$, while the purpose of the second one is to bound the semigroup $e^{T\mathcal{L}_\varepsilon}$, when applied to $\psi(X)e^{in\varepsilon^{-2}X}$.

Lemma 4. *There is a constant $C > 0$ such that, for $T > 0$ and $A \in \mathcal{H}^r$ for $r > \frac{1}{2}$, we have*

$$\sup_{X\in\mathbb{R}} \left| e^{T\mathcal{L}_\varepsilon} A(X)e^{i\varepsilon^{-2}X} - (e^{4T\partial_X^2} A)(X)e^{i\varepsilon^{-2}X} \right| \leq C \|A\|_r \phi_\varepsilon,$$

where ϕ_ε is defined as

$$\phi_\varepsilon^2 = \begin{cases} \varepsilon^4 & \text{if} \quad r > \frac{3}{2} \\ \varepsilon^4 \ln(1/\varepsilon) & \text{if} \quad r = \frac{3}{2} \\ \varepsilon^{4r-2} & \text{if} \quad \frac{1}{2} < r < \frac{3}{2}. \end{cases} \tag{23}$$

Let $n \in \mathbb{R} \setminus \{\pm 1\}$ and $r > \frac{1}{2}$. There are two constants $C > 0$ and $c_n > 0$, depending on n, such that, for $T > 0$ and $\psi \in \mathcal{H}^r$,

$$\sup_{X\in\mathbb{R}} \left| e^{T\mathcal{L}_\varepsilon} \left(\psi(X)e^{in\varepsilon^{-2}X} \right) \right| \leq C \|\psi\|_r \{\varepsilon^{-1} \exp(-\frac{9}{4}\varepsilon^{-4}T) + \varepsilon^{2r-1}\}. \tag{24}$$

Next let us define and bound the residual.

Definition 4. *Define the residual $res(T)$ as*

$$\begin{aligned} res(T) &= v_A(T) - e^{T\mathcal{L}_\varepsilon} v_A(0) - \nu \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}(v_A + \mathcal{Z}_\varepsilon)ds \\ &\quad -\eta \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}(v_A + \mathcal{Z}_\varepsilon)^3 ds + \gamma \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}(v_A + \mathcal{Z}_\varepsilon)^5 ds, \end{aligned} \tag{25}$$

where v_A is defined as

$$v_A(T, X) = A(T, X)e^{i\varepsilon^{-2}X} + c.c, \tag{26}$$

with $A(T, X)$ is a solution of Equation (3).

Lemma 5. *If $\sup_{[0,T_0]} \|A\|_{r+\delta} = \mathcal{O}(1)$, for $r > \frac{1}{2}$ and for $\delta \in \{0, 1\}$, then*

$$\sup_{T\in[0,T_0]} \|res(T)\|_\infty = \mathcal{O}(\varepsilon^{-\kappa}\phi_\varepsilon), \tag{27}$$

for $\kappa > 0$ chosen sufficiently small, where ϕ_ε is defined in Equation (23).

Proof. From Equation (25), we obtain

$$
\begin{aligned}
res(T) \;=\; & A(T)e^{i\varepsilon^{-2}X} - e^{T\mathcal{L}_\varepsilon}A(0)e^{i\varepsilon^{-2}X} \\
& - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}(\nu + 3\eta\mathcal{Z}_\varepsilon^2 - 5\gamma\mathcal{Z}_\varepsilon^4)Ae^{i\varepsilon^{-2}X}ds + c.c. \\
& - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}[(3\eta - 30\gamma\mathcal{Z}_\varepsilon^2)\,|A|^2 A - 10\gamma\,|A|^4 A]e^{i\varepsilon^{-2}X}ds + c.c. \\
& - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}[3\eta - 20\gamma\,|A|^2 - 10\gamma\mathcal{Z}_\varepsilon^2]A^2\mathcal{Z}_\varepsilon e^{2i\varepsilon^{-2}X}ds + c.c. \\
& - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}[\eta - 5\gamma\,|A|^2 - 10\gamma\mathcal{Z}_\varepsilon^2]A^3e^{3i\varepsilon^{-2}X}ds + c.c. \\
& + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}[5\gamma\mathcal{Z}_\varepsilon A^4e^{4ix} + \gamma A^5e^{5ix}]ds + c.c. \\
& - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}[6\eta - 30\gamma\,|A|^2 - 20\mathcal{Z}_\varepsilon^2]\mathcal{Z}_\varepsilon\,|A|^2\,ds + \mathcal{O}(\varepsilon^{4-\kappa}).
\end{aligned}
$$

In this step we used the bound on $\mathcal{Z}_\varepsilon$ (where $\mathcal{Z}_\varepsilon = \mathcal{O}(\varepsilon^{-\kappa_0})$) and the bound terms like $\int_0^T e^{-\varepsilon^{-4}(T-s)}\mathcal{Z}_\varepsilon^3 ds = \mathcal{O}(\varepsilon^{4-\kappa})$.
Now, using Lemma 4 for changing the semigroup and Lemma 5 to bound terms like $\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^5e^{5ix}ds$, yields

$$
\begin{aligned}
res(T) \;=\; & [A(T) - e^{4(T-s)\partial_X^2}A(0) - \int_0^T e^{4(T-s)\partial_X^2}(\nu + 3\eta\mathcal{Z}_\varepsilon^2 - 5\gamma\mathcal{Z}_\varepsilon^4)Ads \\
& - \int_0^T e^{4(T-s)\partial_X^2}[(3\eta - 30\gamma\mathcal{Z}_\varepsilon^2)\,|A|^2 A - 10\gamma\,|A|^4 A]ds]e^{i\varepsilon^{-2}X} \\
& + c.c. + \mathcal{O}(\varepsilon^{-\kappa}\phi_\varepsilon).
\end{aligned}
$$

where ϕ_ε is defined in Equation (23). From the modulation Equation (3) we have

$$
\begin{aligned}
res(T) \;=\; & [-3\eta\int_0^T e^{4(T-s)\partial_X^2}A(\mathcal{Z}_\varepsilon^2 - \frac{\sigma^2}{2})ds + 5\gamma\int_0^T e^{4(T-s)\partial_X^2}A(\mathcal{Z}_\varepsilon^4 - \frac{3\sigma^4}{4})ds \\
& + 30\gamma\int_0^T e^{4(T-s)\partial_X^2}\,|A|^2\,A(\mathcal{Z}_\varepsilon^2 - \frac{\sigma^2}{2})ds]e^{i\varepsilon^{-2}X} + c.c. + \mathcal{O}(\varepsilon^{-\kappa}\phi_\varepsilon).
\end{aligned}
$$

Using Lemmas 2 and 3, yields Equation (27) for $r > \frac{1}{2} + \kappa$, which is always true for $r > \frac{1}{2}$, if we choose κ sufficiently small. □

Definition 5. *Define the set $\Omega_\varepsilon \subset \Omega$ such that for sufficiently small $\kappa > 0$ and for all $\varepsilon \in (0,1)$, all these estimates*

$$
|\int_0^{T_0}\{|\mathcal{Z}_\varepsilon|^2 - \frac{\sigma^2}{2}\}d\tau| < \varepsilon^{1-2\kappa}, \tag{28}
$$

$$
|\int_0^{T_0}\{|\mathcal{Z}_\varepsilon|^4 - \frac{3\sigma^4}{4}\}d\tau| < \varepsilon^{1-2\kappa}, \tag{29}
$$

and

$$
\sup_{T\in[0,T_0]}\|res(T)\|_\infty < \varepsilon^{-2\kappa}\phi_\varepsilon, \tag{30}
$$

hold on Ω_ε.

Proposition 1. *For all $p > 0$, there exist a constant C_p such that on Ω_ε*

$$
\mathbb{P}(\Omega_\varepsilon) \geq 1 - C_p\varepsilon^p \;\; \textit{for all} \;\; \varepsilon \in (0,1). \tag{31}
$$

Proof.

$$\mathbb{P}(\Omega_\varepsilon) \geq 1 - \mathbb{P}\Big(\int_0^{T_0} \{|\mathcal{Z}_\varepsilon|^2 - \frac{\sigma^2}{2}\} d\tau \geq \varepsilon^{1-2\kappa}\Big)$$
$$-\mathbb{P}\Big(\int_0^{T_0} \{|\mathcal{Z}_\varepsilon|^4 - \frac{3\sigma^4}{4}\} d\tau \geq \varepsilon^{1-2\kappa}\Big) - \mathbb{P}(\sup_{[0,T_0]} \|res\|_\infty \geq \varepsilon^{-2\kappa}\phi_\varepsilon).$$

Using Chebychev's inequality

$$\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon^{2q\kappa}\phi_\varepsilon^{-q}\mathbb{E}\sup_{[0,T_0]} \|res\|_\infty^q - \varepsilon^{-q+2q\kappa}\mathbb{E}\Big(\int_0^{T_0} \{|\mathcal{Z}_\varepsilon|^2 - \frac{\sigma^2}{2}\} d\tau\Big)^q$$
$$- \varepsilon^{-q+2q\kappa}\mathbb{E}\Big(\int_0^{T_0} \{|\mathcal{Z}_\varepsilon|^4 - \frac{3\sigma^4}{4}\} d\tau\Big)^q.$$

From Lemmas 2, 3 and 5, we obtain

$$\mathbb{P}(\Omega_\varepsilon) \geq 1 - C_q\varepsilon^{q\kappa}.$$

For sufficiently large q, yields

$$\mathbb{P}(\Omega_\varepsilon) \geq 1 - C_p\varepsilon^{p}.$$

□

The main result of this paper is the following approximation result for solutions of the stochastic Swift–Hohenberg Equation (9) through the solutions of the Ginzburg–Landau Equation (3).

Theorem 1. *(Approximation) Let $U(t,x)$ be a solution of Equation (9), $v_A(T,X)$ the formal approximation defined in Equation (26) such that $A \in C^0([0,T_0],\mathcal{H}^r)$ and $\sup_{[0,T_0]}\|A\|_{r+\delta} = \mathcal{O}(1)$ for $r > \frac{1}{2}$ and for $\delta \in \{0,1\}$. Suppose that for the initial function*

$$\|U(0) - \varepsilon A(0)e^{ix} - \varepsilon\bar{A}(0)e^{-ix}\|_\infty \leq d\varepsilon^{1-2\kappa}\phi_\varepsilon,$$

for some fixed constant $d > 0$ and for some small $0 < \kappa < (r - \frac{1}{2})$. Then for each $T_0 > 0$ such that for all $p > 0$ there exist $C > 0$, depending on $\sup_{[0,T_0]}\|A\|_{r+\delta}$, such that

$$\mathbb{P}\Big\{ \sup_{t\in[0,\varepsilon^{-4}T_0]} \Big\|U(t,x) - \varepsilon v_A(\varepsilon^4 t, \varepsilon^2 x) - \varepsilon\mathcal{Z}_\varepsilon(\varepsilon^4 t)\Big\|_\infty > \varepsilon^{1-2\kappa}\phi_\varepsilon\Big\} \leq C\varepsilon^p, \tag{32}$$

where the fast OU process $\mathcal{Z}_\varepsilon(T)$ defined in Equation (5) and ϕ_ε defined in Equation (23).

Proof. We follow the same steps of the proof of Theorem 3.4 in [14]. □

6. The Effect of Degenerate Additive Noise

In this section we show the effect of degenerate additive noise on the stabilization of the solution of the Ginzburg–Landau Equation (3). Let us first fix $\gamma = 1$, and $\eta = 2$. Therefore, in the modulation Equation (3), we note in this case that:

1. The coefficient of the cubic term is positive for $\sigma^2 < 0.4$ and is a non-positive otherwise,
2. The coefficient of the linear term is positive for $\sigma^2 < 0.4 + \sqrt{\frac{4}{15}(\nu + 0.6)}$ if $\nu \geq -0.6$ or $\sigma^2 > 0.4 - \sqrt{\frac{4}{15}(\nu + 0.6)}$ if $-0.6 \leq \nu < 0$ and is a non-positive otherwise.

Now, some numerical simulations are presented for the fixed parameters $\nu = 0.9$, $\varepsilon = 0.3$ and varying noise intensity σ. We carried out a straightforward semi-implicit time discretization of the

Galerkin spectral method using fast Fourier transforms. For time discretization, we used a constant small time step, for example, $h = 10^{-4}$.

In the Figure 1, we see that the modulation equation solution fluctuates and has a pattern if the noise intensity $\sigma = 0$.

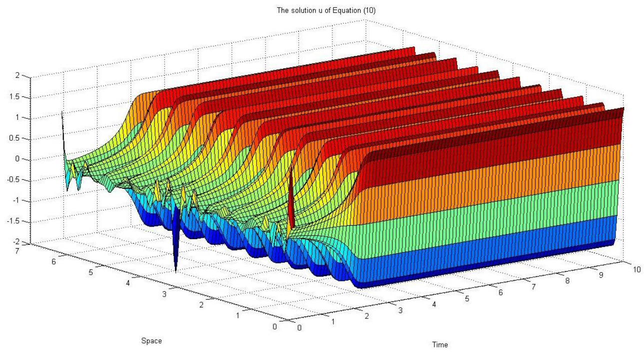

Figure 1. $\sigma = 0$, $\nu = 0.9$ and $\varepsilon = 0.3$.

In Figures 2–5, if the noise intensity of σ increases, the pattern begins to destroy due to the noise effect.

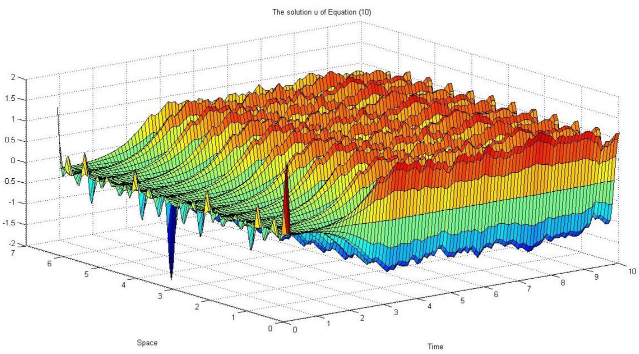

Figure 2. $\sigma = \sqrt{0.4}$, $\nu = 0.9$ and $\varepsilon = 0.3$.

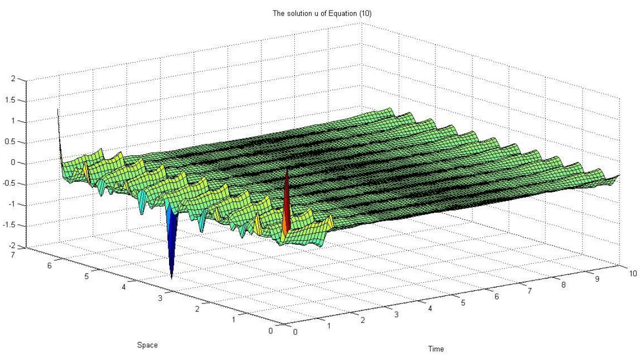

Figure 3. $\sigma = 1$, $\nu = 0.9$ and $\varepsilon = 0.3$.

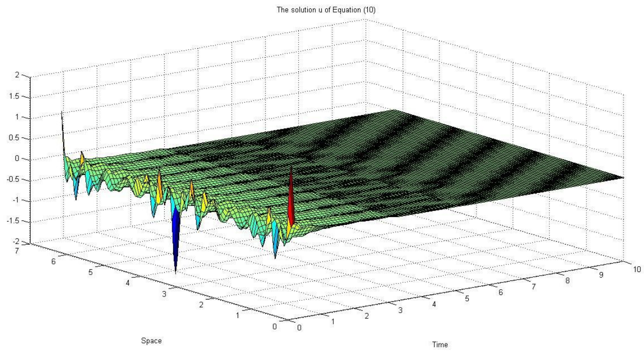

Figure 4. $\sigma = \sqrt{1.2}$, $\nu = 0.9$ and $\varepsilon = 0.3$.

If we choose $\sigma^2 > 0.4 + \sqrt{\frac{4}{15}(\nu + 0.6)}$ if $\nu \geq -0.6$ or $\sigma^2 < 0.4 - \sqrt{\frac{4}{15}(\nu + 0.6)}$ if $-0.6 \leq \nu < 0$, then the coefficient of the linear term is negative. Therefore, we deduce that small global noise has the potential to stabilize the modulation equation, and thus to destroy the dominant pattern.

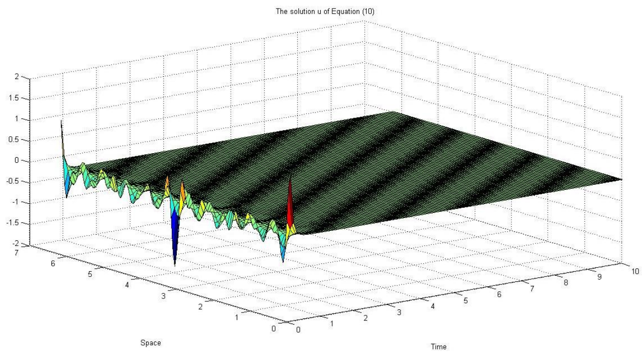

Figure 5. $\sigma = \sqrt{2}$, $\nu = 0.9$ and $\varepsilon = 0.3$.

7. Conclusions

In this paper, we considered the stochastic Swift–Hohenberg Equation (1) with cubic–quintic nonlinearity defined on an unbounded domain and derived rigorously the Ginzburg–Landau equation as a modulation Equation (3). Near change of stability, we approximated the solution of the Swift–Hohenberg Equation (1) by the solution of the Ginzburg–Landau Equation (3). Finally, we showed the effect of the degenerate additive noise on the stabilization of the solution of the modulation Equation (3).

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