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Generalized Parabolic Marcinkiewicz Integral Operators Related to Surfaces of Revolution

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Abstract: In this work, the generalized parametric Marcinkiewicz integral operators with mixed homogeneity related to surfaces of revolution are studied. Under some weak conditions on the kernels, the boundedness of such operators from Triebel–Lizorkin spaces to L^p spaces are established. Our results, with the help of an extrapolation argument, improve and extend some previous known results.

Keywords: parabolic Marcinkiewicz integrals; rough kernels; Triebel–Lizorkin spaces; L^p boundedness; extrapolation

1. Introduction

Throughout this article, the conjugate index of q is denoted by q' , which is defined by the relation $1/q + 1/q' = 1$. Moreover, in the Euclidean space $\mathbf{R}^n$ ($n \geq 2$), the unit sphere $\mathbf{S}^{n-1}$ is equipped with the normalized Lebesgue surface measure $d\mu$.

For fixed $\alpha_j \geq 1$, ($j = 1, 2, \dots, n$), let $\mathcal{H} : \mathbf{R}^n \times \mathbf{R}^+ \rightarrow \mathbf{R}$ be a function given by $\mathcal{H}(u, \rho) = \sum_{j=1}^n \frac{u_j^2}{\rho^{2\alpha_j}}$ with $u = (u_1, u_2, \dots, u_n) \in \mathbf{R}^n$. The mixed homogeneity metric space related to $\{\alpha_j\}_{j=1}^n$ on $\mathbf{R}^n$ is defined by the metric $\rho(u)$, where $\rho(u)$ is the unique solution to $\mathcal{H}(u, \rho) = 1$ (see [1]). For $\rho > 0$, we define the diagonal $n \times n$ matrix A_ρ by

$$A_\rho = \begin{bmatrix} \rho^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \rho^{\alpha_n} \end{bmatrix}.$$

For the space $(\mathbf{R}^n, \rho)$, we use the weighted spherical coordinates

$$\begin{aligned} u_1 &= \rho^{\alpha_1} \cos \theta_1 \cdots \cos \theta_{n-2} \cos \theta_{n-1}, \\ u_2 &= \rho^{\alpha_2} \cos \theta_1 \cdots \cos \theta_{n-2} \sin \theta_{n-1}, \\ &\vdots \\ u_{n-1} &= \rho^{\alpha_{n-1}} \cos \theta_1 \sin \theta_2, \\ u_n &= \rho^{\alpha_n} \sin \theta_1. \end{aligned}$$

This change of variables gives that $du = \rho^{\alpha-1} J(u') d\rho d\mu(u')$, where

$$u' = A_{\rho(u)^{-1}} u \in \mathbf{S}^{n-1}, \quad \alpha = \sum_{j=1}^n \alpha_j, \quad J(u') = \sum_{j=1}^n \alpha_j (u'_j)^2,$$

and $\rho^{\alpha-1}J(u')$ is the Jacobian of the weighted spherical coordinates.

By [1], one can show that $J(u') \in C^\infty(\mathbf{S}^{n-1})$ and

$$1 \leq J(u') \leq C, \quad \forall u' \in \mathbf{S}^{n-1} \quad \text{for some } C \geq 1.$$

For $\tau = \tau_1 + i\tau_2$ ($\tau_1, \tau_2 \in \mathbf{R}$ with $\tau_1 > 0$), let $K_{\Omega,g}$ be the kernel on $\mathbf{R}^n$ given by

$$K_{\Omega,g}(u) = \frac{\Omega(u)g(\rho(u))}{\rho(u)^{\alpha-\tau}},$$

where g is a measurable function on $\mathbf{R}^+$; and Ω is a function in $L^1(\mathbf{S}^{n-1})$ satisfying the conditions

$$\Omega(A_\rho u) = \Omega(u), \quad \forall \rho > 0, \quad (1)$$

$$\int_{\mathbf{S}^{n-1}} \Omega(u')J(u')d\mu(u') = 0. \quad (2)$$

For a nice function $\psi : \mathbf{R}^+ \rightarrow \mathbf{R}$, the generalized parabolic Marcinkiewicz integral operators $\mathcal{M}_{\Omega,\psi,g,\rho}^{(\lambda)}$ is defined by

$$\mathcal{M}_{\Omega,\psi,g,\rho}^{(\lambda)} f(\bar{x}) = \left(\int_0^\infty \left| t^{-\tau} \int_{\rho(u) \leq t} f(x-u, x_{n+1} - \psi(\rho(u))K_{\Omega,g}(u)du \right|^\lambda \frac{dt}{t} \right)^{1/\lambda}$$

for $f \in \mathcal{S}(\mathbf{R}^{n+1})$, where $\bar{x} = (x, x_{n+1})$ and $\lambda > 1$.

It is clear that when $\alpha_1 = \dots = \alpha_n = 1$, then $\alpha = n$ and $\rho(x) = |x|$. Hence, the operator $\mathcal{M}_{\Omega,\psi,g,\rho}^{(\lambda)}$ is just the generalized Marcinkiewicz integral operator, denoted by $\mathcal{M}_{\Omega,\psi,g}^{(\lambda),c}$. Moreover, when $\lambda = 2$, $g \equiv 1$, and $\psi(s) = s$, then the operator $\mathcal{M}_{\Omega,\psi,g}^{(\lambda),c}$, denoted by $\mathcal{M}_{\Omega,\tau}$, becomes the classical parametric Marcinkiewicz integral operator. Historically, Stein, in [2], introduced the operator $\mathcal{M}_{\Omega,1}$. Precisely, he showed that $\mathcal{M}_{\Omega,1}$ is bounded on $L^p(\mathbf{R}^n)$ for $p \in (1, 2]$ provided that $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$ with $0 < \alpha \leq 1$. Subsequently, a considerable amount of research has been done to obtain the L^p boundedness of $\mathcal{M}_{\Omega,1}$ (see for instant [3–6]).

The study of the boundedness of $\mathcal{M}_{\Omega,\tau}$ was started by Hörmander in [7]. In fact, he proved that $\mathcal{M}_{\Omega,\tau}$ is of type (p, p) for $1 < p < \infty$ if $\tau > 0$ and $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$ with $\alpha > 0$. Later on, the study of the operator $\mathcal{M}_{\Omega,\psi,g}^{(2),c}$ under very various conditions on the kernels attracted the attention of many mathematicians. For relevant results, we advice the readers to consult [8–16], among others.

Afterward, the investigation to establish the boundedness of the integral operator $\mathcal{M}_{\Omega,\psi,g}^{(\lambda),c}$ began. In fact, Chen, Fan and Ying introduced the operator $\mathcal{M}_{\Omega,\psi,g}^{(\lambda),c}$ in [17], where the authors showed that if $g \equiv 1$, $\psi(s) = s$, $1 < \lambda < \infty$, and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$, then for all $1 < p < \infty$,

$$\left\| \mathcal{M}_{\Omega,\psi,g}^{(\lambda),c} f \right\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{F_{p,\lambda}^0(\mathbf{R}^n)}. \quad (3)$$

Recently, this result was improved by Le in [18]. Precisely, he satisfied (3) for all $p \in (1, \infty)$ provided that Ω is a function in $L(\log L)(\mathbf{S}^{n-1})$, $1 < \lambda < \infty$ and $g \in \Gamma_{\max\{\lambda', 2\}}(\mathbf{R}^+)$, where $\Gamma_r(\mathbf{R}^+)$ refers to the class of all measurable functions $g : [0, \infty) \rightarrow \mathbf{C}$, which satisfy

$$\|g\|_{\Gamma_r(\mathbf{R}^+)} = \sup_{j \in \mathbf{Z}} \left(\int_{2^j}^{2^{j+1}} |g(\rho)|^r \frac{d\rho}{\rho} \right)^{1/r} < \infty.$$

For background information and recent advances on the study of the operator $\mathcal{M}_{\Omega, \psi, g'}^{(\lambda), c}$ readers may refer to [16,19–22], and the references therein.

It is worth mentioning that the parabolic Marcinkiewicz operator $\mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)}$ was recently introduced by Ali in [23].

For $s > 0$ and a measurable functions g on $\mathbf{R}^+$, define

$$L_s(g) = \sup_{j \in \mathbf{Z}} \left(\int_{2^j}^{2^{j+1}} |g(\rho)| (\log(2 + |g(\rho)|))^s \frac{d\rho}{\rho} \right),$$

$$E_s(g) = \sum_{j=1}^{\infty} 2^j j^s d_j(g),$$

where $d_j(g) = \sup_{m \in \mathbf{Z}} 2^{-m} |V(m, j)|$ with $V(m, j) = \{\rho \in (2^m, 2^{m+1}] : 2^{j-1} < |g(\rho)| \leq 2^j\}$ for $j \geq 2$ and $V(m, 1) = \{\rho \in (2^m, 2^{m+1}] : |g(\rho)| \leq 2\}$.

Now, the class of all such functions with $L_s(g) < \infty$ is denoted by $\mathfrak{L}^s(\mathbf{R}^+)$. However, the class of all such functions with $\mathcal{E}_s(\mathbf{R}^+) < \infty$ is denoted by $\mathcal{E}_s(\mathbf{R}^+)$.

It is easy to see that $\Gamma_r(\mathbf{R}^+) \subset \mathcal{E}_s(\mathbf{R}^+) \subset \mathfrak{L}^s(\mathbf{R}^+)$ for any $r \geq 1$ and $s > 0$; and also for a given $r > 1$, $\mathfrak{L}^{r+s}(\mathbf{R}^+) \subset \mathcal{E}_s(\mathbf{R}^+)$ for any $s > 0$.

Here, we recall some useful spaces defined on $\mathbf{S}^{n-1}$ and related to our work. In these spaces, we only deal with measurable functions. The space $L(\log L)^s(\mathbf{S}^{n-1})$ (for $s > 0$) is the set of all functions Ω on $\mathbf{S}^{n-1}$ such that

$$\|\Omega\|_{L(\log L)^s(\mathbf{S}^{n-1})} = \int_{\mathbf{S}^{n-1}} |\Omega(v)| \log^s(2 + |\Omega(v)|) d\mu(v) < \infty.$$

Moreover, the block space which was introduced in [24] is denoted by $B_q^{(0, \eta)}(\mathbf{S}^{n-1})$ (for $\eta > -1$ and $q > 1$).

The homogeneous Triebel–Lizorkin space $\dot{F}_{p, \lambda}^\beta$ is defined on $\mathbf{R}^n$ as follows: Let $\zeta \in \mathbf{R}^n$ and $0 \leq Y \leq 1$ with $\text{supp } Y \subset \{\zeta : \frac{1}{2} \leq |\zeta| \leq 2\}$ be a radial function satisfying:

- (i) $Y(\zeta) \geq c > 0$ if $\frac{3}{5} \leq |\zeta| \leq \frac{5}{3}$;
- (ii) $\sum_{j \in \mathbf{Z}} Y(2^{-j}\zeta) = 1$ ($\zeta \neq 0$). For $\beta \in \mathbf{R}$ and $1 < p, \lambda \leq \infty$ with $(p \neq \infty)$,

$$\dot{F}_{p, \lambda}^\beta(\mathbf{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{\dot{F}_{p, \lambda}^\beta(\mathbf{R}^n)} = \left\| \left(\sum_{j \in \mathbf{Z}} 2^{j\beta\lambda} |\Psi_j * f|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)} < \infty \right\},$$

where $\mathcal{S}'$ denotes the tempered distribution class on $\mathbf{R}^n$ and $\widehat{\Psi_j}(\zeta) = Y(2^{-j}\zeta)$ for $j \in \mathbf{Z}$.

The following properties of the Triebel–Lizorkin space are well known (for more details see [25]).

- (a) $\mathcal{S}(\mathbf{R}^n)$ is dense in $\dot{F}_{p, \lambda}^\beta(\mathbf{R}^n)$;
- (b) $\dot{F}_{p, 2}^0(\mathbf{R}^n) = L^p(\mathbf{R}^n)$ for $1 < p < \infty$;
- (c) $\dot{F}_{p, \lambda_1}^\beta(\mathbf{R}^n) \subset \dot{F}_{p, \lambda_2}^\alpha(\mathbf{R}^n)$ if $\lambda_1 < \lambda_2$;
- (d) $\left(\dot{F}_{p, \lambda}^\beta(\mathbf{R}^n) \right)^* = \dot{F}_{p', \lambda'}^{-\beta}(\mathbf{R}^n)$.

We formulate our main results as follows:

Theorem 1. Suppose that $g \in \Gamma_r(\mathbf{R}^+)$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ satisfies the conditions of Equations (1) and (2) for some $1 < q, r \leq 2$. Let ψ be in $C^2([0, \infty))$, and is an increasing and convex function with $\psi(0) = 0$. Then there is $C_p > 0$ such that

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq \frac{C_p}{(r-1)(q-1)} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})} \quad (4)$$

for $1 < p < \lambda$ and

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq \frac{C_p}{(q-1)^{1/\lambda}(r-1)^{1/\lambda}} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})} \quad (5)$$

for $\lambda \leq p < \infty$. The constant C_p is independent of Ω, g, r, q and ψ .

Theorem 2. Assume that Ω and ψ are given as in Theorem 1. Let $g \in \Gamma_r(\mathbf{R}^+)$ for some $r > 2$. Then

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p (q-1)^{-1/\lambda} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})}$$

for $r' < p < \infty$ if $2 < r \leq \infty$ and $\lambda > r'$; and for $1 < p < \lambda$ if $2 < r < \infty$ and $\lambda \leq r'$. Where C_p is independent of g, Ω, r, q and ψ .

Applying the extrapolation arguments as in [20] to the conclusions of Theorems 1 and 2, one can find the following:

Theorem 3. Let Ω satisfy the conditions of Equations (1) and (2) and ψ be given as in Theorem 1.

(i) If $\Omega \in B_q^{(0, \frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $g \in \mathcal{E}_{1/\lambda}(\mathbf{R}^+)$, then for $\lambda \leq p < \infty$,

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \left(1 + \|\Omega\|_{B_q^{(0, \frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})} \right) (1 + E_{1/\lambda}(g)) \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})};$$

(ii) If $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $g \in \mathcal{E}_1(\mathbf{R}^+)$, then for $1 < p < \lambda$,

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \left(1 + \|\Omega\|_{B_q^{(0,0)}(\mathbf{S}^{n-1})} \right) (1 + E_1(g)) \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})};$$

(iii) If $\Omega \in L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})$ and $g \in \mathcal{E}_{1/\lambda}(\mathbf{R}^+)$, then for $\lambda \leq p < \infty$,

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \left(1 + \|\Omega\|_{L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})} \right) (1 + E_{1/\lambda}(g)) \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})};$$

(iv) If $\Omega \in L(\log L)(\mathbf{S}^{n-1})$ and $g \in \mathcal{E}_1(\mathbf{R}^+)$, then for $1 < p < \lambda$,

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \left(1 + \|\Omega\|_{L(\log L)(\mathbf{S}^{n-1})} \right) (1 + E_1(g)) \|f\|_{\dot{F}_{p, \Gamma}^0(\mathbf{R}^{n+1})}.$$

The constant $C_p > 0$ is independent of g, Ω , and ψ .

Theorem 4. Suppose that Ω satisfies Equations (1) and (2), $g \in \Gamma_r(\mathbf{R}^+)$ for some $r > 2$ and ψ is given as in Theorem 1.

(i) If $\Omega \in B_q^{(0, \frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})$ for some $q > 1$, then

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \left(1 + \|\Omega\|_{B_q^{(0, \frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})} \right) \|g\|_{\Gamma_r(\mathbf{R}^+)} \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})}$$

for $r' < p < \infty$ if $2 < r \leq \infty$ and $\lambda' < r$; and for $1 < p < \lambda$ if $2 < r < \infty$ and $\lambda' \geq r$.

(ii) If $\Omega \in L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})$, then

$$\left\| \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \left(1 + \|\Omega\|_{L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})} \right) \|g\|_{\Gamma_r(\mathbf{R}^+)} \|f\|_{\dot{F}_{p, \lambda}^0(\mathbf{R}^{n+1})}$$

for $r' < p < \infty$ if $2 < r \leq \infty$ and $\lambda' < r$; and for $1 < p < \lambda$ if $2 < r < \infty$ and $\lambda' \geq r$.

Let us present some results related to the optimality of our main results.

Remark 1. (a) The authors of [3] established that $\mathcal{M}_{\Omega, 1}$ is bounded on $L^p(\mathbf{R}^n)$ for $p \in (1, \infty)$ provided that $\Omega \in B_q^{(0, -1/2)}(\mathbf{S}^{n-1})$ with $q > 1$. Furthermore, they proved that the exponent $-1/2$ in $B_q^{(0, -1/2)}(\mathbf{S}^{n-1})$ cannot be replaced by any number $\nu \in (-1, -1/2)$ so that the operator $\mathcal{M}_{\Omega, 1}$ is bounded on $L^2(\mathbf{R}^n)$.

(b) The author of [6] showed that $\mathcal{M}_{\Omega, 1}$ is of type $(2, 2)$ if $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$. Moreover, he proved that the operator $\mathcal{M}_{\Omega, 1}$ is not bounded on $L^2(\mathbf{R}^n)$ whenever $\Omega \in L(\log L)^\nu(\mathbf{S}^{n-1})$ for some $0 < \nu < 1/2$.

(c) The authors of [20] established the boundedness of the parametric Marcinkiewicz operator $\mathcal{M}_{\Omega, \psi, g}^{(\lambda), c}$ under the same our conditions on Ω, g and λ only when $\psi(s) = s$ and $\alpha_1 = \dots = \alpha_n = 1$.

Here and henceforth, whenever the letter C appears, it refers to a positive constant whose value may be different at each appearance but independent of the basic variables.

2. Some Preliminary Lemmas

This section is devoted to establish some preliminary results and to introduce some notations. For $\vartheta \geq 2$, a suitable function ψ on $\mathbf{R}^+$, and real valued measurable functions Ω on $\mathbf{S}^{n-1}$ and g on $\mathbf{R}^+$, the family of measures $\{\mu_{\Omega, \psi, g, t} := \mu_{g, t} : t \in \mathbf{R}^+\}$ and the corresponding maximal operators μ_g^* and $M_{g, \vartheta}$ on $\mathbf{R}^{n+1}$ are defined by

$$\int_{\mathbf{R}^{n+1}} f d\mu_{g, t} = t^{-\tau} \int_{t/2 \leq \rho(v) \leq t} f(v, \psi(\rho(v))) K_{\Omega, g}(v) dv,$$

$$\mu_g^*(f) = \sup_{t \in \mathbf{R}^+} |\mu_{g, t} * f|,$$

and

$$M_{g, \vartheta}(f) = \sup_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g, t} * f| \frac{dt}{t},$$

where $|\mu_{g, t}|$ is defined in the same way as $\mu_{g, t}$, but with replacing Ω by $|\Omega|$ and g by $|g|$. We write $A^{\pm\kappa} = \min\{A^\kappa, A^{-\kappa}\}$ and $\|\mu_{g, t}\|$ for the total variation of $\mu_{g, t}$.

Let us recall the following lemma due to Cheng and Ding.

Lemma 1. [26] Suppose that $0 \leq \gamma \leq 1$. Let m denote the distinct numbers of $\{\alpha_j\}_{j=1}^n$. Then for $v, \xi \in \mathbf{R}^n$, we have

$$\left| \int_1^2 e^{-iA_{\rho}v \cdot \xi} \frac{d\rho}{\rho} \right| \leq C |v \cdot \xi|^{-\frac{\gamma}{m}},$$

where C is independent of v and ξ .

Lemma 2. Let $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q \in (1, 2]$ satisfy the conditions of Equations (1) and (2), and let $g \in \Gamma_r(\mathbf{R}^+)$ for some $r > 1$. Suppose that ψ is in $C^2([0, \infty))$, increasing and convex function with $\psi(0) = 0$. Then there exist constants C and γ with $0 < 2\gamma \leq \min\{1, \frac{m}{q'}, \frac{m}{\alpha}\}$ such that

$$(i) \quad \|\mu_{g,t}\| \leq C \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})};$$

(ii) If $1 < r \leq 2$, then

$$\int_{2^{q'r'j}}^{2^{q'r'(j+1)}} |\hat{\mu}_{g,t}(\bar{\xi})|^2 \frac{dt}{t} \leq \frac{C}{(q-1)(r-1)} \left| A_{2^{q'r'j}\xi} \right|^{\pm \frac{2\gamma}{mq'r'}} \|g\|_{\Gamma_r(\mathbf{R}^+)}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2;$$

(iii) If $r > 2$, then

$$\int_{2^{q'r'j}}^{2^{q'r'(j+1)}} |\hat{\mu}_{g,t}(\bar{\xi})|^2 \frac{dt}{t} \leq \frac{C}{(q-1)(r-1)} \left| A_{2^{q'r'j}\xi} \right|^{\pm \frac{\gamma}{mq'}} \|g\|_{\Gamma_r(\mathbf{R}^+)}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2;$$

for all $j \in \mathbf{Z}$. The constant C is independent of $j, \bar{\xi}, q$, and r .

Proof. By the definition of $\mu_{g,t}$, it is clear that (i) is true. Consider the case $g \in \Gamma_r(\mathbf{R}^+)$ for some $1 < r \leq 2$. A simple change of variables and Hölder's inequality implies that

$$|\hat{\mu}_{g,t}(\bar{\xi})| \leq C \|g\|_{\Gamma_r(\mathbf{R}^+)} \left(\int_{1/2}^1 |G_t(\rho)|^{r'} \frac{d\rho}{\rho} \right)^{1/r'},$$

where

$$G_t(\rho) = \int_{\mathbf{S}^{n-1}} e^{-i\{A_{t\rho}x \cdot \bar{\xi} + \xi_{n+1}\psi(t\rho)\}} \Omega(x) J(x) d\mu(x).$$

Since $2 \leq r' < \infty$ and $|G_t(\rho)| \leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})}$, then we directly get

$$|\hat{\mu}_{g,t}(\bar{\xi})| \leq C \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/r')} \left(\int_{1/2}^1 |G_t(\rho)|^2 \frac{d\rho}{\rho} \right)^{1/r'}. \quad (6)$$

Thanks to the Schwartz inequality, we have

$$\int_{1/2}^1 |G_t(\rho)|^2 \frac{d\rho}{\rho} \leq C \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}} I(\xi, x, u) \Omega(x) \overline{\Omega(u)} J(x) \overline{J(u)} d\mu(x) d\mu(u),$$

where $I(\xi, x, u) = \int_{1/2}^1 e^{-iA_{t\rho}\xi \cdot (x-u)} \frac{d\rho}{\rho}$. Let $\zeta = \frac{A_{t\rho}\xi}{|A_{t\rho}\xi|}$. So, by Lemma 1, we get

$$|I(\xi, x, u)| \leq C |A_t \xi|^{-\gamma/m} |\zeta \cdot (x - u)|^{-\gamma/m}$$

for any γ with $0 < \gamma < \min\{\frac{1}{2}, \frac{m}{\alpha}\}$. Thus, by Hölder's inequality we reach

$$\int_{1/2}^1 |G_t(\rho)|^2 \frac{d\rho}{\rho} \leq C |A_t \xi|^{-\frac{\gamma}{mq'}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2 \left(\int_{\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}} |\zeta \cdot (x - u)|^{-\gamma q'/m} d\mu(x) d\mu(u) \right)^{1/q'}.$$

Choose $0 < \gamma < m/(2q')$, so we get that the last integral is finite, which leads to

$$\int_{1/2}^1 |G_t(\rho)|^2 \frac{d\rho}{\rho} \leq C |A_t \xi|^{-\frac{\gamma}{mq'}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2 \quad (7)$$

and hence, by using Equation (6), it follows

$$\int_{2^{q'r'j}}^{2^{q'r'(j+1)}} |\hat{\mu}_{g,t}(\bar{\xi})|^2 \frac{dt}{t} \leq C(q-1)^{-1}(r-1)^{-1} \left| A_{2^{q'r'j}} \bar{\xi} \right|^{\frac{2\gamma}{mq'r'}} \|g\|_{\Gamma_r(\mathbf{R}^+)}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2.$$

Using the cancelation property of Equation (2) and a simple change of variables, we deduce that

$$\begin{aligned} \int_{1/2}^1 |G_t(\rho)|^2 \frac{d\rho}{\rho} &\leq C \int_{1/2}^1 \left(\int_{\mathbf{S}^{n-1}} |e^{-iA_t \rho \xi \cdot x} - 1| |\Omega(x)| |J(x)| d\mu(x) \right)^2 \frac{d\rho}{\rho} \\ &\leq C |A_t \xi|^2 \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^2, \end{aligned}$$

which, when combined with the trivial estimate $\int_{1/2}^1 |G_t(\rho)|^2 \frac{d\rho}{\rho} \leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^2$, gives

$$\int_{1/2}^1 |G_t(\rho)|^2 \frac{d\rho}{\rho} \leq C |A_t \xi|^{\frac{\gamma}{mq'}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2.$$

Thus, for $2^{q'r'j} \leq t \leq 2^{q'r'(j+1)}$, we have

$$\begin{aligned} |\hat{\mu}_{g,t}(\bar{\xi})| &\leq C |A_t \bar{\xi}|^{\frac{\gamma}{mq'r'}} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \\ &\leq C |A_{2^{q'r'j}} \bar{\xi}|^{\frac{\gamma}{mq'r'}} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \\ &\leq C 2^{\max\{\alpha_1, \dots, \alpha_n\}} |A_{2^{q'r'j}} \bar{\xi}|^{\frac{\gamma}{mq'r'}} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}. \end{aligned}$$

Therefore,

$$\int_{2^{q'r'j}}^{2^{q'r'(j+1)}} |\hat{\mu}_{g,t}(\bar{\xi})|^2 \frac{dt}{t} \leq C(q-1)^{-1}(r-1)^{-1} \left| A_{2^{q'r'j}} \bar{\xi} \right|^{\pm \frac{2\gamma}{mq'r'}} \|g\|_{\Gamma_r(\mathbf{R}^+)}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2$$

for $1 < r \leq 2$. To prove (iii), we follow the same above procedure and the fact

$$\left(\int_{1/2}^1 |G_t(l)|^{r'} \frac{dl}{l} \right)^{1/r'} \leq C \left(\int_{1/2}^1 |G_t(l)|^2 \frac{dl}{l} \right)^{1/2}$$

for $r > 2$. This completes the proof of Lemma 2. $\square$

We shall need the following lemma from [27].

Lemma 3. Let α'_i s and v'_i s be fixed real numbers, and $P_v(\rho) = (v_1 \rho^{\alpha'_1}, \dots, v_n \rho^{\alpha'_n})$ be a function from $\mathbf{R}^+$ to $\mathbf{R}^n$. Let $\mathcal{M}_{P_v}$ be the maximal function related to the curve P_v given by

$$\mathcal{M}_{P_v} f(x) = \sup_{R>0} \frac{1}{R} \int_0^R |f(x - P_v(\rho))| d\rho.$$

Then for $1 < p \leq \infty$, a constant $C_p > 0$ exists so that

$$\|\mathcal{M}_{P_v}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)}$$

for all $f \in L^p(\mathbf{R}^n)$. The constant C_p is independent of v_i 's and f .

By following a similar argument found in [28], which has its roots in [29], we establish the following lemma.

Lemma 4. Let $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq 2$ satisfying Equations (1) and (2), $g \in \Gamma_r(\mathbf{R}^+)$ for some $1 < r \leq 2$ and $\vartheta = 2^{q'r'}$. Suppose that ψ is given as in Theorem 1. Then for any $1 < p < \infty$, there exists a positive constant C_p such that

$$\|\mu_g^*(f)\|_{L^p(\mathbf{R}^{n+1})} \leq C_p(q-1)^{-1}(r-1)^{-1} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^{n+1})}, \quad (8)$$

$$\|M_{g,\vartheta}(f)\|_{L^p(\mathbf{R}^{n+1})} \leq C_p(q-1)^{-1}(r-1)^{-1} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^{n+1})}. \quad (9)$$

Proof. Without loss of generality we may assume that $g, \Omega \geq 0$. Let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ be a Schwartz function such that $\widehat{\varphi}(\xi) = 1$ for $\rho(\xi) \leq \frac{1}{2}$, and $\widehat{\varphi}(\xi) = 0$ for $\rho(\xi) \geq 1$. Let $\varphi_t(x) = t^{-n}\varphi(A_{t^{-1}}x)$. Define the family of measure τ_t on $\mathbf{R}^{n+1}$ by

$$\widehat{\tau}_t(\xi, \xi_{n+1}) = \widehat{\mu_{g,t}}(\xi, \xi_{n+1}) - \widehat{\varphi}_t(\xi) \widehat{\mu_{g,t}}(0, \xi_{n+1}).$$

Then by the proof of Lemma 2 and a simple calculation, we obtain

$$|\widehat{\tau}_t(\xi, \xi_{n+1})| \leq |A_t \xi|^{\pm \frac{2\gamma}{mq'r'}} \|g\|_{\Gamma_r(\mathbf{R}^+)}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2. \quad (10)$$

Let

$$N_\tau(f)(\bar{x}) = \left(\int_0^\infty |\tau_t * f(\bar{x})|^2 dt \right)^{1/2} \quad \text{and} \quad \tau^*(f) = \sup_{t \in \mathbf{R}^+} |\tau_t * f|. \quad (11)$$

Moreover let $H_\psi(f)$ be the maximal function defined by

$$H_\psi(f)(\rho) = \sup_{t \in \mathbf{R}^+} \left| \frac{1}{t} \int_0^t f(x, x_{n+1} - \psi(t)) dt \right|.$$

Notice that

$$\mu_g^*(f)(\bar{x}) \leq N_\tau(f)(\bar{x}) + C((\mathcal{M}_{P_v} \otimes id_{\mathbf{R}}) \circ H_\psi(f)(\bar{x})) \quad (12)$$

and

$$\tau^*(f)(\bar{x}) \leq N_\tau(f)(\bar{x}) + 2C((\mathcal{M}_{P_v} \otimes id_{\mathbf{R}}) \circ H_\psi(f)(\bar{x})). \quad (13)$$

Therefore, by using a similar argument as in the proof of Lemma in ([29], p. 544) together with Equations (10)–(13), Lemma 3, ([28], Inequality (2.13)), and ([30], Lemma 2.3) we directly establish Equations (8) and (9). $\square$

In order to handle our main results, we shall establish the following lemmas which can be derived by following the same procedure used in [20].

Lemma 5. Let $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq 2$, $g \in \Gamma_r(\mathbf{R}^+)$ for some $1 < r \leq 2$ and $\vartheta = 2^{q' r'}$. Assume that λ is a real number with $\lambda > 1$ and ψ is given as in Theorem 1. Then, there exists a constant $C > 0$ such that

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})} \leq C(q-1)^{-\frac{1}{\lambda}}(r-1)^{-\frac{1}{\lambda}} \\ & \times \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{j \in \mathbf{Z}} |h_j|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})}, \quad \lambda \leq p < \infty; \end{aligned} \quad (14)$$

and

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})} \leq C(q-1)^{-1}(r-1)^{-1} \\ & \times \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{j \in \mathbf{Z}} |h_j|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})}, \quad 1 < p < \lambda \end{aligned} \quad (15)$$

hold for arbitrary functions $\{h_j(\cdot), j \in \mathbf{Z}\}$ on $\mathbf{R}^{n+1}$. The constant $C = C_{n,p,\psi}$ is independent of Ω , g , r , and q .

Proof. Let us first prove Equation (14). For a fixed p with $\lambda \leq p < \infty$, by duality, there is a non-negative function $\mathcal{B} \in L^{(p/\lambda)'}(\mathbf{R}^{n+1})$ with $\|\mathcal{B}\|_{L^{(p/\lambda)'}(\mathbf{R}^{n+1})} \leq 1$ such that

$$\left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})}^\lambda = \int_{\mathbf{R}^{n+1}} \sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j(\bar{x})|^\lambda \frac{dt}{t} \mathcal{B}(\bar{x}) d\bar{x}. \quad (16)$$

Hölder's inequality and a simple change of variable lead to

$$\begin{aligned} |\mu_{g,t} * h_j(\bar{x})|^\lambda & \leq C \|g\|_{\Gamma_1(\mathbf{R}^+)}^{(\lambda/\lambda')} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\lambda/\lambda')} \\ & \times \int_{t/2}^t \int_{\mathbf{S}^{n-1}} |h_j(x - A_\rho v, x_{n+1} - \psi(\rho))|^\lambda |\Omega(v)| J(v) d\mu(v) |h(\rho)| \frac{d\rho}{\rho}. \end{aligned} \quad (17)$$

Thus, by Hölder's inequality and Equations (16) and (17), we get that

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})}^\lambda \\ & \leq C \|g\|_{\Gamma_1(\mathbf{R}^+)}^{(\lambda/\lambda')} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\lambda/\lambda')} \int_{\mathbf{R}^{n+1}} \left(\sum_{j \in \mathbf{Z}} |h_j(\bar{x})|^\lambda \right) M_{|g|,\vartheta} \tilde{\mathcal{B}}(-\bar{x}) d\bar{x} \\ & \leq C \|g\|_{\Gamma_1(\mathbf{R}^+)}^{(\lambda/\lambda')} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\lambda/\lambda')} \left\| \sum_{j \in \mathbf{Z}} |h_j|^\lambda \right\|_{L^{(p/\lambda)}(\mathbf{R}^{n+1})} \left\| M_{|g|,\vartheta}(\tilde{\mathcal{B}}) \right\|_{L^{(p/\lambda)'}(\mathbf{R}^{n+1})}, \end{aligned}$$

where $\tilde{\mathcal{B}}(-\bar{x}) = \mathcal{B}(\bar{x})$. Hence, by the assumption on $\mathcal{B}$ and Lemma 4, we obtain

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})} \leq C(q-1)^{-\frac{1}{\lambda}}(r-1)^{-\frac{1}{\lambda}} \\ & \times \|h\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{j \in \mathbf{Z}} |h_j|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})} \end{aligned} \quad (18)$$

for $\lambda < p < \infty$. Now for the case $p = \lambda$; by Hölder's inequality and Equation (17), we have that

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})}^\lambda \leq C \|g\|_{\Gamma_1(\mathbf{R}^+)}^{(\lambda/\lambda')} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\lambda/\lambda')} \\ & \times \sum_{j \in \mathbf{Z}} \int_{\mathbf{R}^{n+1}} \int_{\vartheta^j}^{\vartheta^{j+1}} \int_{t/2}^t \int_{\mathbf{S}^{n-1}} |h_j(x - A_\rho v, x_{n+1} - \psi(\rho))|^\lambda |\Omega(v)| J(v) |g(\rho)| d\mu(v) \frac{d\rho}{\rho} \frac{dt}{t} d\bar{x} \\ & \leq C(q-1)^{-1}(r-1)^{-1} \|g\|_{\Gamma_1(\mathbf{R}^+)}^{(\lambda/\lambda') + 1} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\lambda/\lambda') + 1} \int_{\mathbf{R}^{n+1}} \left(\sum_{j \in \mathbf{Z}} |h_j(\bar{x})|^\lambda \right)^{p/\lambda} d\bar{x}. \end{aligned}$$

Therefore, Equation (14) is satisfied for the case $p = \lambda$.

Now consider the case $1 < p < \lambda$. By the duality, there exist functions $\{\mathcal{A}_j(\bar{x}, t)\}$ defined on $\mathbf{R}^{n+1} \times \mathbf{R}^+$ with $\left\| \left\| \mathcal{A}_j \right\|_{L^{\lambda'}([\vartheta^j, \vartheta^{j+1}], \frac{dt}{t})} \right\|_{L^{p'}(\mathbf{R}^{n+1})} \leq 1$ such that

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})} = \int_{\mathbf{R}^{n+1}} \sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} (\mu_{g,t} * h_j(\bar{x})) \mathcal{A}_j(\bar{x}, t) \frac{dt}{t} d\bar{x} \\ & \leq C(q-1)^{-1/\lambda}(r-1)^{-1/\lambda} \left\| (H(\mathcal{A}))^{1/\lambda'} \right\|_{L^{p'}(\mathbf{R}^{n+1})} \left\| \left(\sum_{j \in \mathbf{Z}} |h_j|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})}, \end{aligned} \quad (19)$$

where

$$H(\mathcal{A})(x) = \sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * \tilde{\mathcal{A}}_j(\bar{x}, t)|^{\lambda'} \frac{dt}{t} \quad \text{and} \quad \tilde{\mathcal{A}}_j(\bar{x}, t) = \mathcal{A}_j(-\bar{x}, t).$$

Since $p' > \lambda'$, there is a nonnegative function $\phi \in L^{(p'/\lambda')'}(\mathbf{R}^{n+1})$ such that

$$\|H(\mathcal{A})\|_{L^{(p'/\lambda')}(\mathbf{R}^{n+1})} = \sum_{j \in \mathbf{Z}} \int_{\mathbf{R}^{n+1}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * \tilde{\mathcal{A}}_j(\bar{x}, t)|^{\lambda'} \frac{dt}{t} \phi(\bar{x}) d\bar{x}. \quad (20)$$

So, by following the same above argument, we get

$$\begin{aligned} \|H(\mathcal{A})\|_{L^{(p'/\lambda')}(\mathbf{R}^{n+1})} & \leq C \|g\|_{\Gamma_r(\mathbf{R}^+)}^{(\lambda'/\lambda)} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\lambda'/\lambda)} \\ & \times \left\| \mu^*_{|g|}(\tilde{\phi}) \right\|_{L^{(p'/\lambda')'}(\mathbf{R}^{n+1})} \left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mathcal{A}_j(\cdot, t)|^{\lambda'} \frac{dt}{t} \right) \right\|_{L^{(p'/\lambda')}(\mathbf{R}^{n+1})} \\ & \leq C(q-1)^{-1}(r-1)^{-1} \|g\|_{\Gamma_r(\mathbf{R}^+)}^{(\lambda'/\lambda) + 1} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\lambda'/\lambda) + 1} \|\tilde{\phi}\|_{L^{(p'/\lambda')'}(\mathbf{R}^{n+1})}, \end{aligned} \quad (21)$$

where $\tilde{\phi}(\bar{x}) = \phi(-\bar{x})$. Therefore, Equation (15) is satisfied by using Equations (19) and (21). This completes the proof of Lemma 5. $\square$

By combining the proofs of ([20], Lemmas 2.4–2.5) and Lemma 5, we get the following:

Lemma 6. Let $g \in \Gamma_r(\mathbf{R}^+)$ for some $2 \leq r < \infty$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq 2$ and $\vartheta = 2^{q'}$. Assume that ψ is given as in Theorem 1, and λ is a real number. Then for arbitrary functions $\{h_j(\cdot), j \in \mathbf{Z}\}$ on $\mathbf{R}^{n+1}$, a positive constant C exists such that

$$\left\| \left(\sum_{j \in \mathbf{Z}} \int_{\vartheta^j}^{\vartheta^{j+1}} |\mu_{g,t} * h_j|^\lambda \frac{dt}{t} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})} \leq C(q-1)^{-1/\lambda} \|g\|_{\Gamma_r(\mathbf{R}^+)} \times \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{j \in \mathbf{Z}} |h_j|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^{n+1})}$$

for $1 < p < r$ if $r \leq \lambda'$; and also, for $r' < p < \infty$ if $r > \lambda'$.

3. Proof of Theorem 1

The idea of the proof of this theorem is taken from [21], which has its roots in [20]. Thanks to Minkowski's inequality, we have that

$$\begin{aligned} \mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f(\bar{x}) &\leq \sum_{j=0}^{\infty} \left(\int_0^\infty \left| t^{-\tau} \int_{2^{-j-1}t < \rho(v) \leq 2^{-j}t} f(x-v, x_{n+1} - \psi(\rho(v))) K_{\Omega, g}(v) dv \right|^\lambda \frac{dt}{t} \right)^{1/\lambda} \\ &= \frac{2^{\tau_1}}{2^{\tau_1} - 1} \left(\int_0^\infty |\mu_{g,t} * f(\bar{x})|^\lambda \frac{dt}{t} \right)^{1/\lambda}. \end{aligned} \quad (22)$$

Let $\vartheta = 2^{q'}$. Choose a collection of smooth functions $\{\Phi_j\}_{j \in \mathbf{Z}}$ on $(0, \infty)$ satisfying the following:

$$\begin{aligned} \text{supp } \Phi_j &\subseteq \mathcal{I}_{j, \vartheta} = [\vartheta^{-j-1}, \vartheta^{-j+1}]; \quad 0 \leq \Phi_j \leq 1; \\ \sum_{j \in \mathbf{Z}} \Phi_j(t) &= 1; \quad \text{and} \quad \left| \frac{d^m \Phi_j(t)}{dt^m} \right| \leq \frac{C_m}{t^m}. \end{aligned}$$

Define the multiplier operators $\widehat{\Psi}_j$ in $\mathbf{R}^{n+1}$ by

$$(\widehat{\Psi}_j f)(\xi, \xi_{n+1}) = \Phi_j(\rho(\xi)) \widehat{f}(\xi, \xi_{n+1}) \quad \text{for } (\xi, \xi_{n+1}) \in \mathbf{R}^n \times \mathbf{R}.$$

Hence, we deduce that for $f \in \mathcal{S}(\mathbf{R}^{n+1})$,

$$\mathcal{M}_{\Omega, \psi, g, \rho}^{(\lambda)} f(\bar{x}) \leq \frac{2^{\tau_1}}{2^{\tau_1} - 1} \sum_{k \in \mathbf{Z}} \mathcal{J}_{\Omega, \psi, g, k}^{(\lambda)}(f), \quad (23)$$

where

$$\begin{aligned} \mathcal{J}_{\Omega, \psi, g, k}^{(\lambda)} f(\bar{x}) &= \left(\int_0^\infty |\mathcal{U}_{\Omega, \psi, g, k, \vartheta}(\bar{x}, t)|^\lambda \frac{dt}{t} \right)^{1/\lambda}, \\ \mathcal{U}_{\Omega, \psi, g, k, \vartheta}(\bar{x}, t) &= \sum_{j \in \mathbf{Z}} (\Psi_{j+k} * \mu_{g,t} * f)(\bar{x}) \chi_{[\vartheta^j, \vartheta^{j+1})}(t). \end{aligned}$$

Let us now estimate the L^p -norm of $\mathcal{J}_{\Omega,\psi,g,k}^{(\lambda)}(f)$. First, we consider the case $p = \lambda = 2$. Notice that $\|f\|_{F_{2,2}^0(\mathbf{R}^{n+1})} = \|f\|_{L^2(\mathbf{R}^{n+1})}$. Thus, by Plancherel's theorem and Lemma 2, we have that

$$\begin{aligned} \left\| \mathcal{J}_{\Omega,\psi,g,k}^{(2)}(f) \right\|_{L^2(\mathbf{R}^{n+1})}^2 &\leq \sum_{j \in \mathbf{Z}} \int_{\mathcal{L}_{j+k,\emptyset}} \left(\int_{\emptyset^j}^{\emptyset^{j+1}} |\hat{\mu}_{g,t}(\bar{\xi})|^2 \frac{dt}{t} \right) |\hat{f}(\bar{\xi})|^2 d\bar{\xi} \\ &\leq C_{r,q,g,\Omega} \sum_{j \in \mathbf{Z}} \int_{\mathcal{L}_{j+k,\emptyset}} \left(|A_{2^{q'j}} \bar{\xi}|^{\pm \frac{2\gamma}{mq'r'}} \right) |\hat{f}(\bar{\xi})|^2 d\bar{\xi} \\ &\leq C_{r,q,g,\Omega} 2^{-\varepsilon|k|} \sum_{j \in \mathbf{Z}} \int_{\mathcal{L}_{j+k,\emptyset}} |\hat{f}(\bar{\xi})|^2 d\bar{\xi} \\ &\leq C_{r,q,g,\Omega} 2^{-\varepsilon|k|} \|f\|_{L^2(\mathbf{R}^{n+1})}^2, \end{aligned}$$

where $C_{r,q,g,\Omega} = C(r-1)^{-1}(q-1)^{-1} \|g\|_{\Gamma_r(\mathbf{R}^+)}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2$, $\mathcal{L}_{j,\emptyset} = \{\bar{\xi} \in \mathbf{R}^{n+1} : \rho(\bar{\xi}) \in \mathcal{I}_{j,\emptyset}\}$ and $0 < \varepsilon < 1$. Therefore,

$$\begin{aligned} &\left\| \mathcal{J}_{\Omega,\psi,g,k}^{(2)}(f) \right\|_{L^2(\mathbf{R}^{n+1})} \\ &\leq C 2^{-\frac{\varepsilon}{2}|k|} (r-1)^{-1/2} (q-1)^{-1/2} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{F_{2,2}^0(\mathbf{R}^{n+1})}. \end{aligned} \quad (24)$$

Now consider the case $\lambda \leq p < \infty$. By Lemma 5, we obtain that

$$\begin{aligned} &\left\| \mathcal{J}_{\Omega,\psi,g,k}^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^{n+1})} \\ &\leq C (q-1)^{-1/\lambda} (r-1)^{-1/\lambda} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{F_{p,\lambda}^0(\mathbf{R}^{n+1})}. \end{aligned} \quad (25)$$

However, for the case for $1 < p < \lambda$, we have

$$\begin{aligned} &\left\| \mathcal{J}_{\Omega,\psi,g,k}^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^{n+1})} \\ &\leq C (q-1)^{-1} (r-1)^{-1} \|g\|_{\Gamma_r(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{F_{p,\lambda}^0(\mathbf{R}^n)}. \end{aligned} \quad (26)$$

Therefore, by interpolating Equation (24) with Equations (25) and (26) and using Equation (23), we finish the proof of Theorem 1.

It should be noticed that the proof of Theorem 2 can be done in the same manner as in the above argument, whereas Lemma 6 with $\vartheta = 2^{q'}$ is needed instead of Lemma 5.

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