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Some Applications of a New Integral Operator in q-Analog for Multivalent Functions

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Abstract: This paper introduces a new integral operator in q-analog for multivalent functions. Using as an application of this operator, we study a novel class of multivalent functions and define them. Furthermore, we present many new properties of these functions. These include distortion bounds, sufficiency criteria, extreme points, radius of both starlikness and convexity, weighted mean and partial sum for this newly defined subclass of multivalent functions are discussed. Various integral operators are obtained by putting particular values to the parameters used in the newly defined operator.

Keywords: p-valent analytic function; Hadamard product; q-integral operator

MSC: Primary 30C45; Secondary 30C50

1. Introduction

The study of q-extension of calculus or q-analysis motivated the researchers due to its recent use in different applications. In [1,2], Jackson introduced the theory of q-calculus. We have seen applications of q-analysis in Geometric Function Theory (GFT). They were introduced and applied systematically to the generalized q-hypergeometric functions in [3]. Later, Ismail et al. [4] used the q-differential operator to examine the geometry of starlike function in q-analog. This theory was later extended to the family of q-starlike function with some order by Agrawal and Sahoo [5]. Due to this development in function theory, many researchers were motivated, as we have seen by Srivastava in [6]. They added significant contributions, which has slowly made this research area more attractive to forthcoming researchers. We direct the attention of our readers to [7–12] for more information. Moreover, Kanas et al. [13] used Hadamad product to define the q-extension of the Ruscheweyh operator. They also discussed in detail some intricate applications of this operator.

Mohammad and Darus [14] conducted an elaborate study of this operator. We have also seen similar work by Mahmood and Sokół [15] and Ahmad et al. [16]. Recently, new thoughts by Maslina in [17] were used to create a novel differential operator called generalized q-differential operator with the help of q-hypergeometric functions where the authors conducted an in-depth study of applications of this operator. For further information on the extensions of different operators in q-analog, we direct the readers to [18–22]. The aim of the present article is to introduce a new integral operator in q-analog for multivalent functions using Hadamard product and then study some of its useful applications.

Let $\mathfrak{A}_p$ $(p \in \mathbb{N} = \{1, 2, \ldots\})$ contain multivalent functions of all forms f that can be defined as holomorphic and/or analytic in any given subset $\mathbb{D} = \{z : |z| < 1\}$ that is part of a complex plane $\mathbb{C}$ which also has the series form shown as:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \ (z \in \mathbb{D}). \tag{1}$$

For any two given functions that are analytic in form f and g in $\mathbb{D}$, then we can clearly state that f is subordinate to g, mostly symbolically if it is presented clearly as $f \prec g$ or $f(z) \prec g(z)$, if and only if there exists an analytic function w with the given properties as $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{D})$. Moreover, if and only if g can be seen as univalent in $\mathbb{D}$, then we can clearly have:

$$f(z) \prec g(z) \quad (z \in \mathbb{D}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

For analytic functions f of the form Equation (1) and g of the form

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}, \ (z \in \mathbb{D}), \tag{2}$$

the convolution or Hadamard product is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p}, \ (z \in \mathbb{D}).$$

For given $q \in (0, 1)$, the derivative in q-analog of f is given by

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \ (z \neq 0, \ q \neq 1). \tag{3}$$

Making use of Equations (1) and (3), we can easily obtain for $n \in \mathbb{N}$ and $z \in \mathbb{D}$

$$\mathcal{D}_q \left\{ \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \right\} = \sum_{n=1}^{\infty} [n+p]_q \, a_{n+p} z^{n+p-1}, \tag{4}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{k=1}^{n-1} q^k, \ [0]_q = 0. \tag{5}$$

For $n \in \mathbb{Z}^+ := \mathbb{Z} \backslash \{-1, -2, \ldots\}$, the q-factorial is given as:

$$[n]_q! = \begin{cases} 1, \ n = 0, \\ [1]_q [2]_q \ldots [n]_q, \ n \in \mathbb{N}. \end{cases}$$

In addition, with $t > 0$, the q-Pochhammer symbol has the form:

$$[t]_{q,n} = \left([t]_q\right)_n = \begin{cases} 1, \ n = 0, \\ [t]_q [t+1]_q \cdots [t+n-1]_q, \ n \in \mathbb{N}, \end{cases}$$

where $[t]_q$ is given by Equation (5).

For $t > 0$, the gamma function in q-analog is presented as

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \text{ and } \Gamma_q(1) = 1.$$

We now consider a function

$$\mathcal{F}^{-1}_{q,\lambda+p}(z) = z^p + \sum_{n=1}^{\infty} \Psi_{n-p} a_{n+p} z^{n+p}, \ (\lambda > -p, \ z \in \mathbb{D}), \tag{6}$$

with

$$\Psi_{n-p} = \frac{[\lambda+p]_{q,n+1-p}}{[n+1-p]_q!}. \tag{7}$$

We can see that the series given in Equation (6) is absolutely convergent in $\mathbb{D}$. Now, we introduce the integral operator $\mathcal{J}_q^{\lambda+p-1} : \mathfrak{A}_p \to \mathfrak{A}_p$ by

$$\mathcal{J}_q^{\lambda+p-1} f(z) = (\mathcal{F}^{-1}_{q,\lambda+p} * f)(z) = z^p + \sum_{n=1}^{\infty} \Psi_{n-p} a_{n+p} z^{n+p} \quad (z \in \mathbb{D}), \tag{8}$$

where $\lambda > -p$. We note that

$$\lim_{q\to 1^-} \mathcal{F}^{-1}_{q,\lambda+p}(z) = \frac{z^p}{(1-z)^{\lambda+1}} \quad \text{and} \quad \lim_{q\to 1^-} \mathcal{J}_q^{\lambda+p-1} f(z) = f(z) * \frac{z^p}{(1-z)^{\lambda+1}}.$$

Various integral operators were obtained by putting particular values to the parameters used in the newly defined operator as given by Equation (7):

(i). Making $p = 1$ in our newly defined operator $\mathcal{J}_q^{\lambda+p-1} f$, we obtain the operator $\mathcal{J}_q^{\lambda} f$ which was introduced by Arif et al. [20] and is given by

$$\mathcal{J}_q^{\lambda} f(z) = (\mathcal{F}^{-1}_{q,\lambda+1} * f)(z) = z + \sum_{n=1}^{\infty} \Psi_{n-1} a_{n+1} z^{n+1} \quad (z \in \mathbb{D}).$$

(ii). When $q \to 1^-$, the operator defined in Equation (7) leads to the following well-known Noor integral operator for multivalent functions introduced in [23].

$$\mathcal{J}^{\lambda+p-1} f(z) = (\mathcal{F}^{-1}_{\lambda+p} * f)(z) = z^p + \sum_{n=1}^{\infty} \Psi_{n-p} a_{n+p} z^{n+p}, \ (z \in \mathbb{D}).$$

(iii). If we set $p = 1$ along with $q \to 1^-$ in Equation (7), then the operator $\mathcal{J}^{\lambda+p-1} f$ reduced to the following familiar Noor integral operator studied in [24,25].

$$\mathcal{I}^{\lambda} f(z) = (\mathcal{F}^{-1}_{\lambda+1} * f)(z) = z + \sum_{n=1}^{\infty} \Psi_{n-1} a_{n+1} z^{n+1}, \ (z \in \mathbb{D}).$$

For more details on the q-analog of differential and integral operators, see [17,26,27].

Motivated from the work in [28–35], we now introduce a subfamily $\mathcal{H}^{\lambda}_{p,q}(\alpha, \mu, \beta)$ of $\mathfrak{A}_p$ by using $\mathcal{J}_q^{\lambda+p-1}$ as follows:

Definition 1. *Let $f \in \mathfrak{A}_p$. Then, $f \in \mathcal{H}^{\lambda}_{p,q}(\alpha, \mu, \beta)$, if it satisfies the relation*

$$\left| \frac{z^{1-p}\mathcal{D}_q \mathcal{J}_q^{\lambda+p-1} f(z) - [p]_q}{2\beta \left[z^{1-p}\mathcal{D}_q \mathcal{J}_q^{\lambda+p-1} f(z) - \alpha [p]_q \right] - \left[z^{1-p}\mathcal{D}_q \mathcal{J}_q^{\lambda+p-1} f(z) - [p]_q \right]} \right| < \mu, \tag{9}$$

where $\frac{1}{2} \leqslant \beta < 1$, $0 \leqslant \alpha < \frac{1}{2}$, $0 < \mu \leqslant 1$ and $0 < q < 1$.

By varying the parameters values in the class $\mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$, we get many new classes; we list some of them.

(i). For $p=1$, we have $\mathcal{H}_{1,q}^{\lambda}(\alpha,\mu,\beta)\equiv\mathcal{H}_{q}^{\lambda}(\alpha,\mu,\beta)$.

(ii). Taking the limit $q\to 1^{-}$, we get the class $\mathcal{H}_{p}^{\lambda}(\alpha,\mu,\beta)$.

(iii). Putting $\beta=\frac{1}{2}$, $\mu=1$ and $\alpha=0$, we obtain $\mathcal{H}_{p,q}^{\lambda}\left(0,1,\frac{1}{2}\right)$.

(iv). Further, if we put $p=1$ and $q\to 1^{-}$ in $\mathcal{H}_{p,q}^{\lambda}\left(0,1,\frac{1}{2}\right)$, we have the class $\mathcal{H}_{1}^{\lambda}\left(0,1,\frac{1}{2}\right)$.

Note that we assume throughout our discussion, unless otherwise stated,

$$\frac{1}{2}\leq\beta\leq 1,\ 0\leq\alpha<1,\ 0<\mu\leq 1,\ \lambda>-p,\ 0<q<1$$

and all coefficients a_k are positive.

2. The Main Results and Their Consequences

Theorem 1. *If $f\in\mathfrak{A}_p$ has the form of Equation (1) and satisfies the inequality*

$$\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,(1+\mu\,(2\beta-1))\,|a_{n+p}|\leq 2\mu\beta\,[p]_q\,(1-\alpha)\,, \tag{10}$$

then $f\in\mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$.

Proof. To show that $f\in\mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$, we just need to prove Equation (9). For this, consider

$$\begin{aligned} L &= \left|\frac{\frac{1}{[p]_q}z^{1-p}\mathcal{D}_q\mathcal{J}_q^{\lambda+p-1}f(z)-1}{2\beta\left[\frac{1}{[p]_q}z^{1-p}\mathcal{D}_q\mathcal{J}_q^{\lambda+p-1}f(z)-\alpha\right]-\left[\frac{1}{[p]_q}z^{1-p}\mathcal{D}_q\mathcal{J}_q^{\lambda+p-1}f(z)-1\right]}\right| \\ &= \left|\frac{z^{1-p}\mathcal{D}_q\mathcal{J}_q^{\lambda+p-1}f(z)-[p]_q}{2\beta\left[z^{1-p}\mathcal{D}_q\mathcal{J}_q^{\lambda+p-1}f(z)-\alpha[p]_q\right]-\left[z^{1-p}\mathcal{D}_q\mathcal{J}_q^{\lambda+p-1}f(z)-[p]_q\right]}\right|. \end{aligned}$$

Using Equation (8) with the help of Equations (3) and (4), we can easily obtain

$$\begin{aligned} L &= \left|\frac{\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,a_{n+p}z^n}{2\beta\,[p]_q\,(1-\alpha)+2\beta\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,a_{n+p}z^n-\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,a_{n+p}z^n}\right| \\ &= \left|\frac{\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,a_{n+p}z^n}{2\beta\,[p]_q\,(1-\alpha)+(2\beta-1)\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,a_{n+p}z^n}\right| \\ &\leq \frac{\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,|a_{n+P}|}{2\beta\,[p]_q\,(1-\alpha)-(2\beta-1)\sum_{n=1}^{\infty}\Psi_{n-p}\,[n+p]_q\,|a_{n+P}|}<\mu, \end{aligned}$$

where we have used the inequality in Equation (10) and this completes the proof. □

Making $\beta=\frac{1}{2}$, $p=1$ along with $q\to 1^{-}$, we get the following result.

Corollary 1. *If $f \in \mathfrak{A}$ and satisfies the inequality*

$$\sum_{n=1}^{\infty} \Psi_{n-1} \ (n+1) \left|a_{n+1}\right| \leq \mu \left(1-\alpha\right),$$

then $f \in \mathcal{H}^{\lambda}\left(\alpha, \mu, \frac{1}{2}\right)$.

Theorem 2. *If $f \in \mathcal{H}^{\lambda}_{p,q}\left(\alpha, \mu, \beta\right)$ has the form of Equation (1), then*

$$r^p - \xi r \leq |f(z)| \leq r^p + \xi r, |z| = r < 1,$$

where

$$\xi = \frac{2\mu\beta\left[p\right]_q \left(1-\alpha\right)}{\left(1+\mu\left(2\beta-1\right)\right)\Psi_{1-p}\left[1+p\right]_q}.$$

Proof. Consider

$$\begin{aligned} |f(z)| &= \left| z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \right| \\ &\leq |z|^p + \sum_{n=1}^{\infty} \left|a_{n+p}\right| |z|^{n+p} \\ &= r^p + \sum_{n=1}^{\infty} \left|a_{n+p}\right| r^{n+p}. \end{aligned}$$

Since $0 < r < 1$ and $r^{n+p} < r$,

$$|f(z)| \leq r^p + r\sum_{n=1}^{\infty} \left|a_{n+p}\right|. \tag{11}$$

Similarly,

$$|f(z)| \geq r^p - r\sum_{n=1}^{\infty} \left|a_{n+p}\right|. \tag{12}$$

It can easily be seen that

$$\Psi_{1-p}\left(1+\mu\left(2\beta-1\right)\right)\left[1+p\right]_q \sum_{n=1}^{\infty} \left|a_{n+p}\right| \leq \sum_{n=1}^{\infty} \Psi_{n-p}\left(1+\mu\left(2\beta-1\right)\right)\left[n+p\right]_q \left|a_{n+p}\right|.$$

By using the relation in Equation (10), we obtain

$$\Psi_{1-p}\left(1+\mu\left(2\beta-1\right)\right)\left[1+p\right]_q \sum_{n=1}^{\infty} \left|a_{n+p}\right| \quad \leq \quad 2\mu\beta\left[p\right]_q \left(1-\alpha\right),$$

which gives

$$\sum_{n=1}^{\infty} \left|a_{n+p}\right| \quad \leq \quad \frac{2\mu\beta[p]_q(1-\alpha)}{\Psi_{1-p}(1+\mu(2\beta-1))[1+p]_q}.$$

Now, by using the above relation in Equations (11) and (12), we obtain the result. □

Setting $\beta = \frac{1}{2}$, $p = 1$ along with $q \to 1^-$ in the last theorem, we have

Corollary 2. *If $f \in \mathcal{H}^{\lambda}\left(\alpha, \mu, \frac{1}{2}\right)$, then for $|z| = r < 1$*

$$\left(1 - \frac{\mu(1-\alpha)}{2(\lambda+1)}\right) r \le |f(z)| \le \left(1 + \frac{\mu(1-\alpha)}{2(\lambda+1)}\right) r.$$

Theorem 3. *If $f \in \mathcal{H}^{\lambda}_{p,q}(\alpha, \mu, \beta)$ has the form of Equation (1), then*

$$[p]_q r^{p-1} - \vartheta r \quad \le \quad \left|\mathcal{D}_q f(z)\right| \quad \le \quad [p]_q r^{p-1} + \vartheta r, |z| = r < 1,$$

where $\vartheta = \frac{2\mu\beta[p]_q(1-\alpha)}{(1+\mu(2\beta-1))\Psi_{1-p}}$.

Proof. By using Equations (3) and (4), we can have

$$\mathcal{D}_q f(z) = [p]_q z^{p-1} + \sum_{n=1}^{\infty} [n+p]_q a_{n+p} z^{n+p-1}.$$

Since $|z|^{p-1} = r^{p-1} < 1$, $r^{n+p-1} \le r$ and

$$\left|\mathcal{D}_q f(z)\right| \le [p]_q r^{p-1} + r \sum_{n=1}^{\infty} [n+p]_q \left|a_{n+p}\right|. \tag{13}$$

Similarly,

$$\left|\mathcal{D}_q f(z)\right| \ge [p]_q r^{p-1} - r \sum_{n=1}^{\infty} [n+p]_q \left|a_{n+p}\right|. \tag{14}$$

Now, by using Equation (10), we get

$$\Psi_{1-p}(1+\mu(2\beta-1)) \sum_{n=1}^{\infty} [n+p]_q \left|a_{n+p}\right| \quad \le \quad \sum_{n=1}^{\infty} \Psi_{n-p} [n+p]_q (1+\mu(2\beta-1)) |a_{n+p}|.$$

This implies that

$$\sum_{n=1}^{\infty} [n+p]_q \left|a_{n+p}\right| \quad \le \quad \frac{2\mu\beta[p]_q(1-\alpha)}{(1+\mu(2\beta-1))\Psi_{1-p}}.$$

Finally, by using above relation in Equations (13) and (14), we have the result. □

For $q \to 1^-$, we have the following corollary.

Corollary 3. *If $f \in \mathcal{H}^{\lambda}_{p}(\alpha, \mu, \beta)$, then for $|z| = r < 1$*

$$pr^{p-1} - \vartheta r \le \left|f'(z)\right| \le pr^{p-1} + \vartheta r,$$

where

$$\vartheta = \frac{2\mu\beta(1-\alpha)p}{(1+\mu(2\beta-1))\Psi_{1-p}}.$$

Theorem 4. *If $f \in \mathcal{H}^{\lambda}_{p,q}(\alpha, \mu, \beta)$, then $f \in \mathcal{S}^*_p(\delta)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{(p-\delta)(1+\mu(2\beta-1))\Psi_{n-p}[n+p]_q}{2\mu\beta(1-\alpha)(\delta-p)[p]_q}\right)^{\frac{1}{n}}.$$

Proof. Let $f \in \mathcal{H}_{p,q}^{\lambda}(\alpha, \mu, \beta)$. To show that $f \in \mathcal{S}_p^*(\delta)$, we have to prove that

$$\left| \frac{zf'(z) - pf(z)}{zf'(z) + (p - 2\delta)f(z)} \right| < 1.$$

Using Equation (1), we conclude that

$$\sum_{n=1}^{\infty} \left(\frac{\delta - p}{p - \delta} \right) |a_{n+p}| \, |z|^n < 1. \tag{15}$$

From Equation (10), it can easily be obtained that

$$\sum_{n=1}^{\infty} \left(\frac{\Psi_{n-p}[n+p]_q (1+\mu(2\beta-1))|}{[p]_q 2\mu\beta(1-\alpha)} \right) |a_{n+p}| \quad < \quad 1.$$

The relation in Equation (15) is true, if the following holds

$$\sum_{n=1}^{\infty} \left(\frac{\delta-p}{p-\delta} \right) |a_{n+p}| \, |z|^n \quad < \quad \sum_{n=1}^{\infty} \left(\frac{\Psi_{n-p}[n+p]_q (1+\mu(2\beta-1))|}{[p]_q 2\mu\beta(1-\alpha)} \right) |a_{n+p}|,$$

which implies that

$$|z|^n < \frac{(p-\delta)(1+\mu(2\beta-1)) \Psi_{n-p} [n+p]_q}{2\mu\beta(1-\alpha)(\delta-p)[p]_q}.$$

Therefore,

$$|z| < \left(\frac{(p-\delta)(1+\mu(2\beta-1)) \Psi_{n-p} [n+p]_q}{2\mu\beta(1-\alpha)(\delta-p)[p]_q} \right)^{\frac{1}{n}} = r_1.$$

Hence, we get the required result. □

Letting $p = 1$ and $q \to 1^-$ in the last theorem, we get the result below.

Corollary 4. *If $f \in \mathcal{H}^{\lambda}(\alpha, \mu, \beta)$, then $f \in \mathcal{S}^*(\delta)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{(1-\delta)(1+\mu(2\beta-1)) [(n+1) \Psi_{n-1}}{2\mu\beta(1-\alpha)(\delta-1)} \right)^{\frac{1}{n}}.$$

Theorem 5. *If $f \in \mathcal{H}_{p,q}^{\lambda}(\alpha, \mu, \beta)$, then $f \in \mathcal{C}_p(\delta)$ for $|z| < r_2$, where*

$$r_2 = \left(\frac{p(p-\delta)(1+\mu(2\beta-1)) \Psi_{n-p} [n+p]_q}{2\mu\beta(1-\alpha)[p]_q (\delta-p)(n+p)} \right)^{\frac{1}{n}}.$$

Proof. Since $f \in \mathcal{C}_p(\delta)$,

$$\left| \frac{zf''(z) - (p-1) f'(z)}{zf''(z) + (1 - 2\delta + p) f'(z)} \right| < 1.$$

By using Equation (1) and after some simplifications, we get

$$\sum_{n=1}^{\infty} \left(\frac{(\delta-p)(n+p)}{p(p-\delta)} \right) |a_{n+p}| \, |z|^n < 1. \tag{16}$$

Now, from Equation (10), we can easily obtain that

$$\sum_{n=1}^{\infty} \left(\frac{(1+\mu(2\beta-1)) \Psi_{n-p}[n+p]_q}{2\mu\beta(1-\alpha)[p]_q} \right) |a_{n+p}| \quad < \quad 1.$$

The relation in Equation (16) is true if

$$\sum_{n=1}^{\infty}\left(\frac{(\delta-p)(n+p)}{p(p-\delta)}\right)\left|a_{n+p}\right||z|^{n} \quad < \quad \sum_{n=1}^{\infty}\left(\frac{(1+\mu(2\beta-1))\Psi_{n-p}[n+p]_q}{2\mu\beta(1-\alpha)[p]_q}\right)\left|a_{n+p}\right|,$$

which gives

$$|z|^{n}<\left(\frac{p\,(p-\delta)\,(1+\mu\,(2\beta-1))\,\Psi_{n-p}\,[n+p]_q}{2\mu\beta\,(1-\alpha)\,[p]_q\,(\delta-p)\,(n+p)}\right).$$

Hence,

$$|z|<\left(\frac{p\,(p-\delta)\,(1+\mu\,(2\beta-1))\,\Psi_{n-p}\,[n+p]_q}{2\mu\beta\,(1-\alpha)\,[p]_q\,(\delta-p)\,(n+p)}\right)^{\frac{1}{n}}=r_2.$$

Thus, we obtain the required result. □

Substituting $p=1$ and taking $q\to 1^{-}$ in the last theorem, we get the corollary below.

Corollary 5. *If $f\in\mathcal{H}^{\lambda}\,(\alpha,\mu,\beta)$, then $f\in\mathcal{C}\,(\delta)$ for $|z|<r_2$, where*

$$r_2=\left(\frac{(1-\delta)\,(1+\mu\,(2\beta-1))\,\Psi_{n-1}}{2\mu\beta\,(1-\alpha)\,(\delta-1)}\right)^{\frac{1}{n}}.$$

Theorem 6. *Let $f_p\,(z)=z^p$ and*

$$f_k\,(z)=z^p+\frac{2\mu\beta\,(1-\alpha)\,[p]_q}{(1+\mu\,(2\beta-1))\,\Psi_{k-2p}\,[k]_q\,a_k}z^k,\ \ (k\geq n+p)\,. \tag{17}$$

Then, $f\in\mathcal{H}^{\lambda}_{p,q}\,(\alpha,\mu,\beta)$, if and only if it can be expressed in the form

$$f\,(z)=\lambda_p z^p+\sum_{k=n+p}^{\infty}\lambda_k f_k\,(z)\,, \tag{18}$$

where $\lambda_p\geq 0$, $\lambda_k\geq 0$, $k\geq n+p$ and $\lambda_p+\sum\limits_{k=n+p}^{\infty}\lambda_k=1$.

Proof. We suppose that f can be written of the form of Equation (18), thus

$$\begin{aligned} f\,(z) &= \lambda_p z^p+\sum_{k=n+p}^{\infty}\lambda_k f_k\,(z)\,,\\ &= \lambda_p z^p+\sum_{k=n+p}^{\infty}\lambda_k\left(z^p+\frac{2\mu\beta\,(1-\alpha)\,[p]_q}{(1+\mu\,(2\beta-1))\,\Psi_{k-2p}\,[k]_q\,a_k}z^k\right)\\ &= z^p+\sum_{k=n+p}^{\infty}\frac{2\mu\beta\,(1-\alpha)\,[p]_q\,\lambda_k}{(1+\mu\,(2\beta-1))\,\Psi_{k-2p}\,[k]_q\,a_k}z^k. \end{aligned}$$

This implies that

$$\begin{aligned} &\sum_{k=n+p}^{\infty}\frac{(1+\mu(2\beta-1))\Psi_{k-2p}[k]_q a_k}{2\mu\beta(1-\alpha)[p]_q}\times\frac{2\mu\beta(1-\alpha)[p]_q}{(1+\mu(2\beta-1))\Psi_{k-2p}[k]_q a_k}\lambda_k\\ &=\sum_{k=n+p}^{\infty}\lambda_k,\\ &=1-\lambda_p\leq 1. \end{aligned}$$

Conversely, we suppose that $f_n \in \mathcal{H}_{p,q}^{\lambda}(\alpha, \mu, \beta)$. Then, by using Equation (10), we have

$$|a_k| \leq \frac{2\mu\beta(1-\alpha)[p]_q}{(1+\mu(2\beta-1))\Psi_{k-2p}[k]_q}, \quad k \geq p+n.$$

By setting

$$\lambda_k = \frac{(1+\mu(2\beta-1))\Psi_{k-2p}[k]_q}{2\mu\beta(1-\alpha)[p]_q} a_k,$$

then

$$\begin{aligned} f(z) &= z^p + \sum_{k=n+p}^{\infty} a_k z^k \\ &= z^p + \sum_{k=n+p}^{\infty} \frac{2\mu\beta(1-\alpha)[p]_q}{(1+\mu(2\beta-1))\Psi_{k-2p}[k]_q a_k} \lambda_k z^k \\ &= z^p + \sum_{k=n+p}^{\infty} [z^p - f_k(z)]\lambda_k \\ &= \left(1 - \sum_{k=n+p}^{\infty} \lambda_k\right) z^p + \sum_{k=n+p}^{\infty} \lambda_k f_k(z) \\ &= \lambda_p z^p + \sum_{k=n+p}^{\infty} \lambda_k f_k(z). \end{aligned}$$

This complete the result. □

Putting $p = 1$, $\beta = \frac{1}{2}$ and $q \to 1^-$ in the above theorem, we obtain the upcoming corollary.

Corollary 6. *Let $f(z) = z$ and*

$$f_k(z) = z + \frac{\mu(1-\alpha)}{k\Psi_{k-2}a_k} z^k, \quad (k \geq n+1). \tag{19}$$

Then, $f \in \mathcal{H}_p^*\left(\alpha, \mu, \frac{1}{2}\right)$, if and only if it can be expressed in the form

$$f(z) = \lambda z + \sum_{k=n+1}^{\infty} \lambda_k f_k(z), \tag{20}$$

where $\lambda \geq 0$, $\lambda_k \geq 0$, $k \geq n+1$ and $\lambda + \sum_{k=n+1}^{\infty} \lambda_k = 1$.

Theorem 7. *If $f, g \in \mathcal{H}_{p,q}^{\lambda}(\alpha, \mu, \beta)$, then $(f * g) \in \mathcal{H}_{p,q}^{\lambda}(\alpha, \mu, \beta)$, where*

$$l \geq \mu\left(\frac{2l\beta(1-\alpha)[p]_q}{\Psi_{n-p}[n+p]_q} - l(2\beta-1)\right). \tag{21}$$

Proof. We have to find largest μ such that

$$\sum_{n=1}^{\infty} \frac{(1+\mu(2\beta-1))\Psi_{n-p}[n+p]_q}{2\mu\beta(1-\alpha)[p]_q} a_{n+p} b_{n+p} \leq 1.$$

Let $f,\ g\in\mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$. Then, using Equation (10), we obtain

$$\sum_{n=1}^{\infty}\frac{(1+l(2\beta-1))\Psi_{n-p}[n+p]_q}{2l\beta(1-\alpha)[p]_q}a_{n+p}\le 1 \tag{22}$$

and

$$\sum_{n=1}^{\infty}\frac{(1+l(2\beta-1))\Psi_{n-p}[n+p]_q}{2l\beta(1-\alpha)[p]_q}b_{n+p}\le 1. \tag{23}$$

By using Cauchy–Schwarz inequality, we have

$$\sum_{n=1}^{\infty}\frac{(1+l(2\beta-1))\Psi_{n-p}[n+p]_q}{2l\beta(1-\alpha)[p]_q}\sqrt{a_{n+p}b_{n+p}}\le 1. \tag{24}$$

Thus, we have to show that

$$\begin{aligned}&\frac{(1+l(2\beta-1))\Psi_{n-p}[n+p]_q}{2l\beta(1-\alpha)[p]_q}a_{n+p}b_{n+p}\\ \le\ &\frac{(1+\mu(2\beta-1))\Psi_{n-p}[n+p]_q}{2\mu\beta(1-\alpha)[p]_q}\sqrt{a_{n+p}b_{n+p}},\end{aligned}$$

that is

$$\sqrt{a_{n+p}b_{n+p}}\le\frac{l(1+\mu(2\beta-1))}{\mu(1+l(2\beta-1))}. \tag{25}$$

In addition, from Equation (24), we can write

$$\sqrt{a_{n+p}b_{n+p}}\le\frac{2l\beta(1-\alpha)[p]_q}{(1+l(2\beta-1))\Psi_{n-p}[n+p]_q}. \tag{26}$$

Consequently, we have to show that

$$\frac{2l\beta(1-\alpha)[p]_q}{(1+l(2\beta-1))\Psi_{n-p}[n+p]_q}\le\frac{l(1+\mu(2\beta-1))}{\mu(1+l(2\beta-1))}.$$

By simple calculation, we get

$$l\ge\mu\left(\frac{2l\beta(1-\alpha)[p]_q}{\Psi_{n-p}[n+p]_q}-l(2\beta-1)\right),$$

which completes the required result. □

Theorem 8. *Let f_1 and f_2 be in the class $\mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$. Then, the weighted mean w_q of f_1 and f_2 is also in the class $\mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$.*

Proof. From the definition of weighted mean, we have

$$\begin{aligned}w_q &= \frac{1}{2}[(1-q)f_1(z)+(1+q)f_2(z)]\\ &= \left[(1-q)\left(z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\right)+(1+q)\left(z^p+\sum_{n=1}^{\infty}b_{n+p}z^{n+p}\right)\right]\\ &= z^p+\sum_{n=1}^{\infty}\frac{1}{2}\left[(1-q)a_{n+p}+(1+q)b_{n+p}\right]z^{n+p}.\end{aligned}$$

Now, using Equation (10), we have

$$\sum_{n=1}^{n} \Psi_{n-p} [n+p]_q (1+\mu(2\beta-1)) a_{n+p} \leq 2\mu\beta [p]_q (1-\alpha)$$

and

$$\sum_{n=1}^{n} \Psi_{n-p} [n+p]_q (1+\mu(1-2\beta)) b_{n+p} \leq 2\mu\beta [p]_q (1-\alpha).$$

Consider

$$\begin{aligned}
& \left[\sum_{n=1}^{\infty} \Psi_{n-p} [n+p]_q (1+\mu(2\beta-1))\right] \left[\frac{1}{2}\left[(1-q) a_{n+p} + (1+q) b_{n+p}\right]\right] \\
= \; & \frac{1}{2}(1-q) \sum_{n=1}^{\infty} \Psi_{n-p} [n+p]_q (1+\mu(2\beta-1)) a_{n+p} \\
& + \frac{1}{2}(1+q) \sum_{n=1}^{\infty} \Psi_{n-p} [n+p]_q (1+\mu(2\beta-1)) b_{n+p} \\
\leq \; & \frac{1}{2}(1-q) 2\mu\beta [p]_q (1-\alpha) + \frac{1}{2}(1+q) 2\mu\beta [p]_q (1-\alpha).
\end{aligned}$$

This shows that $w_q \in \mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$. □

3. Applications

The q-calculus has played an important role in the study of almost every branch of mathematics and physics, for example, in the theory of special functions, differential equations, combinatorics, analytic number theory, quantum theory, quantum group, special polynomials, numerical analysis, operator theory and other related theories. Quantum calculus is considered as one of the most active research areas in mathematics and physics. For more details, please refer to [22,36–41].

4. Concluding Remarks and Observations

In this paper, we introduce a new integral operator $\mathcal{J}_q^{\lambda+p-1}$ in q-analog and define the class $\mathcal{H}_{p,q}^{\lambda}(\alpha,\mu,\beta)$ of multivalent functions by using this operator. Several useful properties such as sufficiency criteria, distortion bounds, radius of starlikness and radius of convexity, extreme points, weighted mean and partial sum for this newly defined subclass of multivalent functions are investigated. In addition, we observe that, if we take some suitable parameters p, q, α, μ, β in the results involved, we get the corresponding properties for q-analog of differential and integral operators mentioned in the Introduction.

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